The Theory and Computation of Optimal Transportation

Spherical Optimal Transportation

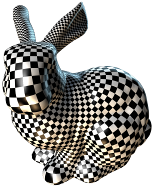

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图: Theory and Computation of Optimal Transportation

最优传输理论口诀

最优传输的几何观点口诀:

- 代价变换支撑
- 支撑包络势能
- 势能微分映射
- 映射对偶凸壳

Minkowski Problem

Given k unit vectors $n_1, \cdots, n_k$, not contained in any half-space of $\mathbb{R}^n$, $A_1, \cdots, A_k > 0$, satisfying

$$\sum_{i=1}^{k} A_i \mathbf{n}_i = \mathbf{0},$$

find a compact convex polyhedron P, with k co-dimension 1 facets $F_1, \cdots, F_k$, such that the volume of F_i equals to A_i, the normal to F_i is $\mathbf{n}_i$.

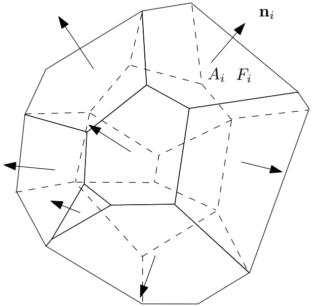


图: Minkowski problem.

定理 (Minkowsi)

Such kind of P exists, and is unique up to a translation.

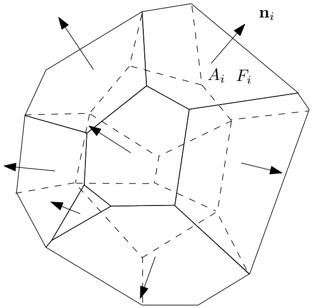


图: Minkowski Problem.

定理 (Alexadnrov 1950)

Suppose Ω *is a compact convex domain in* $\mathbb{R}^n$, $p_1, \dots, p_k$ *are distinct vectors in* $\mathbb{R}^n$, $A_1, \dots, A_k > 0$, *satisfying* $\sum A_i = vol(\Omega)$, *then there exists a convex piecewise linear function, unique up to a constant,*

$$u(x) = \max_{i=1}^{k} \langle p_i, x \rangle - h_i,$$

such that

$$vol(W_i) = A_i, \quad W_i = \{x | u(x) = p_i\}.$$

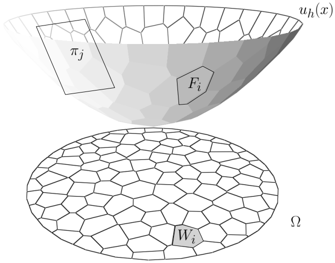


图: Alexandrov Theorem.

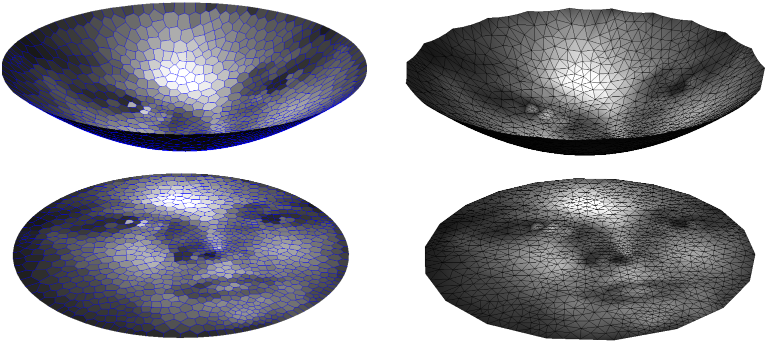


图: Semi-discrete Optimal Transportation, initial stage.

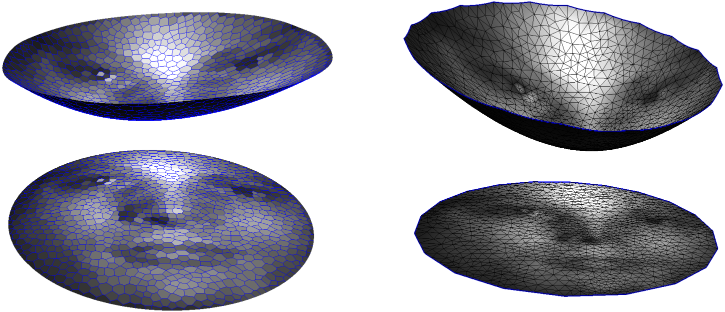
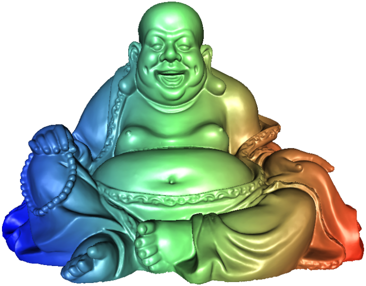
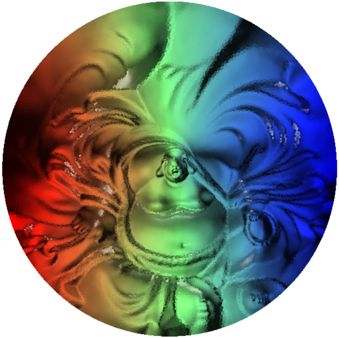


图: Semi-discrete Optimal Transportation, final stage.

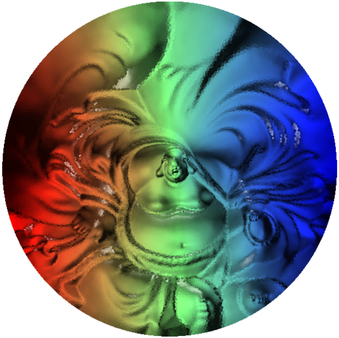


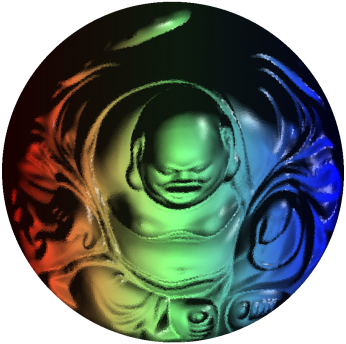
Buddha surface



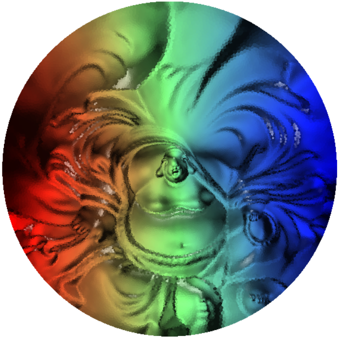
Riemann mapping

图: Conformal mapping.


