Riemann-Roch Theorem

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Abel Differential

An Abel differential has local Laurent series:

$$\omega = \left(\underbrace{\frac{a_{-n}}{z^n} + \frac{a_{-(n-1)}}{z^{n-1}} + \cdots + \frac{a_{-2}}{z^2} + \frac{a_{-1}}{z}}_{principle\ (singular)\ part} + \underbrace{a_0 + a_1 z + \cdots a_k z^k + \cdots}_{holomorphic\ part} \right) dz$$

$$= \omega_2 + \omega_3 + \omega_1$$

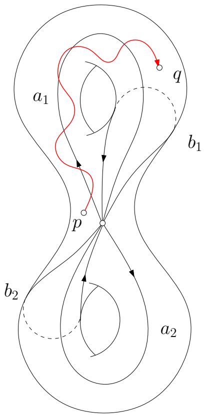
where

$$Type\ 1: \quad \omega_1 = (a_0 + a_1 + \cdots + a_k z^k + \cdots) dz$$

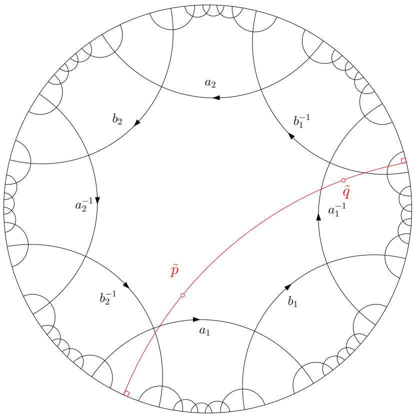
$$Type\ 2: \quad \omega_2 = \left(\frac{a_{-n}}{z^n} + \frac{a_{-(n-1)}}{z^{n-1}} + \cdots + \frac{a_{-2}}{z^2} \right) dz$$

$$Type\ 3: \quad \omega_3 = \left(\frac{a_{-1}}{z-p} - \frac{a_{-1}}{z-q} \right) dz$$

Hyperbolic Geodesic



geodesic on surface



Poincaré's disk model

Bilinear Relation

Definition (Normalized Abel Differential)

Suppose C is a genus g compact Riemann surface, $\{a_1, b_1, \cdots, a_g, b_g\}$ is a set of canonical basis of $\pi_1(C, p)$, $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$ is a set of basis of $\Omega^1(C)$, the period matrix is $(I\ Z)$. Suppose ω is an meromorphic differential. The normalization of ω is given by

$$\omega_0 = \omega - (A_1\varphi_1 + A_2\varphi_2 + \cdots + A_g\varphi_g),$$

where $A_k = \int_{a_k} \omega$ is the A-period of ω on a_k, $k = 1, 2, \cdots, g$.

Theorem (Bilinear Relation)

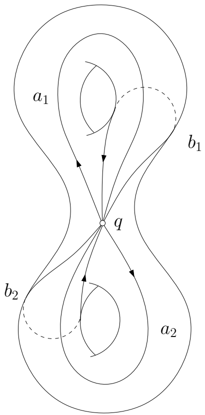
Suppose ω_1 is a holomorphic differential (Abel differential of the first type), the A-periods of ω_1 are $A_1, A_2, \cdots, A_g$, B-periods are $B_1, B_2, \cdots, B_g$; ω_3 is an Abel differential of the third type, ω_3 is with simple poles at $p_1, p_2, \cdots, p_m$ and corresponding residues $c_1, c_2, \cdots, c_m$, namely the principle part of ω_3 at p_k is $\frac{c_k}{z}dz$ $(1 \leq k \leq m)$, the A-periods are $A'_1, A'_2, \cdots, A'_g$, the B-periods are $B'_1, B_2, \cdots, B'_g$.
Suppose Ω is a fundamental polygon of C,

$$\partial\Omega = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1},$$

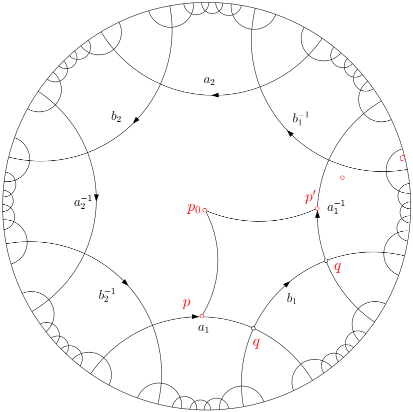
the boundary of Ω doesn't go through any p_k, $(1 \leq k \leq m)$. Select a base point $p_0 \in \Omega$, l_k is the path connecting p_0 and p_k, then

$$\sum_{i=1}^{g} (A_i Bi' - A'_i B_i) = 2\pi\sqrt{-1} \sum_{k=1}^{m} c_k \int_{l_k} \omega_1.$$

