Abel Differential

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Quad-Mesh

The number of singularities, and the layouts of separatrices are different.

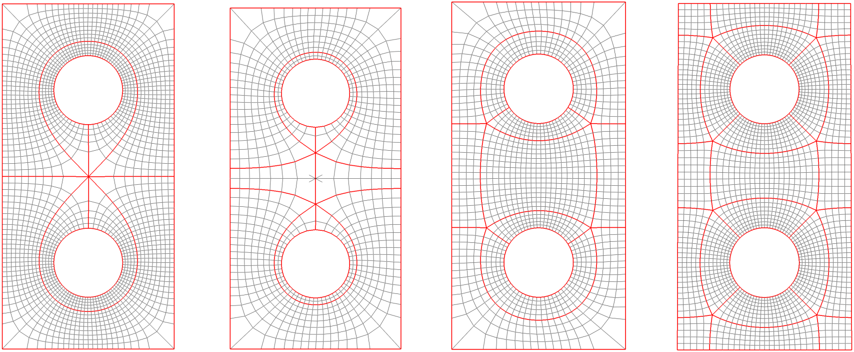

Figure: Quad-meshes with different number of singularities.

The number of singularities, and the layouts of separatrices are different.

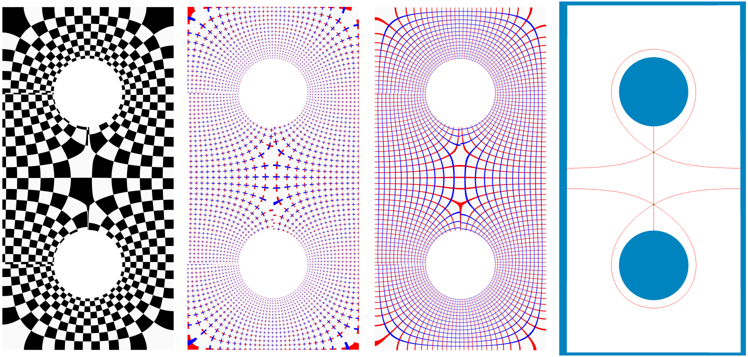

Figure: A quad-mesh induced by a holomorphic quadratic differential.

Quad-Meshes

Aim

Establish complete mathematical theory for structural mesh.

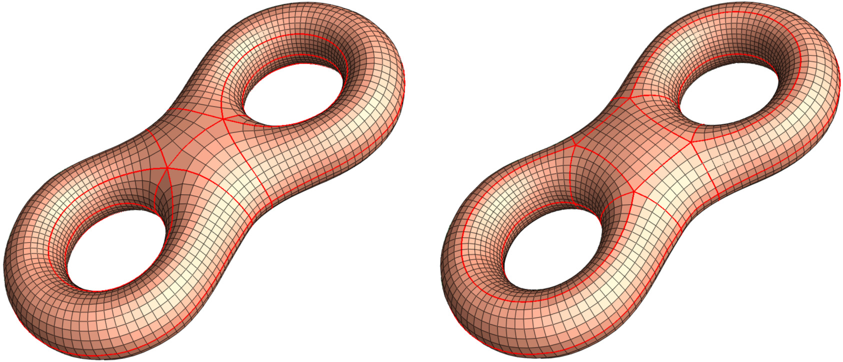

Figure: A quad-mesh of a genus two surface with different number of singularities.

Hodge Theory and Abel Differential Theory

Smooth Manifold

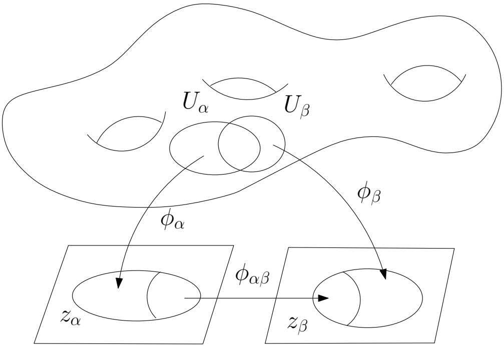

Figure: A manifold.