Riemann Mapping

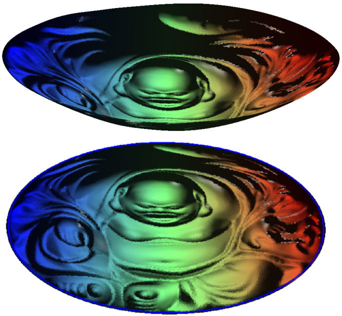

OT Mapping

Semi-discrete Worst Transportation

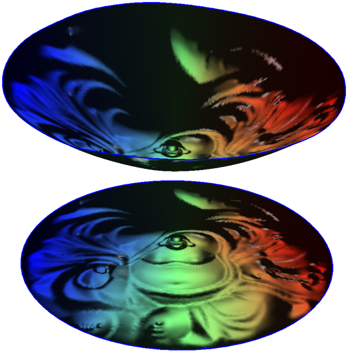


Riemann Mapping

Worst Transportation

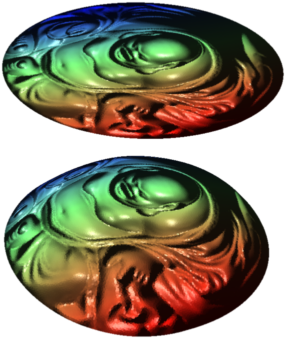


Brenier potential

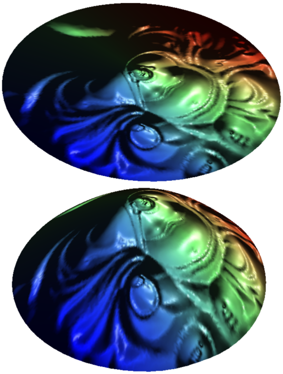


Legendre dual

Semi-discrete Transportation Map



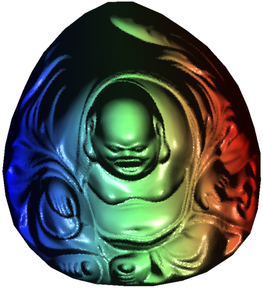
Brenier potential



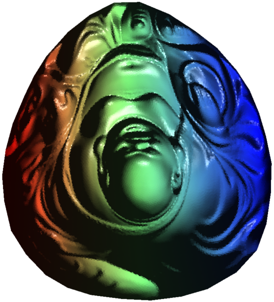
Legendre dual

Optimal, worst transportation comparison

Riemann Mapping



OT mapping



WT mapping

Suppose $K \subset \mathbb{R}^d$ is a bounded open convex domain, containing the origin, the boundary ∂K is parameterized by polar coordinates:

$$\partial K = \{\rho(x)x : x \in \mathbb{S}^{d-1}, \rho : \mathbb{S}^{d-1} \to \mathbb{R}^+\}.$$

定义 (sub-normal map)

For any point $z \in \partial K$, the *sub-normal map* maps a point $z$to a closed set on the unit sphere, $z \mapsto N_K(z)$,

$$N_K(z) := \left\{ y \in \mathbb{S}^{d-1} : K \subset \{w : \langle y, w - z \rangle \le 0\} \right\}. \quad \blacklozenge \qquad (1)$$

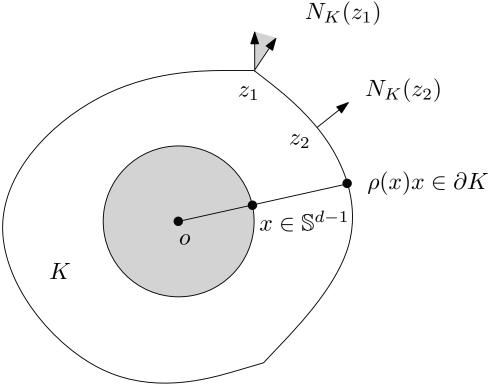


图: Given a convex $K \ni 0$, the boundary ∂K is parameterized by polar coordinates, represented as $\rho : \mathbb{S}^{d-1} \to \mathbb{R}^+$. Given a point $z \in \partial K$, the set $N_K(z)$ consists of all the exterior normals at z. When K has a unique tangent plane at z (such as z_2), $N_k(z)$ is a singleton. If z is a corner point, then $N_K(z)$ consists of multiple elements (such as z_1).

定义 (Gauss Map)

Multi-valuedGauss map $G_K : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is defined by:

$$G_K(x) := N_K(\rho(x)x).$$

The Gauss curvature measure is defined as:

$$\mu_K(E) := \mathcal{H}^{d-1}(G_K(E)), \quad \forall \text{ Borel 集合 } E \subset \mathbb{S}^{d-1}.$$

where $\mathcal{H}^{d-1}$ represents the $d-1$ dimensional Hausdorff measure on $\mathbb{S}^{d-1}$. □

It can be shown that μ_K is a Borel measure.

问题 (Minkowski I)

Given a Borel measure ν defined on the sphere $\mathbb{S}^{d-1}$, can we find a bounded convex open set $K \ni 0$, such that $\nu = \mu_K$? □

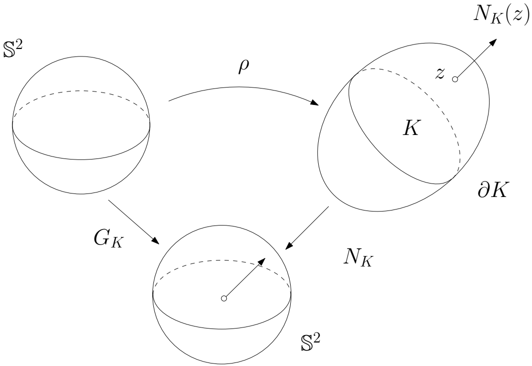


图: Minkowski Problem I.