Hyperbolic Geodesic



homology basis on surface



fundamental polygon

Bilinear Relation

Proof.

Ω is simply connected, define a holomorphic function

$$f(p) = \int_{p_0}^{p} \omega_1,$$

the integration path is any analytic path inside Ω connecting p_0 and p. For each point $p \in a_j$, the corresponding point is $p' \in a_j^{-1}$,

$$f(p') = \int_{p_0}^{p'} \omega_1 = \int_{p_0}^{p} \omega_1 + \int_{p}^{q} \omega_1 + \int_{b_j} \omega_1 + \int_{q}^{p'} \omega_1$$
$$= f(p) + \int_{b_j} \omega_1 = f(p) + B_j.$$

Similarly, for $p \in b_j$, the corresponding equivalence point $p' \in b_j^{-1}$, we have

$$f(p') = f(p) - A_j.$$

Bilinear Relation

continued.

For $a_j b_j a_j^{-1} b_j^{-1}$, we have

$$\begin{aligned}
\int_{a_j b_j a_j^{-1} b_j^{-1}} f\omega_3 &= int_{a_j} f\omega_3 + \int_{b_j} f\omega_3 + \int_{a_j^{-1}} f\omega_3 + \int_{b_j^{-1}} f\omega_3 \\
&= \int_{a_j} (f(p) - f(p'))\omega_3 + \int_{b_j} (f(p) - f(p'))\omega_3 \\
&= A_j \int_{b_j} \omega_3 - B_j \int_{a_j} \omega_3 \\
&= A_j B_j' - A_j' B_j.
\end{aligned}$$

□

Bilinear Relation

continued.

By residue theorem,

$$\sum_{j=1}^{g} \int_{a_j b_j a_j^{-1} b_j^{-1}} f\omega_3 = \int_{\partial\Omega} f\omega_3 = 2\pi\sqrt{-1}\sum_{k=1}^{m} \mathrm{Res}(f\omega_3, p_k)$$

$$\sum_{j=1}^{g} (A_j B_j' - B_j A_j') = 2\pi\sqrt{-1}\sum_{k=1}^{m} f(p_k)c_k.$$

This is the desired bilinear relation. □

Bilinear Relation

Corollary

If ω_3 *is a normalized Abel differential of the third type,* $\varphi_1, \cdots, \varphi_g$ *is a canonical basis of holomorphic differentials, (period matrix is* $(I \ Z)$*), then*

$$B_k' = \int_{b_k} \omega_3 = 2\pi\sqrt{-1}\sum_{j=1}^{m} c_j \int_{l_j} \varphi_k.$$

Bilinear Relation

Proof.

Let $\omega_1 \leftarrow \varphi_k$, then $A_i(\varphi_k) = \delta_i^k$; ω_3 is normalized, then $A'_j = 0$, $(1 \leq j \leq g)$:

$$\sum_{j=1}^{g}(A_j B'_j - B_j A'_j) = 2\pi\sqrt{-1}\sum_{j=1}^{m} f(p_j)c_j$$

$$\sum_{j=1}^{g} A_j B'_j = 2\pi\sqrt{-1}\sum_{j=1}^{m} f(p_j)c_j$$

$$B'_k = 2\pi\sqrt{-1}\sum_{j=1}^{m} f(p_j)c_j$$

□

Bilinear Relation

Corollary

If ω_3 is a normalized Abel differential of the third type with simple poles at p_1, p_2 and the corresponding residues $+1, -1$; $\varphi_1, \cdots, \varphi_g$ is a canonical basis of holomorphic differentials, (period matrix is $(I\ Z)$), then

$$B'_k = \int_{b_k} \omega_3 = 2\pi\sqrt{-1}\sum_{j=1}^{m} c_j \int_{l_j} \varphi_k = -pi \int_{p_1}^{p_2} \varphi_k.$$