Smooth Manifold

Definition (Manifold)

A manifold is a topological space M covered by a set of open sets $\{U_\alpha\}$. A homeomorphism $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ maps U_α to the Euclidean space $\mathbb{R}^n$. (U_α, ϕ_α) is called a coordinate chart of M. The set of all charts $\{(U_\alpha, \phi_\alpha)\}$ form the atlas of M. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a transition map.

Riemann Surface

Definition (Riemann Surface)

A two dimensional manifold S is a Riemann surface, if the chart transition maps

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

are biholomorphic. On each local chart $(U_\alpha, \varphi_\alpha)$, we use z_α to denote the local complex coordinate. The atlas $\{(U_\alpha, z_\alpha)\}$ is called a conformal structure of the surface S.

Holomorphic Differential

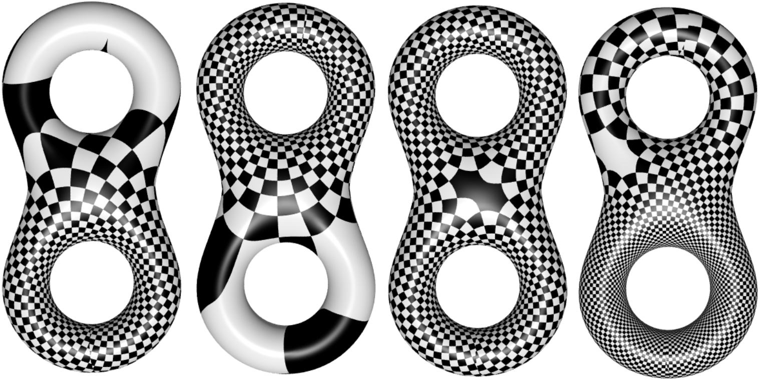

Figure: Holomorphic 1-form on a genus two surface.

Riemann Surface

Definition (Meromorphic Differential)

Suppose S is a Riemann surface with a conformal structure $\{(U_\alpha, z_\alpha)\}$, a complex differential 1-form ω is called a meromorphic (holomorphic) 1-form (meromorphic differential), if on each local chart $(U_\alpha, \varphi_\alpha)$, its local representation is

$$\omega = f_\alpha(z_\alpha) dz_\alpha,$$

where f_α is a meromorphic (holomorphic) function, and on the other chart $\omega = f_\beta(z_\beta) dz_\beta$,

$$f_\alpha(z_\alpha) = f_\beta(z_\beta(z_\alpha)) \frac{dz_\beta}{dz_\alpha}.$$

The zeros and poles of ω are those of f_α's.

All the meromrophic (holomorhic) 1-forms on C is denoted as $K^1(C)(\Omega^1(C))$.

Residue Theorem

Definition (Residue)

Let C be a Riemann surface, $\omega \in K^1(C)$, $p \in C$, γ_p is a small circle around the point p, ω has no other pole except p (p itself may be or may be not a pole). Then the residue of ω at p is defined as

$$\text{Res}_p(\omega) = \frac{1}{2\pi i} \oint_{\gamma_p} \omega.$$

Locally, $p \in U_j$, $\gamma_p \subset U_j$, we have

$$\text{Res}_p(\omega) = \frac{1}{2\pi i} \oint_{\gamma_p} \omega = \frac{1}{2\pi i} \oint f_j(z_j) dz_j = \text{Res}_p(f_j(z_j) dz_j).$$

Residue Theorem

Theorem (Residue)

Suppose C is a compact Riemann surface, for $\omega \in K^1(C)$, we have

$$\sum_{p\in C} Res_p(\omega) = 0.$$

Proof.