定义 (Spherical Convex Set and Polar Set)

Given a spherical set $\omega \subset \mathbb{S}^{d-1}$, we say ω is convex, if the cone

$$\mathbb{R}^+\omega := \{tx : t > 0, x \in \omega\}$$

is convex. The polar set of ω is defined as

$$\omega^* := \{y \in \mathbb{S}^{d-1} : \langle x, y\rangle \leq 0, \forall x \in \omega\}. \quad \square$$

定理 (Minkowski I)

Let ν be a Borel measure on $\mathbb{S}^{d-1}$, then there exists a bounded convex open set K, such that

$$\nu = \mu_K \iff \begin{cases} \text{(a) } \nu(\mathbb{S}^{d-1}) = \mathcal{H}^{d-1}(\mathbb{S}^{d-1}); \\ \text{(b) } \nu(\mathbb{S}^{d-1} \setminus \omega) > \mathcal{H}^{d-1}(\omega^*), \forall \omega \subsetneq \mathbb{S}^{d-1} \text{ compact conv} \end{cases}$$

If K exists, then different solutions differ by a dilation. $\square$

定理 (Regularity of the Solution to Minkowski Problem)

Suppose $K \subset \mathbb{R}^3$ *is a convex open set containing the origin, if* $\mu_K = f d\mathcal{H}^2$*, the density function* $f : \mathbb{S}^2 \to \mathbb{R}^+$ *is bounded, then* ∂K *is* C^1*.* □

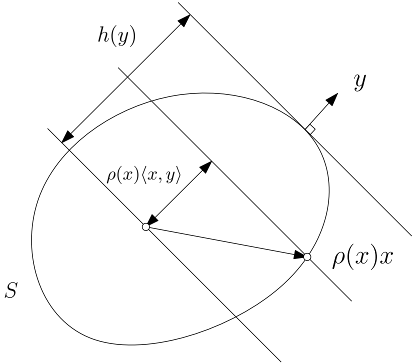


图: Generalized Legendre Transform,
$h(y) = \max\{\rho(x)\langle x, y\rangle, x \in \mathbb{S}^{d-1}\}$.

定义 (Spherical Legendre Dual)

Given a hyper-surface in $\mathbb{R}^d$, with polar representation $S := \{\rho(x)x : x \in \mathbb{S}^{d-1}, \rho : \mathbb{S}^{d-1} \to \mathbb{R}^+\}$, its spherical Legendre dual is $S^* := \{h(y)y : y \in \mathbb{S}^{d-1}, h : \mathbb{S}^{d-1} \to \mathbb{R}^+\}$, where

$$h(y) := \sup_{x \in \mathbb{S}^{d-1}} \rho(x)\langle x, y\rangle. \tag{2}$$

symmetrically, $S = (S^*)^*$, furthermore

$$\rho(x) = \inf_{y \in \mathbb{S}^{d-1}} \frac{h(y)}{\langle x, y\rangle}, \tag{3}$$

or equivalently

$$\rho^{-1}(x) = \sup_{y \in \mathbb{S}^{d-1}} h^{-1}(y)\langle x, y\rangle. \quad \square$$

Formula （口诀）

cost determines support, support envelopes potential （代价变换支撑，支撑包络势能）;

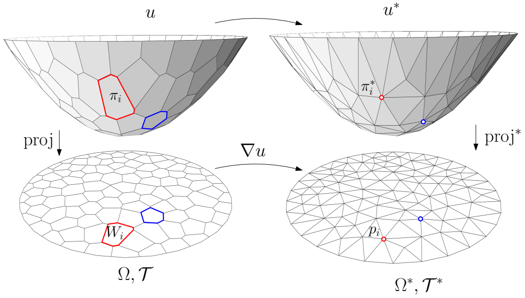


图: Legendre Dual in Euclidean Space.

Formula （口诀）

Differentiation of Potential gives maps; maps is dual to convex hull （势能微分映射；映射对偶凸形。）

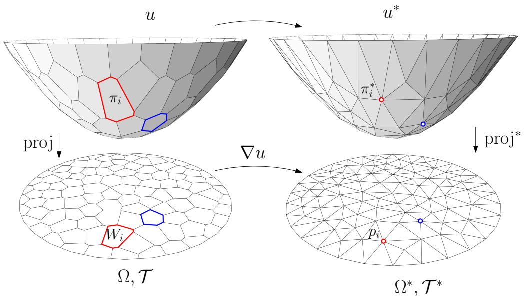


图: Euclidean Legendre dual. Support plane $\langle \mathbf{p}, x \rangle - h = 0$, dual point $(\mathbf{p}, h)$.

Formula

cost transformed to support, support envelopes potential, potential differentiates map, map dual to convex hull.

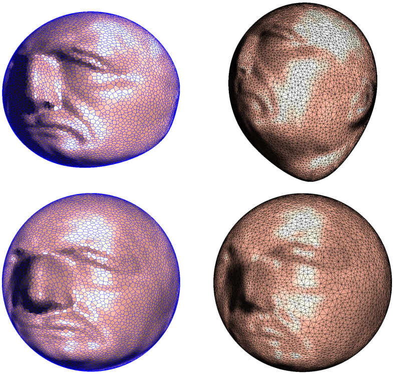


图: Legendre dual. support plane $\rho(x) = h/\langle x, \mathbf{y}\rangle$, dual point $h^{-1}\mathbf{y}$

Take logarithm of spherical Legendre duality formula,

$$\log \rho(x) = \inf_{y} \left\{ -\log\langle x, y\rangle - \log \frac{1}{h(y)} \right\}, \tag{4}$$

and

$$\log \frac{1}{h(y)} = \inf_{x} \left\{ -\log\langle x, y\rangle - \log \rho(x) \right\}. \tag{5}$$

Define cost function $c : \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \to \mathbb{R}^{+} \cup \{0\}$,

$$c(x, y) := -\log\langle x, y\rangle, \tag{6}$$

then $\log \rho(x)$ and $-\log h(y)$ are c-transform of each other:

$$(\log \rho(x))^{c} = \log \frac{1}{h(y)} \quad 和 \quad \left(\log \frac{1}{h(y)}\right)^{\bar{c}} = \log \rho(x).$$

证明.

Minkowski problem I can be rephrased as an optimal transportation problem: given a Borel measure ν on $\mathbb{S}^{d-1}$, find an optimal transportation map $T\colon (\mathbb{S}^{d-1}, \mathcal{H}^{d-1}) \to (\mathbb{S}^{d-1}, \nu)$,

$$\min_{T_{\#}\mathcal{H}^{d-1}=\nu} \int_{\mathbb{S}^{d-1}} -\log\langle x, T(x)\rangle \, d\mathcal{H}^{d-1}.$$

this is equivalent to the dual problem:

$$\max\left\{\int_{\mathbb{S}^{d-1}} \varphi(x)\, d\mathcal{H}^{d-1}(x) + \int_{\mathbb{S}^{d-1}} \varphi^c(y)\, d\nu(y), \quad \varphi \in c\text{-conv}\left(\mathbb{S}^{d-1}\right)\right\}.$$

the cost function $-\log\langle x, y\rangle$ is continuous, $\mathbb{S}^{d-1}$ is a compact metric space, by (DP) theory, there exists a solution $(\varphi, \varphi^c) = (\rho(x), 1/h(y))$. □