Bilinear Relation

Theorem

Suppose ω_2 *is an Abel differential of the second type with a single pole at* p_0*, the Laurent series of* ω_2 *in a local parameter neighborhood of* p_0 *is*

$$\frac{dz}{z^n} \quad (n \geq 2).$$

where $z = \varphi(p)$ *is the local parameter,* $\varphi(p_0) = 0$*.* ω_1 *is a holomorphic differential with local representation*

$$\omega_1 = (c_0 + c_1 z + \cdots + c_{n-2} z^{n-2} + c_{n-1} z^{n-1} + c_n z^n \cdots) dz,$$

The A-period, B-period of ω_1 *are* A_j, B_j*; those of* ω_2 *are* A'_j *and* B'_j*, then we have*

$$\sum_{j=1}^{g} (A_j B'_j - A'_j B_j) = 2\pi\sqrt{-1}\frac{c_{n-2}}{n-1}. \tag{1}$$

Proof.

Similar to the last proof, in Ω define $f(p) = \int_{p_0}^{p} \omega_1$, the integration path is inside . Then in the neighborhood of p_0,

$$f(z) = c_0 z + \frac{c_1}{2} z^2 + \cdots + \frac{c_{n-2}}{n-1} z^{n-1} + \cdots ,$$

$$f\omega_2(z) = c_0 z^{-(n-1)} + \frac{c_1}{2} c_0 z^{-(n-2)} + \cdots \frac{c_{n-2}}{n-1} z^{-1} + \cdots$$

then we obtain

$$\sum_{j=1}^{g} (A_j B_j' - A_j' B_j) = 2\pi\sqrt{-1}\text{Res}(f\omega_2, p_0),$$

hence

$$2\pi\sqrt{-1}\text{Res}(f\omega_2, p_0) = 2\pi\sqrt{-1}\frac{c_{n-2}}{n-1}$$

Bilinear Relation

Corollary

Suppose ω_2 is a normalized Abel differential of the second type with a single pole at p_0, the Laurent series of ω_2 in a local parameter neighborhood of p_0 is

$$\frac{dz}{z^n} \quad (n \geq 2).$$

where $z = \varphi(p)$ is the local parameter, $\varphi(p_0) = 0$. $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$ is the canonical basis of holomorphic differentials (period matrix $(I \ Z)$) with local representation

$$\varphi_k = (a_{k,0} + a_{k,1}z + \cdots + a_{k,n-2}z^{n-2} + a_{k,n-1}z^{n-1} + a_{k,n}z^n \cdots)dz,$$

Then

$$B'_k = \int_{b_k} \omega_2 = 2\pi\sqrt{-1}\frac{a_{k,n-2}}{n-1}. \tag{2}$$

Bilinear Relation

Proof.

The A-periods of ω_2, A'_j are zeros. The A-period of φ_k are $A_j(\varphi_k) = \delta_i^k$. Set $\omega_1 \leftarrow \varphi_k$, then we have

$$\sum_{j=1}^{g}(A_j(\varphi_k)B'_j - A'_jB_j) = \sum_{j=1}^{g} A_j(\varphi_k)B'_j = B'_k = 2\pi\sqrt{-1}\frac{a_{k,n-2}}{n-1}. \quad (3)$$

□

Riemann Roch

Definition

Let $K(C)$ represent all the meromorphic functions defined on the Riemann surface C;

$$L(D) := \{f \in K((C) : (f) \geq D\};$$

$\Omega(C)$ represents all the meromorphic differentials on C,

$$\Omega(D) := \{\omega \in \Omega(C) : (\omega) \geq D\}.$$

For example

$$\Omega(0) = \{\text{holomorphic differentials}\}.$$

By Laurent series, it is easy to show $\nu_p(f+g) \geq \min\{\nu_p(f), \nu_p(g)\}$, then $L(D)$ and $\Omega(D)$ are linear spaces.