Since C is compact, ω has finite number of poles on C, denoted as $p_1, p_2, \dots, p_m$. Choose small disks $\Delta_1, \Delta_2, \dots, \Delta_m$ surrounding these poles. Denote

$$\Omega = C \setminus \bigcup_i \Delta_i, \quad \partial\Omega = -\bigcup_i \partial\Delta_i.$$

By Stokes, we have

$$2\pi i \sum_{p\in C} \text{Res}_p(\omega) = 2\pi i \sum_{j=1}^{m} \text{Res}_{p_j}(\omega) = \sum_{j=1}^{m} \int_{\partial\Delta_j} \omega = -\int_{\partial\Omega} \omega = -\int_{\Omega} d\omega = 0.$$

Abel Differential

Definition (Measurable Differential)

A measurable differential form ω on a Riemann surface has local representation

$$\omega = u(z)dz + v(z)d\bar{z},$$

where $u(z)$ and $v(z)$ are Lebegue measurable functions.

Definition (Hodge Star)

Given a measurable differential $\omega = u(z)dz + v(z)d\bar{z}$, the Hodge star (conjugate) of ω is defined as

$$^*\omega = -iu(z)dz + iv(z)d\bar{z}.$$

Abelian Differential

Definition (Abelian Differential)

A differential ω on the Riemann surface is called an Abelian differential of the first kind, if it is a holomorphic differential.

Definition (Abelian Differential)

A meromorphic differential ω on the Riemann surface is called an Abelian differential of the second kind, if the residues at all poles are equal to zero. A memorphic diffrential with non-zero residues is called an Abelian differential of the third kind.

Abel Differential

Definition (Inner Product)

Given a measurable differential ω on a Riemann surface C, the norm of ω is defined as

$$\|\omega\|^2 := (\omega, \omega) = \int\int_C \omega \wedge \overline{{}^*\omega}.$$

Note that $\overline{{}^*\omega} = {}^*\overline{\omega}$,

$$\begin{aligned} \omega \wedge \overline{{}^*\omega} &= (udz + vd\bar{z}) \wedge \overline{{}^*(udz + vd\bar{z})} \\ &= (udz + vd\bar{z}) \wedge \overline{(-iudz + ivd\bar{z})} \\ &= i(u\bar{u} + v\bar{v}) dz \wedge d\bar{z} \\ &= 2(u^2 + v^2) dx \wedge dy. \end{aligned}$$

Abel Differential

Definition (Measurable Differential Space)

The measurable differential space on the Riemann surface C is defined as

$$L^2(C) := \{\omega : \text{measurable } \|\omega\|^2 < \infty\}.$$

$L^2(C)$ is a linear space, with a norm

$$\|\omega\| = \sqrt{(\omega, \omega)}.$$

Definition (Inner Product)

For any $\omega_1, \omega_2 \in L^2(C)$, $\omega_1 = u_1 dz + v_1 d\bar{z}$, $\omega_2 = u_2 dz + v_2 d\bar{z}$, define inner product as

$$(\omega_1, \omega_2) = \int\int_C \omega_1 \wedge \overline{{}^*\omega_2} = i \int\int_C (u_1 \bar{u}_2 + v_1 \bar{v}_2) dz \wedge d\bar{z},$$

$L^2(C)$ is a Hilbert space.

Properties of Inner Product in $L^2(C)$

$$(\omega_1,\omega_2) = \int\int_C \omega_1 \wedge \overline{{}^*\omega_2} = \overline{\int\int_C \omega_2 \wedge {}^*\omega_1} = \overline{(\omega_2,\omega_1)}$$

$$({}^*\omega_1, {}^*\omega_2) = \overline{\int\int {}^*\omega \wedge -\overline{\omega_2}} = \int\int \overline{\omega \wedge \overline{{}^*\omega_1}} = \overline{(\omega_2,\omega_1)} = (\omega_1,\omega_2)$$

Hodge Decomposition

Definition

Define subspaces E and E^* as the closure in $L^2(C)$:

$$E := \overline{\{df : f \in C_0^\infty(C)\}}$$

and

$$E^* := \overline{\{{}^*df : f \in C_0^\infty(C)\}}.$$

Namely, $\omega \in E$ if and only there is a sequence $\{f_n \in C_0^\infty(C)\}$, such that

$$\lim_{n\to\infty} \|\omega - df_n\| = 0.$$

By $\|\omega - df_n\| = \|{}^*\omega - {}^*df_n\|$, we obtain $\omega \in E$ iff ${}^*\omega \in E^*$.