Minkowski Problem II

Assume S is a smooth strictly convex surface, its Gauss map $N_k : S \to \mathbb{S}^2$ is invertible. We can use Gauss sphere to parameterize the surface, denoted as $S(y)$, $y \in \mathbb{S}^2$. The normal to the surface at $S(y)$ is y, the Gaussian curvature is $\mathcal{K}(y)$. The Gaussian curvature satisfies:

$$\int_{\mathbb{S}^2} \frac{y}{\mathcal{K}(y)} dA_{\mathbb{S}^2}(y) = 0.$$

The surface area element is:

$$d\nu = dA_S(y) = \frac{1}{\mathcal{K}(y)} dA_{\mathbb{S}^2}(y).$$

Namely, the Gauss map pushes the area element dA_S to measure ν on the Gauss sphere , the density is $\mathcal{K}(y)^{-1}$.

问题 (Minkowski II)

Given measure ν *on the sphere，satisfying*

$$\int_{\mathbb{S}^2} y d\nu(y) = \mathbf{0},$$

Find a convex surface $S(y)$，*such that* $d\nu$ *is the area element of* S, *where the density of* ν *is* $d\nu = \frac{1}{\mathcal{K}(y)} dA_{\mathbb{S}^2}$，*the normal to the surface at* $S(y)$ *is* y，*and the Gaussian curvature is* $\mathcal{K}(y)$。

In Minkowski problem I，the surface has polar representation $\rho(x)x$, $x \in \mathbb{S}^2$; in Minkowski problem II, surface is parameterized by the Gauss sphere, namely parameterized by the normals.

We use the sum of Dirac distributions to approximate the measure ν. Construct a cell decomposition of the sphere $\mathcal{D}$,

$$\mathbb{S}^2 = \bigcup_{i=1}^{n} W_i,$$

for each cell W_i, compute a vector

$$v_i = \int_{W_i} \frac{y}{\mathcal{K}(y)} dA_{\mathbb{S}^2},$$

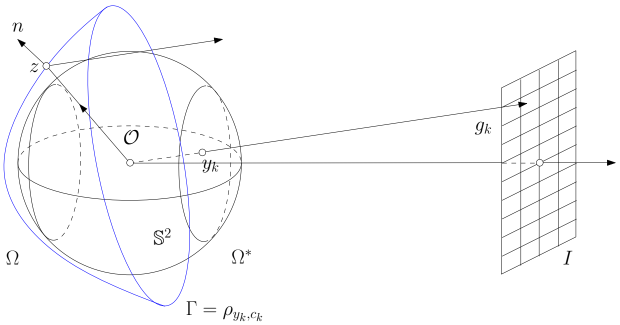
let $A_i = |v_i|$ and $y_i = v_i/A_i$, then use $\{(A_i, y_i)\}_{i=1}^n$ to solve discrete Minkowski problem to obtain the discrete convex polyhedron P, the normal to the i-th face is y_i, the area of the i-th face is A_i.

Construct a sequence of cell decompositions $\mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n, \dots,$ if the diameters of the cells uniformly monotonously converge to 0, then there is a subsequence of convex polyhedra $P_1, P_2, \dots, P_n, \dots$ converge to the smooth convex surface S.

A illumination system consists of a point light source at $\mathcal{O}$ and a reflector surface Γ with polar representation,

$$\Gamma_\rho = \{x\rho(x); x \in \Omega\}, \quad \rho > 0, \tag{7}$$

all the incidence light rays fall inside the input domain Ω.



If we only consider the far field problem, then we can only care about the directions of the reflected rays. All the reflected rays fall in the output domain Ω^*.

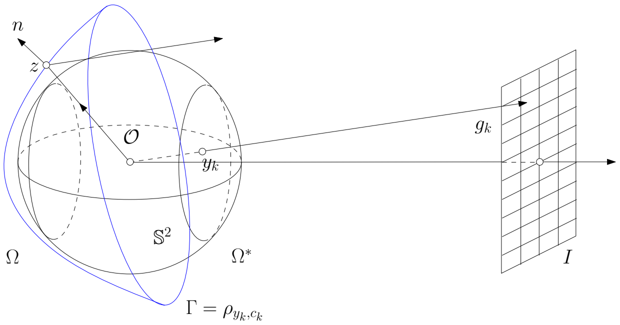


图: Illumination system.

Reflector Design

图: Left: the desired far field image, Lena; Right: the simulated reflected image.

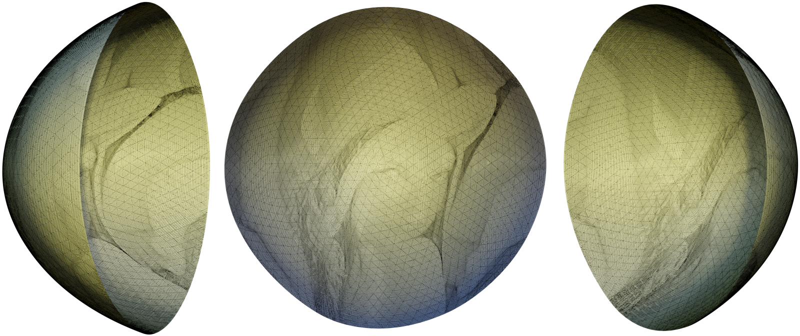


图: The reflector surface for the Lena image.

图: Left: the desired far field image, Monge; Right: the simulated reflected image.

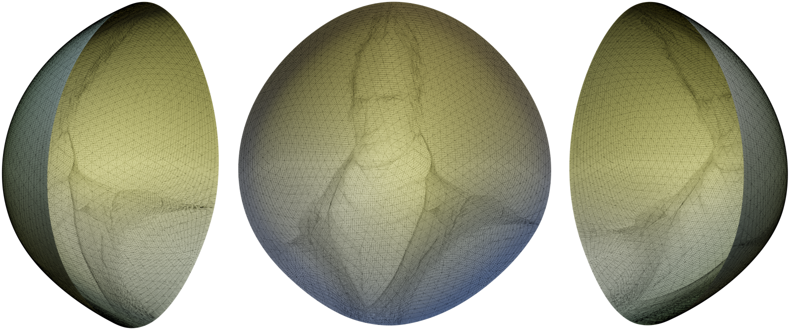


图: The reflector surface for the Lena image.