Divisors

Definition (Effective Divisor)

A divisor D is called effective, if $D \geq 0$, namely

$$D = \sum_{k=1}^{m} n_k p_k, \quad n_k \geq 0.$$

Definition (Multiple Divisor)

A divisor D_1 is called a multiple divisor of D_2 if $D_1 - D_2 \geq 0$.

Riemann Roch

Given a divisor $D = \sum_{k=1}^{m} n_k p_k$, $D = D^+ + D^-$, where

$$D^+ = \sum_{k=1}^{m} \max\{n_k, 0\} p_k$$
$$D^- = \sum_{k=1}^{m} \min\{n_k, 0\} p_k$$

If $D_1 \leq D_2$, then $\dim L(D_2) \leq \dim L(D_1)$. So

$$\dim L(D) \leq \dim L(D^-) < \infty.$$

Riemann Roch

Suppose

$$D^{-} = -\sum_{k=1}^{m} n_k p_k \quad n_k \in \mathbb{Z}^{+}.$$

Suppose at the neighborhood of p_k, the principle part of f is given by

$$f_k(z) := \frac{a_{k,n_k}}{z^{n_k}} + \frac{a_{k,n_k-1}}{z^{n_k-1}} + \cdots \frac{a_{k,2}}{z^2} + \frac{a_{k,1}}{z^1}$$

Then

$$f - f_1 - f_2 - \cdots - f_m \equiv c$$

is a holomorphic function, thereefore equals to a constant c. So there are

$$c, a_{1,1}, \cdots, a_{1,n_1}, a_{2,1}, \cdots, a_{2,n_2}, \cdots, a_{m,1}, \cdots, a_{m,n_m}$$

Therefore

$$\dim L(D^{-}) = 1 + \sum_{j=1} n_j = -deg(D^{-}) + 1 < \infty.$$

Riemann Roch

Theorem

Suppose ω_0 *is a meromorphic differential,* $\omega_0 \not\equiv 0$*, then for any divisor* D

$$dim\Omega(D) = dimL(D - (\omega_0)).$$

Proof.

For any $\omega \in \Omega(D)$, $(\omega) \geq D$

$$\left(\frac{\omega}{\omega_0}\right) = (\omega) - (\omega_0) \geq D - (\omega_0),$$

therefore $\frac{\omega}{\omega_0} \in L(D - (\omega_0))$. Inversely, if $f \in L(D - (\omega_0))$, then

$$(f\omega_0) = (f) + (\omega_0) \geq D - (\omega_0) + (\omega_0) = D,$$

$f\omega_0 \in \Omega(D)$. So $\omega \mapsto \frac{\omega}{\omega_0}$ is an isomorphism. □

Riemann-Roch Theorem

Theorem (Riemann-Roch)

Suppose C is a genus g compact Riemann surface, given a divisor D, then

$$dimL(-D) = dim\Omega(D) + deg(D) - g + 1. \tag{4}$$

Riemann-Roch Theorem

Proof.

First, we prove the theorem for $D = 0$, then $L(0)$ is the space of holomorphic functions, which are constants globally. $\dim L(0) = 1$; $\Omega(0)$ is the space of holomorphic 1-forms, $\dim\Omega(D) = g$, therefore

$$\dim L(0) = \dim\Omega(0) + deg(0) - g + 1.$$

□

Riemann-Roch Theorem

continued.

Second, we prove the theorem for effective divisor $D > 0$.

$$D = \sum_{k=1}^{m} n_k p_k, \quad n_k > 0.$$

By definition, $f \in L(-D)$ iff f has a pole at p_k with an order no greater than n_k. Take a local parameter disk V_k centered at p_k, the local parameter is

$$z = z(p), \quad z(p_k) = 0.$$

Take a set of canonical homology basis $(a_1, \cdots, a_g, b_1, \cdots, b_g)$, which don't go through p_k. $\forall f \in L(-D)$, df has Laurent series

$$df = \left(\sum_{j=2}^{n_k+1} \frac{c_j(p_k)}{z^j} + \sum_{j=0}^{\infty} A_j(p_k) z^j \right) dz.$$

Riemann-Roch Theorem

continued.