Hodge Decomposition

Definition (Orthogonal Complementary Space)

Define the orthogonal complementary subspaces $E^\perp$ and $(E^*)^\perp$ as :

$$E^\perp := \{\omega \in L^2(C) : (\omega, \varphi) = 0, \forall \varphi \in E\}$$

and

$$(E^*)^\perp := \{\omega \in L^2(C) : (\omega, \varphi) = 0, \forall \varphi \in E^*\}$$

Definition (Harmonic Differential Space)

Define the harmonic differential subspace as :

$$H := E^\perp \cap (E^*)^\perp,$$

namely

$$H = \{\omega \in L^2(C) : (, df) = 0,\ (\omega, {}^*df) = 0, \forall f \in C_0^\infty(C)\}.$$

Hodge Decomposition

Theorem (Hodge Decomposition)

E, E^* and H are pairwise orthogonal, and with decomposition

$$L^2(C) = E \oplus E^* \oplus H. \quad (1)$$

Proof.

First, we show $E \perp E^*$. Suppose $\gamma \in E$, $\pi \in E^*$, by definition, there is a sequence $f_n, g_n \in C_0^\infty(C)$, such that in $L^2(C)$,

$$\lim_{n\to\infty} df_n = \gamma, \quad \lim_{n\to\infty} {}^*dg_n = \pi.$$

By the continuity of the inner product

$$(\gamma, \pi) = \lim_{n\to\infty} (df_n, {}^*dg_n) = \lim_{n\to\infty} \int\int_C df_n \wedge \overline{{}^{**}dg_n} = -\lim_{n\to\infty} \int\int_{G_n} df_n \wedge d\bar{g}_n$$

□

Hodge Decomposition

Proof.

$$
\begin{aligned}
(\gamma, \pi) &= -\lim_{n \to \infty} \int\int_{G_n} df_n \wedge d\bar{g}_n \\
&= -\lim_{n \to \infty} \left(\int_{\partial G_n} f_n d\bar{g}_n - \int\int_{G_n} f_n d^2 \bar{g}_n \right) = 0
\end{aligned}
$$

where we use Stokes, $G_n \subset W$ are relatively compact domain and $f_n = 0$, $g_n = 0$ on the boundary ∂G_n. Hence $E \perp E^*$.
Because $E \oplus E^*$ is a sublinear space of $L^2(C)$, we have the decomposition

$$
L^2(C) = E \oplus E^* \oplus (E \oplus E^*)^{\perp}.
$$

Next, we want to prove

$$
(E \oplus E^*)^{\perp} = E^{\perp} \cap (E^*)^{\perp}.
$$

Hodge Decomposition

Proof.

Next, we want to prove

$$(E \oplus E^*)^\perp = E^\perp \cap (E^*)^\perp.$$

If $\omega \in E^\perp \cap (E^*)^\perp$, then for any $\gamma \in E$ and $\pi \in E^*$, we have

$$(\omega, \gamma + \pi) = (\omega, \gamma) + (\omega, \pi) = 0, \tag{2}$$

therefore $\omega \in (E \oplus E^*)^\perp$, namely $E^\perp \cap (E^*)^\perp \subset (E \oplus E^*)^\perp$. Reversely, if $\omega \in (E \oplus E^*)^\perp$, then for any $\gamma \in E$ and $\pi \in E^*$, we have $(\omega, \gamma + \pi) = 0$, especially

$$(\omega, \gamma) = 0, \quad (\omega, \pi) = 0,$$

hence $\omega \in E^\perp$ and $\omega \in (E^*)^\perp$, then $\omega \in E^\perp \cap (E^*)^\perp$, namely $(E \oplus E^*)^\perp \subset E^\perp \cap (E^*)^\perp$. Therefore Eqn. (2) holds. □