Suppose f is the illumination intensity defined on the input domain Ω, namely the distribution of the incidence rays emanating from $\mathcal{O}$, g is the illumination intensity in the output domain Ω^*. Assume there is no energy loss, then according energy conservation law, we have

$$\int_{\Omega} f = \int_{\Omega^*} g. \tag{8}$$

A ray emanates from $\mathcal{O}$, propagates along a direction $x \in \Omega$, intersects the mirror at $z = x\rho(x) \in \Gamma_\rho$, the reflection direction is determined by the reflection law,

$$T(x) = T_\rho(x) = \partial\rho(x) = x - 2\langle x, n\rangle n, \tag{9}$$

where n is the exterior normal to the reflector surface Γ_ρ at point z, $\langle x, n\rangle$ represents the inner product. By energy conservation, T is measure preserving,

$$\int_{T^{-1}(E)} f = \int_E g, \quad \forall \text{ Borel 集合 } E \subset \Omega^*. \tag{10}$$

satisfying the natural boundary condition

$$T_\rho(\Omega) = \partial\rho(\Omega) = \Omega^*. \tag{11}$$

By measure preserving condition, we can obtain the PDE for the reflector. In fact, at $x \in \Omega$, the Jacobi of T equals to $f(x)/g(T(x))$, in a local ortha-normal coordinates of $\mathbb{S}^2$, the local representation of the PDE is

$$\mathcal{L}\rho = \eta^{-2} \det(-\nabla_i\nabla_j\rho + 2\rho^{-1}\nabla_i\rho\nabla_j\rho + (\rho - \eta)\delta_{ij}) = f(x)/g(T(x)), \tag{12}$$

where ∇ is the covariant differential operator, $\eta = (|\nabla\rho|^2 + \rho^2)/2\rho$, and δ_{ij} is the Kronecker function. This is a non-linear Monge-Ampère PDE, a natural boundary condition is

$$T_\rho(\Omega) = \partial\rho(\Omega) = \Omega^*. \tag{13}$$

问题 (Reflector Design)

Given spherical domains $\Omega,\ \Omega^* \subset \mathbb{S}^2$, *and density functions* $f: \Omega \to \mathbb{R}_+$ *and* $g: \Omega^* \to \mathbb{R}_+$, *find a reflector surface* Γ_ρ, *such that the reflection map* T_ρ *satisfies the measure-preserving condition and the natural boundary condition.* □

Surface with uniform reflection property

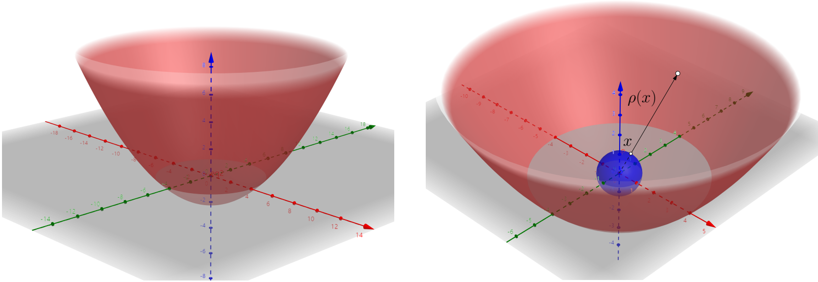


图: A paraboloid of revolution about the axis of direction y, with radial representation $\rho(x) = C/(1 - \langle x, y \rangle)$.

The uniform reflection property of a paraboloid of revolution: all the reflected rays of the incidence rays parallel to the rotation axis intersect at the focal point, vice versa.

定义 (Supporting Paraboloid)

Let $\rho \in C(\Omega)$ be a positive function, $\Gamma_\rho = \{x\rho(x) : x \in \Omega\}$ represents the radial graph of ρ. We say Γ_p is a supporting paraboloid of ρ at $x_0\rho(x_0) \in \Gamma_\rho$, where $p = p_{y,C}$, if

$$\begin{cases} \rho(x_0) = p_{y,C}(x_0), \\ \rho(x) \le p_{y,C}(x), \quad \forall x \in \Omega. \end{cases} \quad \blacklozenge \tag{14}$$

定义 (Admissible Function)

We say ρ is an *admissible function*, if its radial graph Γ_ρ has a supporting paraboloid at every point. □

定义 (Subdifferential)

Let ρ be an admissible function, the subdifferential is a set-valued map $\partial\rho : \Omega \to \mathbb{S}^2$: for any $x_0 \in \Omega$, $\partial\rho(x_0)$ is set of y_0, such thast there exists a $C > 0$, $p_{y_0,C}$ is the supporting parabolid of ρ at x_0,

$$\partial\rho(x) = \left\{ y \in \Omega^* : \exists\ C > 0 \text{ s.t. paraboloid } p_{y,C} \text{ supports } \rho \text{ at } x \right\}. \quad \square$$

定义 (Generalized Alexandrov Measure)

The subdifferential $\partial\rho$ induces a measure $\mu = \mu_{\rho,g}$ on Ω, where $g \in L^1(\Omega^*)$ is a non-negative measurable function on $\mathbb{S}^2$, such that for any Borel set $E \subset \Omega$,

$$\mu_{\rho,g}(E) = \int_{\partial\rho(E)} g(x)\, dx. \tag{15}$$

定义 (Generalized Solution)

Admissible function ρ is called the generalized solution to the spherical Monge-Ampère equation for reflection system, if as measures $\mu_{\rho,g} = fdx$. Equivalently, for any Borel set $E \subset \Omega$, we have

$$\int_E f = \int_{\partial\rho(E)} g. \tag{16}$$

Furthermore, if ρ satisfies

$$\Omega^* \subset \partial\rho(\Omega), \quad |\{x \in \Omega : f(x) > 0 \text{ and } \partial\rho(x) - \overline{\Omega^*} \neq \emptyset\}| = 0, \tag{17}$$

then ρ is the generalized solution to the spherical Monge-Ampère equation for the OT map $\mathcal{L}\rho = f/g \circ T$ with natural boundary condition $T_\rho(\Omega) = \Omega^*$. □

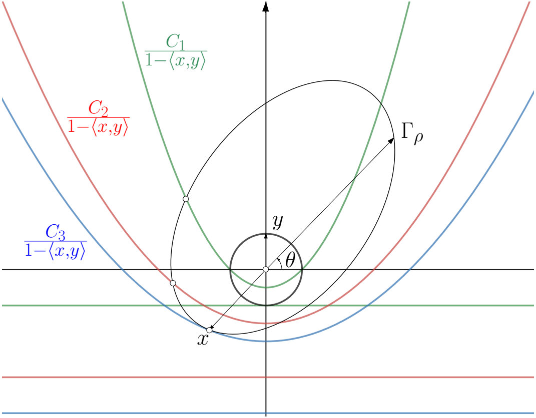


图: Generalized Legendre transformation.