Let

$$D_1 = \sum_{k=1}^{m} (n_k + 1) p_k,$$

then $df \in \Omega(-D_1)$. The differential operator d defines a homomorphism $d : L(-D) \to \Omega(-D_1)$, $f \mapsto df$, the image space of $L(-D)$ is $dL(-D)$, which is a sublinear space of $\Omega(-D_1)$.
Consider $dL(-D)$, $\forall p_k$, $1 \le k \le m$ and $2 \le n \le n_k + 1$, let ω_k^n be the normalized Abel differential of the second type, which has zero A-periods and a single pole at p_k with order n, and principle part $\frac{dz}{z^n}$ in the local parameter disk V_k, $(\omega_k^n)_\infty = -np_k$, $\forall f \in L(-D)$,

$$df = \sum_{k=1}^{m} \sum_{j=2}^{n_k+1} c_j(p_k) \omega_k^j + \varphi,$$

where φ is a holomorphic differential. □

continued.

$$\int_{a_j} df = 0, \int_{a_j} \omega_k^n = 0, 1 \leq j \leq g \implies \int_{a_j} \varphi = 0$$

$$\|\varphi\|^2 = \sqrt{-1}\sum_{j=1}^{g}(A_j\bar{B}_j - B_j\bar{A}_j) = 0 \implies \varphi = 0.$$

$\{\omega_k^n\}$, $1 \leq k \leq m, 2 \leq n \leq n_k + 1$ are linearly independent, they form a basis of $dL(-D)$, $|\{\omega_k^n\}| = \deg(D)$. $d : L(-D) \to \mathbb{C}^{\deg D}$,

$$f \mapsto df = (c_j(p_k) : 1 \leq k \leq m, 2 \leq j \leq n_k + 1).$$

□

Riemann-Roch Theorem

continued.

For any $(c_j(p_k)) \in \mathbb{C}^{\deg D}$, there is a $f \in L(-D)$, such that $df \mapsto (c_j(p_k))$, if and only if

$$\sum_{k=1}^{m} \sum_{j=2}^{n_k+2} c_j(p_k)\omega_k^j$$

is exact, hence the B-periods for the above differential are zeros (since ω_k^n are normalized, the A-periods are automatically zeros): for any b_l, $1 \leq l \leq g$,

$$\sum_{k=1}^{m} \sum_{j=2}^{n_k+2} c_j(p_k) \int_{b_l} \omega_k^j = 0, \quad l = 1, 2, \cdots, g. \tag{5}$$

□

Riemann-Roch Theorem

continued.

The dimension of $dL(-D)$ equals to the dimension of the solution space of above linear equation group. The coefficient matrix is

$$\left(\int_{b_l} \omega_k^n\right)_{g\times \deg(D)}$$

assume its rank is r, then $\dim(dL(-D)) = \deg D - r$. On the other hand, the kernel of d is

$$d^{-1}(0) = \{f \in L(-D) : df = 0\} = \mathbb{C},$$

therefore $\dim(d^{-1}(0)) = 1$. By $L(-D)/d^{-1}(0) \cong dL(D)$, we have

$$\dim L(-D) = \dim(dL(-D)) + 1 = \deg D - r + 1. \tag{6}$$

□

continued.

The holomorphic differential space is $\Omega(0) = A$ with canonical basis $\{\varphi_1, \varphi_2, \cdots, \varphi_g\}$, $\int_{a_j} \varphi_i = \delta_i^j$. The local representations for φ_l in V_k, $1 \leq k \leq m$

$$\varphi_l = a_{l,0}(p_k) + a_{l,1}(p_k)z + a_{l,2}(p_k)z^2 + \cdots + a_{l,n_k-1}(p_k)z^{n_k-1} + \cdots$$

for any $\omega \in \Omega(D)$, $D > 0$ then ω is a holomorpic differential, $\omega \in A$, there is a set of complex numbers $(\lambda_1, \lambda_2, \cdots, \lambda_g)$

$$\begin{aligned}\omega &= \lambda_1\varphi_1 + \lambda_2\varphi_2 + \cdots + \lambda_g\varphi_g \\ &= \sum_{l=1}^{g} \lambda_l \left(\sum_{i=1}^{n_k-1} a_{l,i}(p_k)z^i + \sum_{i=n_k}^{\infty} a_{l,i}(p_k)z^i \right)\end{aligned}$$

□

Riemann-Roch Theorem

continued.