Hodge Decomposition

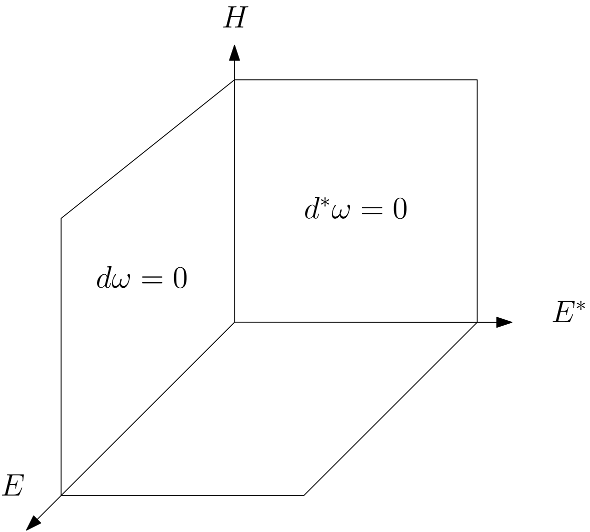

Figure: Hodge Decomposition.

Hodge Decomposition

Lemma

Suppose $\omega \in C^1(C)$*, then*

1. $\omega \in (E^*)^\perp \iff d\omega = 0$*, namely* ω *is closed;*
2. $\omega \in E^\perp \iff d^*\omega = 0$*, namely* ω *is co-closed.*

Proof.

$\omega \in (E^*)^\perp$ if and only if for any $f \in C_0^\infty(C)$, we have $(\omega, {}^*df) = 0$, namely

$$(\omega, {}^*df) = -\int\int_C \omega \wedge d\bar{f} = \int\int_C \bar{f} d\omega = 0,$$

because $f \in C_0^\infty(C)$ is arbitrary, hence we must have $d\omega = 0$. □

Hodge Decomposition

Theorem (Harmonic Differential)

Suppose $\omega \in C^1(C)$, *then*

$$\omega \in H \iff d\omega = 0, \quad d^*\omega = 0,$$

namely ω *is a harmonic differential (derivative of a harmonic function).*

Regularity Theory

Mollifier

Definition (Mollifier)

Suppose a function

$$\chi(z) = \chi(|z|) := \begin{cases} \frac{1}{k} e^{-\frac{1}{1-|z|^2}}, & |z| < 1 \\ 0, & |z| \geq 1 \end{cases}$$

$\chi(z) \in C_0^\infty(\mathbb{C})$, $\chi(z) > 0$ on the unit disk $D = \{|z| < 1\}$, k is chosen such that

$$\int \int_{\mathbb{C}} \chi(z) d\sigma_z = 1,$$

where $d\sigma_z = dxdy$.

Mollifier

Definition (Smoothing Operator)

For any $\varepsilon > 0$, let

$$\chi_\varepsilon(z) = \frac{1}{\varepsilon^2}\chi\left(\frac{z}{\varepsilon}\right)$$

$\chi_\varepsilon(z) \in C_0^\infty(\mathbb{C})$, $\chi_\varepsilon(z) > 0$ on the unit disk $D_\varepsilon = \{|z| < \varepsilon\}$ and

$$\int\int_{\mathbb{C}} \chi_\varepsilon(z) dxdy = 1.$$

For $f \in L^2(D)$, $f = 0$ outside the unit disk D_1, define the convolution

$$(M_\varepsilon f)(z) = \int\int_C f(\zeta)\chi_\varepsilon(\zeta - z) d\sigma_\zeta.$$

$(M_\varepsilon f)$ is zero outside $D_{1+\varepsilon}$, or equivalently

$$(M_\varepsilon f)(z) = \int\int_C f(z+\zeta)\chi_\varepsilon(\zeta) d\sigma_\zeta.$$

Mollifier

Lemma (Mollifier)