Suppose ρ is admissible, fix a direction $y \in \mathbb{S}^2$, there exists a paraboloid of revolution about the axis of direction y, represented as $p_{y,c}$ with radial representation $\frac{c}{1-\langle x,y\rangle}$, which supports Γ_ρ at point $\rho(x)x$. As shown in the figure, for any paraboloid of revolution about the axis of direction y $p_{y,\tilde{c}}$, which intersects Γ_ρ, we have $\tilde{c} \leq c$. Assume Γ_ρ intersects $p_{y,\tilde{c}}$ at $\rho(x)x$, then $\rho(x) = \frac{\tilde{c}}{1-\langle x,y\rangle}$, $\tilde{c} = \rho(x)(1 - \langle x, y\rangle)$. Hence we have

$$c(y) = \sup_{x\in\Omega} \rho(x)(1 - \langle x, y\rangle) \iff \frac{1}{c(y)} = \inf_{x\in\Omega} \frac{1}{\rho(x)(1 - \langle x, y\rangle)},$$

We represent it as $\eta : \Omega^* \to \mathbb{R}_+$, $\eta(y) = 1/c(y)$.

定义 (Generalized Legendre Transform)

Suppose ρ is an admissible function defined on $\Omega \subset \mathbb{S}^2$, the generalized Legendre transform of ρ with respect to the function $\frac{1}{1-\langle x,y\rangle}$ is a function η defined on $\mathbb{S}^2$,

$$\eta(y) = \inf_{x\in\Omega} \frac{1}{\rho(x)(1-\langle x,y\rangle)}. \quad \blacklozenge \tag{18}$$

For any fixed $y_0 \in \Omega^*$, suppose the infimum is reached at $x_0 \in \Omega$, hence we have

$$\eta(y_0)\rho(x_0) = \frac{1}{1 - \langle x_0, y_0 \rangle}, \tag{19}$$

for arbitrary $x \in \Omega$ and $y \in \Omega^*$,

$$\rho(x)\eta(y) \leq \frac{1}{1 - \langle x, y \rangle}, \tag{20}$$

and the paraboloid $p_{y_0,C}(x) = \frac{C}{1-\langle x, y_0 \rangle}$ supports ρ at x_0, and $p_{x_0,C}(y) = \frac{C}{1-\langle x_0, y \rangle}$ supports η at y_0.

Furthermore:

$$y_0 \in \partial\rho(x_0) \iff x_0 \in \partial\eta(y_0).$$

especially, when the generalized Legendre transform of η is restricted on Ω, it is exactly ρ,

$$\rho^{**} = \rho.$$

If ρ is smooth and satisfies the Monge-Ampère equation(12), then the subdifferential $\partial\eta$ is the inverse map of $\partial\rho$. Hence, η satisfies the equation

$$\mathcal{L}\rho = \frac{f(x)}{g(\partial\rho(x))}, \quad \mathcal{L}\eta = \frac{g(y)}{f(\partial\eta(x))}, \tag{21}$$

定理 (Reflector Design)

Suppose Ω *and* Ω^* *are domains contained in the north and the south hemi-sphere respectively,* f *and* g *are bounded positive functions,* $\int_\Omega f(x) = \int_{\Omega^*}$*, then there exist a pair of functions* (φ_1, ψ_1) *maximizing the following energy,*

$$\sup\left\{\int_\Omega \varphi(x) f(x)\,dx + \int_{\Omega^*} \psi(y) g(y)\,dy, \varphi(x) + \psi(y) \le c(x,y)\right\}, \tag{22}$$

where

$$c(x,y) = -\log(1 - \langle x, y\rangle), \tag{23}$$

$\langle x, y\rangle$ *is the inner product in* $\mathbb{R}^3$*, such that* $\rho = e^\varphi$ *is the solution to the spherical Monge-Ampère equation* $\mathcal{L}\rho = f/g \circ \partial\rho$ *satisfying the natural boundary condition* $\partial\rho(\Omega) = \Omega^*$*, and all such solutions* ϕ *differ by a constant.* □

Proof.

Reflector design is an optimal transport problem. By the existence and the uniqueness of the solution to the dual problem (DP), we get that there exist a pair of Kantorovich potentials (φ, ψ), $\psi = \varphi^c$, $\varphi = \psi^{\bar{c}}$, and φ is unique up to a constant. Let $x_0 \in \Omega$ be a differentiable point of φ, let $y_0 \in \overline{\Omega_*}$, such that

$$\begin{cases} \varphi(x_0) & = & c(x_0, y_0) - \psi(y_0) \\ \varphi(x) & \leq & c(x, y_0) - \psi(y_0), \quad \forall x \in \Omega. \end{cases}$$

now let $\rho = e^{\varphi}$, the paraboloid is given by

$$p(x) = \exp(c(x, y_0) - \psi(y_0)) = \frac{C}{1 - \langle x, y_0 \rangle}, C = \exp(-\psi(y_0)).$$

then $p(x)$ supports Γ_ρ at x_0. $\square$

continued.

Γ_ρ is the inner envelope of the supporting paraboloids, ρ is almost everywhere differentiable. At the differentiable points of ρ, the supporting paraboloid is unique, hence y_0 is unique. Hence, the optimal transport plan becomes an optimal transport map $T_\rho : \Omega \to \Omega^*$.

The paraboloid $p(x)$ and Γ_ρ share the same normal vector at the tangential point, by the uniform reflection property of the paraboloid, we have

$$y_0 = T_\rho(x_0) = T_p(x_0) = x_0 - 2\langle x_0, n\rangle n.$$

T_ρ is measure preserving, satisfies the spherical Monge-Ampère equation, $\mathcal{L}\rho = f/g \circ \partial\rho$, with the natural boundary condition $T_\rho(\Omega) = \Omega^*$. □

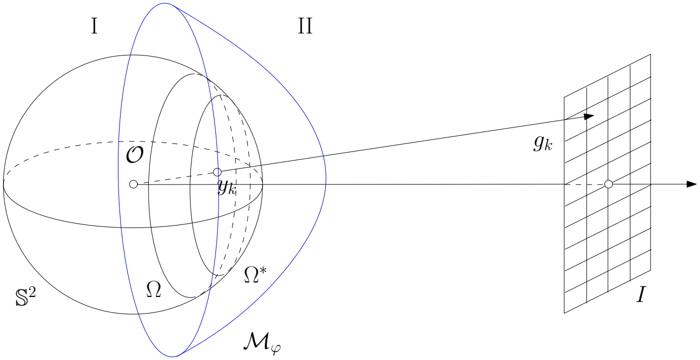


图: Refractive lens system.