$\forall p_k$, $1 \leq k \leq m$, ω has zero at p_k with order $\geq n_k$, we obtain the linear system

$$\sum_{l=1}^{g} a_{l,j}(p_k)\lambda_l = 0, \quad k = 1,2,\cdots,m, j = 0,1,\cdots,n_k - 1, \tag{7}$$

reversely, if $(\lambda_1,\cdots,\lambda_g)$ is a solution to the above linear system, then $\omega \in \Omega(D)$.

Define linear operator $T : \Omega(D) \to \mathbb{C}^g$, $\omega \mapsto (\lambda_1,\lambda_2,\cdots,\lambda_g)$, then $\Omega(D)$ is isomorpic to the solution space of the linear system Eqn. (7), whose coefficient matrix is

$$(a_{l,j}(p_k))_{\text{deg}D\times g}.$$

Assume its rank is ρ, the dimension of the solution space of Eqn. (7) is $g-\rho$, hence

$$\text{dim}\Omega(D) = g - \rho.$$

continued.

We claim that $r = \rho$. By the bilinear relation between the Abel differential of the first type and the Abel differential of the second type, we have

$$\left(\int_{b_l} \omega_k^j\right)_{g\times \deg D} = \left(\frac{2\pi\sqrt{-1}a_{l,j-2}(p_k)}{j-1}\right)$$

where $l = 1, \cdots, g$, $k = 1, \cdots, m$, $j = 2, 3, \cdots, n_k + 1$. The left hand side is

$$\begin{bmatrix} \langle b_1, \omega_1^2\rangle & \cdots & \langle b_1, \omega_1^{n_1+1}\rangle & \cdots & \langle b_1, \omega_m^2\rangle & \cdots & \langle b_1, \omega_m^{n_m+1}\rangle \\ \langle b_2, \omega_1^2\rangle & \cdots & \langle b_2, \omega_1^{n_1+1}\rangle & \cdots & \langle b_2, \omega_m^2\rangle & \cdots & \langle b_2, \omega_m^{n_m+1}\rangle \\ \vdots & & \vdots & & \vdots & & \vdots \\ \langle b_g, \omega_1^2\rangle & \cdots & \langle b_g, \omega_1^{n_1+1}\rangle & \cdots & \langle b_g, \omega_m^2\rangle & \cdots & \langle b_g, \omega_m^{n_m+1}\rangle \end{bmatrix}$$

□

continued.

The right hand side is given by

$$2\pi\sqrt{-1}(a_{l,j-2}(p_k))\begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_m \end{bmatrix}, D_k = \begin{bmatrix} \frac{1}{2} & & & \\ & \frac{1}{3} & & \\ & & \ddots & \\ & & & \frac{1}{n_k+1} \end{bmatrix},$$

Since $diag(D_1, \cdots, D_m)$ is full rank, so the rank of the LHS r equals to that of the RHS ρ, therefore

$$r = \rho.$$

□

Riemann-Roch Theorem

continued.

$$\left\{ \begin{array}{rcl} \dim L(-D) & = & (\deg D - r) + 1 \\ \dim \Omega(D) & = & g - \rho \\ \rho & = & r \end{array} \right.$$

Therefore, we obtain if $D \geq 0$

$$\dim L(-D) = \dim \Omega(D) + \deg D - g + 1$$

□

Riemann-Roch Theorem

Proof.

Suppose ω is a meromorphic differential, then $\deg(\omega) = 2g - 2$,

$$\begin{cases} \dim\Omega(D) & = & \dim L(D - (\omega)) \\ \deg(D - (\omega)) & = & \deg D - \deg(\omega) \\ \deg(\omega) & = & 2g - 2 \end{cases}$$

$$\dim L(-D) = \dim\Omega(D) + \deg D - g + 1$$

$$\dim L(-D) + \frac{1}{2}\deg(-D) = \dim\Omega(D) + \frac{1}{2}\deg D - \frac{1}{2}\deg(\omega)$$

$$\dim L(-D) + \frac{1}{2}\deg(-D) = \dim L(D - (\omega)) + \frac{1}{2}\deg(D - (\omega))$$

$$\dim L(-D) + \frac{1}{2}\deg(-D) = \dim L(-((\omega) - D)) + \frac{1}{2}\deg(-((\omega) - D))$$

□

continued.