1. $(M_\varepsilon f) \in C_0^\infty(\mathbb{C})$;
2. *if* $f \in C^1(D)$, *then on* $D_{1-\varepsilon} = \{|z| < 1-\varepsilon\}$, *we have*
$$\frac{\partial M_\varepsilon f}{\partial x} = M_\varepsilon\left(\frac{\partial f}{\partial x}\right), \frac{\partial M_\varepsilon f}{\partial y} = M_\varepsilon\left(\frac{\partial f}{\partial y}\right).$$
3. *when* $\varepsilon \to 0$, $\|M_\varepsilon f - f\|_{L^2(D)} \to 0$.
4. *if* f *is harmonic on* D_1, *then on* $D_{1-\varepsilon}$, $M_\varepsilon f = f$.
5. *for any* $\varphi \in L^2(D)$, φ *is zero outside* D_1, *then*
$$\int\int_D (M_\varepsilon f)\varphi d\sigma_z = \int\int_D f(M_\varepsilon \varphi) d\sigma_z$$
6. $M_\delta M_\varepsilon f = M_\varepsilon M_\delta f$.

Mollifier

Definition (Mollifier for Differentials)

Suppose the differential $\omega \in L^2(D)$ is defined on D, assume

$$\omega = p(z)dx + q(z)dy,$$

where $p, q \in L^2(D)$, define

$$M_\varepsilon \omega = (M_\varepsilon p)dx + (M_\varepsilon q)dy.$$

Mollifier

Lemma (Mollifier for Differentials)

1. $(M_\varepsilon \omega)$ *is a* C_0^∞ *differential, zero outside* $D_{1+\varepsilon}$;
2. *if* ω *is a* C^1 *differential, then*
$$dM_\varepsilon \omega = M_\varepsilon d\omega.$$
3. *when* $\varepsilon \to 0$, $\|M_\varepsilon \omega - \omega\|_{L^2(D)} \to 0$.
4. *if* ω *is a harmonic differential, then* $M_\varepsilon \omega = \omega$.
5. *for any differential* $\gamma \in L^2(D)$, γ *is zero outside* $D_{1-\varepsilon}$, *then*
$$(M_\varepsilon \omega, \gamma)_{L^2(D)} = (\omega, M_\varepsilon \gamma)_{L^2(D)}.$$
6. *on* D $M_\delta M_\varepsilon \omega = M_\varepsilon M_\delta \omega$.

Weyl's Lemma

Weyl's lemma shows weak solutions to the elliptic differential operators are classical solutions.

Lemma (Weyl)

Suppose $\omega \in L^2(D)$, $D = \{|z| < 1\}$, *and for any* $f \in C_0^\infty(D)$, *we have*

$$(\omega, df)_{L^2(D)} = (\omega, d^*\omega)_{L^2(D)} = 0,$$

then ω *is (possibly after modification on an measure zero set) automatically in* $C^1(D)$, *hence* ω *is a harmonic 1-form.*

Weyl's Lemma

Proof.

Consider $M_\varepsilon\omega$, by lemma of millifier on differentials, we have

$$(M_\varepsilon\omega, df)_{L^2(D)} = (\omega, M_\varepsilon df)_{L^2(D)} = (\omega, dM_\varepsilon f)_{L^2(D)} = 0,$$

$$(M_\varepsilon\omega, d^*f)_{L^2(D)} = (\omega, M_\varepsilon d^*f)_{L^2(D)} = (\omega, d^*M_\varepsilon f)_{L^2(D)} = 0,$$

$M_\varepsilon\omega$ is C_0^∞, $M_\varepsilon\omega \in E^\perp \cap (E^*)^\perp$, $M_\varepsilon\omega$ is a harmonic differential. By the millifier lemma 4),

$$M_\delta M_\varepsilon\omega = M_\varepsilon\omega, \quad M_\varepsilon M_\delta\omega = M_\delta\omega$$

by lemma 6)

$$M_\delta M_\varepsilon\omega = M_\varepsilon M_\delta\omega \implies M_\delta\omega = M_\varepsilon\omega.$$

□

Weyl's Lemma

Proof.