Suppose n_1 and n_2 are refractive indices of two homogeneous, isotropic media I and II. Suppose the light source is at a point $\mathcal{O}$ in the medium I, along a direction $x \in \Omega \subset \mathbb{S}^2$, the light intensity is $f(x)$.
We want to construct a refractive surface with radial representation Γ_ρ,

$$\Gamma_\rho = \{x\rho(x); x \in \Omega\}, \quad \rho > 0, \tag{24}$$

Γ_ρ separates the media I and II, such that all the directions of the refracted rays in the medium II are inside $\Omega^* \subset \mathbb{S}^2$, and the intensity of the ray along $y \in \Omega^*$ equals to $g(y)$, where the spherical function $g : \Omega^* \to \mathbb{R}$ is prescribed.

Suppose the refraction has no energy loss, by energy conservation law,

$$\int_{\Omega} f(x)\,dx = \int_{\Omega^*} g(y)\,dy. \tag{25}$$

A ray starts from $\mathcal{O}$ and arrives at $x\rho(x) \in \Gamma_\rho$, where $x \in \Omega$. It is refracted, the direction of the refracted ray is

$$T(x) = T_\rho(x) = \partial\rho(x). \tag{26}$$

By energy conservation, T is measure preserving, namely

$$\int_{T^{-1}(E)} f(x)\,dx = \int_E g(y)\,dy, \quad \forall \text{ Borel set } E \subset \Omega^*, \tag{27}$$

with natural boundary condition

$$T_\rho(\Omega) = \partial\rho(\Omega) = \Omega^*. \tag{28}$$

问题 (Refractor Design)

Suppose n_1 *and* n_2 *are refractive indices of two homogeneous, isotropic media. Given spherical domains* $\Omega, \Omega^* \subset \mathbb{S}^2$, *density functions* $f: \Omega \to \mathbb{R}_+$ *and* $g: \Omega^* \to \mathbb{R}_+$, *find refractive surface* Γ_ρ *separates the two media, the refraction map* T_ρ *(26) satisfies the measure preserving condition (27) and the natural boundary condition (28).* □

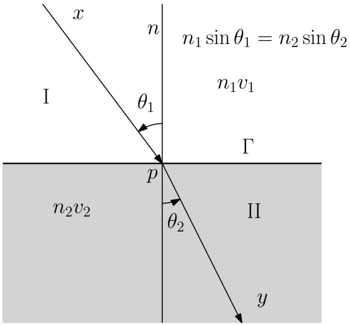


图: Snell refraction law.

v_1 and v_2 are the light speeds in the media I and II, $n_1 = c/v_1$, $n_2 = c/v_2$ are the *refractive indices*. Suppose a ray along the direction $x \in \mathbb{S}^{n-1}$ travels in medium I, and hits a boundary point $p \in \Gamma$ and enters the medium II, the refracted ray is along the direction $y \in \mathbb{S}^{n-1}$.

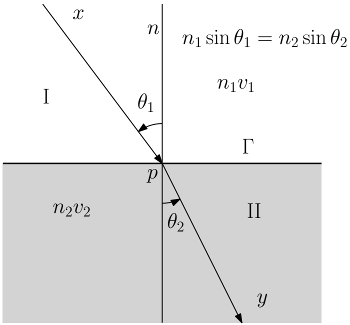


图: Snell refraction law.

Snell law claims

$$n_1 \sin\theta_1 = n_2 \sin\theta_2,$$

where θ_1 is the *angle of incidence*, θ_2 is the *angle of refraction*, n is normal to the interface surface Γ, pointing to the medium II. The vectors x, n and y are co-planar.

定义 (Surface with uniform refraction property)

If the interface surface Γ of the media I and II refracts all the rays of light emanating from the origin $\mathcal{O}$ inside medium I into rays parallel to a fixed $y \in \mathbb{S}^2$, then Γ is called a surface with uniform refraction property.

$\kappa = n_2/n_1$, when $\kappa < 1$, Γ is an ellipsoid of revolution about the axis of direction y, denoted as $e_{y,b}$

$$e_{y,b} = \left\{ \rho(x)x : \rho(x) = \frac{b}{1 - \kappa\langle y, x\rangle}, x \in \mathbb{S}^{n-1}, \langle x, y\rangle \geq \kappa \right\}. \quad (29)$$

when $\kappa > 1$, by physics constraint $\langle x, y\rangle > 1/k$, Γ is a the sheet with opening in direction y of a hyperboloid of revolution of two sheets about the axis of direction y,

$$h_{y,b} = \left\{ \rho(x)x : \rho(x) = \frac{b}{\kappa\langle y, x\rangle - 1}, x \in \mathbb{S}^{n-1}, \langle x, y\rangle \geq 1/\kappa \right\}. \quad (30)$$

引理 (Lemma)

Suppose n_1 *and* n_2 *are the refractive indices of two media I and II respectively, and* $\kappa = n_2/n_1$*. The origin* $\mathcal{O}$ *is in medium I,* $e_{y,b}$ *and* $h_{y,b}$ *are interface surface between media I and II, defined by (29) and (30) respectively, we have*

if $\kappa < 1$*, then* $e_{y,b}$ *refracts all the rays emanating from the origin* $\mathcal{O}$ *in medium I into rays in medium II with refraction direction* y*;*

if $\kappa > 1$*, then* $h_{y,b}$ *refracts all the rays emanating from the origin* $\mathcal{O}$ *in medium I into rays in medium II with refraction direction* y*.* □

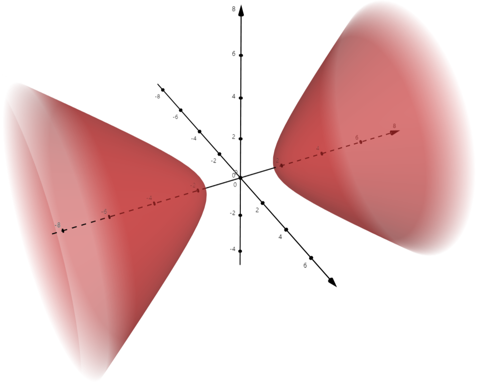


图: hyperboloid of revolution of two sheets.