We have obtained another symmetric formula of Riemann-Roch

$$\boxed{\dim L(-D) + \frac{1}{2}\deg(-D) = \dim L(-((\omega) - D)) + \frac{1}{2}\deg(-((\omega) - D))}$$

If $D \geq 0$ or $(\omega) - D \geq 0$ (D is equivalent to an effective divisor, or $(\omega) - D$ is equivalent to an effective divisor), then the RR has been proven. Otherwise we claim

1. $\dim L(-D) = 0$
2. $\dim L(-((\omega) - D)) = 0$
3. $\deg(D) = g - 1$

□

continued.

1. If $\dim L(-D) \neq 0$, then $\exists f \in L(-D)$, $(f) + D \geq 0$. Let $D_1 = (f) + D0$, $D_1 - D = (f)$, hence $D_1 \sim D$, D is equivalent to an effective divisor, contradiction. Therefore $\dim L(-D) = 0$.
2. Similarly $\dim L(D - (\omega)) = 0$.

Riemann inequality: by $r \leq g$

$$\dim L(-D) = \dim(dL(-D)) + 1 = \deg D - r + 1 \geq \deg D - g + 1$$

We decompose $D = D_1 - D_2$, where $D_1 > 0$ and $D_2 > 0$, therefore $\deg D = \deg D_1 - \deg D_2$. By Riemann inequality $\dim L(-D_1) \geq \deg D_1 - g + 1$,

$$\boxed{\dim L(-D_1) \geq \deg D + \deg D_2 - g + 1}$$

□

Riemann-Roch Theorem

continued.

Claim: $\boxed{\deg D \leq g - 1}$.

Otherwise if $\deg D \geq g$, then $\dim L(-D_1) \geq \deg D_2 + 1 = n$, there are $\deg D_2 + 1$ linearly independent meromorphic functions in $L(-D_1)$,

$$f_1, f_2, \cdots, f_n, n = \deg D_2 + 1.$$

$$D_2 = \sum_{k=1}^{m} n_k p_k, \quad n_k > 0,$$

find $(\lambda_1, \cdots, \lambda_n) \neq 0$, such that

$$f = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_n f_n,$$

$f \in L(-D) = L(-D_1 + D_2)$. □

continued.

It suffices to make f to have zeros at $p_k (1 \leq k \leq m)$ with order at least n_k, namely

$$(f) + D_1 - D_2 \geq (f) - D_2 \geq 0.$$

as previous proof

$$f_i = \sum_{j=0}^{n_k} a_{i,j}(p_k) z^j + \sum_{j=n_k+1}^{\infty} a_{i,j}(p_k) z^j$$

$$0 = \sum_{i=1}^{n} \lambda_i a_{i,j}(p_k), \quad 1 \leq k \leq m, 1 \leq j \leq n_k$$

There are $n = \deg D_2 + 1$ unknowns λ_i, and $\deg D_2$ equations. Therefore, there exists a non-zero solution $(\lambda_1, \lambda_2, \cdots, \lambda_n) \neq 0$, hence $f \not\equiv 0$, $f \in L(-D)$, contradict to $\dim L(-D) = 0$. So we obtain $\deg D \leq g - 1$, similarly $\deg((\omega) - D) \leq g - 1$. □

Riemann-Roch Theorem

continued.

But we know

$$\deg D + \deg((\omega) - D) = \deg() = 2g - 2$$

hence

$$\deg D = g - 1, \quad \deg((\omega) - D) = g - 1.$$

By three claims we obtain: if D and $(\omega) - D$ are not (equivalent to) effective divisors, then RR still holds

$$\underbrace{\dim L(-D)}_{0} + \frac{1}{2}\underbrace{\deg(-D)}_{g-1} = \underbrace{\dim L(-((\omega) - D))}_{0} + \frac{1}{2}\underbrace{\deg(-((\omega) - D))}_{g-1}$$

□