By lemma 3), when $\varepsilon < \delta$, $\varepsilon \to 0$,

$$\|M_\delta \omega - \omega\|_{L^2(D)} = \|M_\varepsilon \omega - \omega\|_{L^2(D)} \to 0.$$

therefore $\|M_\delta \omega - \omega\|_{L^2(D)} = 0$. Therefore on D, $\omega = M_\delta \omega$ almost everywhere, therefore possibly after modification on an measure zero set ω is a $C^1(D)$ differential. □

Weyl's Lemma

Theorem

The space

$$H = E^{\perp} \cap (E^{*})^{\perp}$$

is the space of harmonic differentials.

Proof.

Suppose $\omega \in H$, on any local parameter disk V, the local parameter map $\varphi(p) = z$ maps V to the disk $D = \{|z| < 1\}$, for any $f \in C_0^{\infty}(V)$, in $L^2(D)$ we have

$$(\omega, df) = (\omega, d^{*}f) = 0.$$

By Weyl's lemma, ω is harmonic (in the classical sense, ω is C_0^{∞}) in V, hence ω is harmonic on the whole Riemann surface C. □

Hodge Decomposition

Theorem (Hodge Decomposition)

For any $\omega \in C^3(C) \cap L^2(C)$, ω *has a unique decomposition*

$$\omega = \omega_h + df + {}^*dg$$

where $f, g \in C^2(C)$, $\omega_h \in H$ *and* $d^*g \in E^*$.

Abel Differential of The Third Type

Theorem

Given a Riemann surface C, there is a differential ω satisfying

1. ω *is harmonic on* $C - \{q_0, q_1\}$;
2. *on a local parameter disk* D,
$$\omega - d\log\frac{z-a}{z-b}$$
is harmonic, where z *is the local parameter;*
3. *for any* $h \in C_0^\infty(C)$, $h = 0$ *outside* D, *we have*
$$(\omega, dh) = (\omega, d^*h) = 0,$$
4. ω *is an exact harmonic differential on* $C - D$, *and* $\omega - d\left(\log\frac{z-a}{z-b}\right)$ *is an exact harmonic differential on* D.

Abel Differential of The Third Type

proof: Construct a $C_0^4(D)$ function,

$$e(z) = \begin{cases} 1 & |z| \leq r, \\ e^{\frac{1}{(r_1-r)^6} - \frac{1}{(r_1-r)^6-(|z|-r)^6}} & r < |z| < r_1, \\ 0 & |z| \geq r_1. \end{cases}$$

construct a differential

$$\psi(p) = \begin{cases} d\left(e(z)\log\frac{z-a}{z-b}\right), & p \in D_1, z = z(p), \\ 0 & p \notin D_1. \end{cases}$$

Then ψ is C_0^3 on $C - \{q_0, q_1\}$, and on D

$$\psi = d\left(\log\frac{z-a}{z-b}\right) = \frac{dz}{z-a} - \frac{dz}{z-b},$$

ψ is holomorphic on $D - \{q_0, q_1\}$, therefore ${}^*\psi = -i\psi$, namely $i^*\psi = \psi$.

Proof.

Therefore $\psi - i^*\psi = 0$ on D, therefore $\psi - i^*\psi$ is a differential on C, and $\psi - i^*\psi \in C_0^3(C) \cap L^2(C)$. By Hodge decomposition,

$$\psi - i^*\psi = \omega_h + df + {}^*dg,$$

where $\omega_h \in H$, $f, g \in C^2(C)$, $df \in E$, $d^*g \in E^*$. Define

$$\omega := \psi - df = i^*\psi + \omega_h + {}^*dg.$$

then ω satisfies all the conditions:
1. From $\omega \in C^1(C - \{q_0, q_1\})$ and ψ is exact on $C - \{q_0, q_1\})$, we know $\omega = \psi - df$ is exact on $C - \{q_0, q_1\}$, $d\omega = 0$, and

$$d^*\omega = d^*i^*\psi + d^*\omega_h + d^{**}dg = -id\psi + d^*\omega_h - d^2g = 0.$$

therefore ω is harmonic on $C - \{q_0, q_1\}$. □

Proof.

2. On D,

$$\psi = i^*\psi = d\left(\log\frac{z-a}{z-b}\right),$$

then

$$\omega - d\left(\log\frac{z-a}{z-b}\right) = -df = \omega_h + {}^*dg,$$

Hence

$$d\left(\omega - d\left(\log\frac{z-a}{z-b}\right)\right) = -d^2 f = 0.$$

$$d^*\left(\omega - d\left(\log\frac{z-a}{z-b}\right)\right) = d^*\omega_h + d^{**}dg = 0.$$

therefore $\omega - d\left(\log\frac{z-a}{z-b}\right)$ is harmonic on $C - \{q_0, q_1\}$. □

Abel Differential of The Third Type

Proof.

3. Suppose $h \in C_0^\infty(C)$, $h = 0$ in a neighborhood of q_0 and a neighborhood of q_1, then

$$(\omega, dh) = i({}^*\psi, dh) + (\omega_h, dh) + (* dg, dh) = 0,$$

because $dh \in E$, $H \perp E$, $(\omega_h, dh) = 0$; $E \perp E^*$, $({}^*dg, dh) = 0$. By Stokes

$$({}^*\psi, dh) = -\int\int \psi \wedge d\bar{h} = \int\int \bar{h} d\psi = 0.$$

Similarly, we can obtain

$$(\omega, {}^*dh) = (\psi, {}^*dh) + (df, {}^*dh) = 0.$$

□

Abel Differential of The Third Type

Theorem

Suppose q_0 and q_1 are two points on a Riemann surface C, then there exists a meromorphic differential ω with q_0 and q_1 as holes, the singular part at q_0 is $\frac{dz}{z}$, and that at q_1 is $-\frac{dz}{z}$.

Proof.

Draw a path $\sigma : [0,1] \to C$ connecting q_0 and q_1, $\sigma(0) = q_0$, $\sigma(1) = q_1$. Define a subdivision of σ:

$$[0,1] = \cup_{i=0}^{n} [t_i, t_{i+1}], t_0 = 0, t_i < t_{i+1}, t_{n+1} = 1.$$

such that $\sigma([t_i, t_{i+1}]$ is contained by a parameter disk D_i. For every $i (0 \leq i \leq n)$, there is a meromorphic differential ω_i with poles at $\sigma(t_i)$ and $\sigma(t_{i+1})$, and singular parts $\frac{dz}{z}$ and $-\frac{dz}{z}$ respectively. Then let

$$\omega = \omega_1 + \omega_2 + \cdots + \omega_n.$$

Abelian Differential of the First Type

Algorithm

Input: A genus g closed triangle mesh M;

Output: Holomorphic 1-form group basis $\{\omega_1, \omega_2, \cdots, \omega_g\}$;

1. Compute the homology group basis $H_1(M, \mathbb{Z})$;
$$\{\gamma_1, \gamma_2, \cdots, \gamma_{2g}\}$$
2. Compute the dual cohomology graph basis $H^1(M, \mathbb{R})$;
$$\{\tau_1, \tau_2, \cdots, \tau_{2g}\}$$
3. Compute the harmonic 1-form group basis $H^1_\Delta(M, \mathbb{R})$;
$$\{\omega_1, \omega_2, \cdots, \omega_{2g}\}$$
4. Compute the holomorphic 1-form group basis $\Omega^1(M)$.

Abelian Differential of the First Type

1. τ_i and γ_j are dual to each other:

$$\gamma \cdot \gamma_k = \int_\gamma \tau_k, \forall \gamma, k = 1, 2, \dots, 2g;$$

2. harmonic form ω_k is homologous to τ_k

$$\omega_k = \tau_k + df_k, \delta\omega_k = 0.$$

3. The holomorhic 1-form

$$\omega_k + \sqrt{-1}^* \omega_k.$$