定义 (Supporting Ellipsoid)

Suppose $\rho \in C(\Omega)$ is a positive function, and $\Gamma_\rho = \{x\rho(x) : x \in \Omega\}$ is the radial graph of ρ. Let $e = e_{y,c}$ be an ellipsolid of revolution, its radial graph be Γ_e. If

$$\begin{cases} \rho(x_0) = e_{y,c}(x_0), \\ \rho(x) \leq e_{y,c}(x), \quad \forall x \in \Omega, \end{cases} \tag{31}$$

then we say Γ_e is a *supporting ellipsoid* of ρ at the point $x_0\rho(x_0) \in \Gamma_\rho$.
If the radial graph Γ_ρ has a supporting ellipsoid at every point, then we say ρ is *admissible*.

定义 (sub-differential)

Let ρ be an admissible function. We define a set-valued map $\partial\rho : \Omega \to \mathbb{S}^2$, the so-called *sub-differential*. For any $x_0 \in \Omega$, $\partial\rho(x_0)$ is the set of y_0's, such that $\exists c > 0$, $e_{y_0,c}$ is the supporting ellipsoid of ρ at x_0,

$$\partial\rho(x_0) := \{y_0 \in \mathbb{S}^2 : \exists c > 0, e_{y_0,c} \text{ supports } \rho \text{ at } x_0\}.$$

For any subset $E \subset \Omega$, we define

$$\partial\rho(E) = \bigcup_{x \in E} \partial\rho(x).$$

定义 (Generalized Alexandrov Measure)

Suppose ρ is an admissible function defined on $\Omega \subset \mathbb{S}^2$, $g \in L^1(\Omega^*)$ is a non-negative measurable function defined on $\Omega^* \subset \mathbb{S}^2$, the generalized Alexandrov measure induced by ρ and g, denoted as $\mu_{\rho,g}$, is defined as

$$\mu_{\rho,g}(E) = \int_{\partial\rho(E)} g(x)\,dx, \quad \forall \text{ Borel } E \subset \Omega. \tag{32}$$

定义 (Generalized Solution)

Given spherical measures $f \in L^1(\Omega)$ and $g \in L^1(\Omega^*)$, such that $\int_\Omega f dx = \int_{\Omega^*} g dy$. Suppose ρ is a spherical admissible function. If the generalized Alexandrov measure induced by ρ satisfies $\mu_{\rho,g} = f dx$, namely

$$\int_E f = \int_{\partial\rho(E)} g, \quad \forall \text{ Borel } E \subset \Omega \tag{33}$$

furthermore, if ρ satisfies

$$\Omega^* \subset \partial\rho(\Omega), \quad |\{x \in \Omega : f(x) > 0 \text{ and } \partial\rho(x) - \overline{\Omega^*} \neq \emptyset\}| = 0, \tag{34}$$

then we say ρ is a generalized solution to the spherical Monge-Ampère equation with natural boundary condition. □

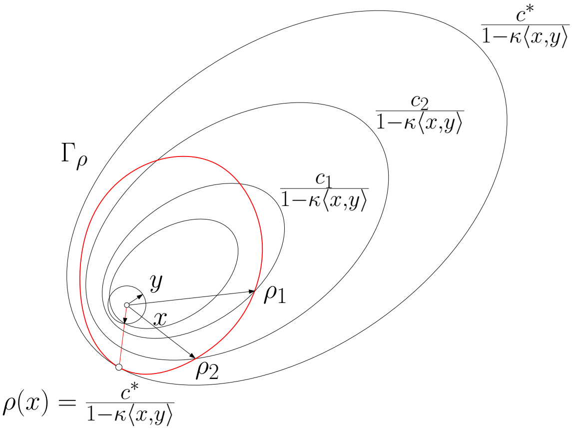


图: Generalized Legendre transform.

Among all ellipsoids $e_{y,c}$'s of revolution about the axis of direction y intersecting with Γ_ρ, $c \leq c^*$. If Γ_ρ intersects $e_{y,c}$ at $\rho(x) = \frac{c}{1-\kappa\langle x,y\rangle}$, $c = \rho(x)(1-\kappa\langle x,y\rangle)$, thus we obtain

$$c^*(y) = \sup_{x\in\Omega} \rho(x)(1-\kappa\langle x,y\rangle) \iff \frac{1}{c^*(y)} = \inf_{x\in\Omega} \frac{1}{\rho(x)(1-\kappa\langle x,y\rangle)}.$$

$1/c^*(y)$ is the function of y, denoted as $\eta : \Omega^* \to \mathbb{R}_+$.

定义 (Generalized Legendre Transform)

Suppose ρ is an admissible function defined on Ω. The generalized Legendre transform of ρ with respect to the function $\frac{1}{1-\kappa\langle x,y\rangle}$ is a function η defined on the sphere $\mathbb{S}^2$ 上的函数 η, given by

$$\eta(y) = \inf_{x\in\Omega} \frac{1}{\rho(x)(1-\kappa\langle x, y\rangle)}. \quad \blacklozenge \tag{35}$$

Denote $\Omega^* = \partial\rho(\Omega)$. For any fixed point $y_0 \in \Omega^*$, (35) reaches the infimum at $x_0 \in \Omega$, then

$$\eta(y_0)\rho(x_0) = \frac{1}{1 - \kappa\langle x_0, y_0\rangle}, \tag{36}$$

For arbitrary $x \in \Omega$ and $y \in \Omega^*$,

$$\rho(x)\eta(y) \leq \frac{1}{1 - \kappa\langle x, y\rangle}. \tag{37}$$

we have

$$y_0 \in \partial\rho(x_0) \iff x_0 \in \partial\eta(y_0).$$

Especially, the generalized Legendre transform of η, restricted on Ω, is ρ itself,

$$\eta^{**} = \eta, \quad (\partial\eta)^{-1} = \partial\rho$$

$$\rho^{**} = \rho, \quad (\partial\rho)^{-1} = \partial\eta$$

定理

Suppose Ω *and* Ω^* *are domains in* $\mathbb{S}^{n-1}$*, the illumination intensity of the emanating ray lights is represented by a positive bounded function* $f(x)$ *defined on* Ω*, the illumination intensity of the refracted rays is represented by a positive bounded function* $g(y)$ *on* $\overline{\Omega^*}$*. Suppose* $|\partial\Omega| = 0$ *and satisfies the physical constraint*

$$\inf_{x\in\Omega, y\in\Omega^*} \langle x, y\rangle \geq \kappa. \tag{38}$$

furthermore, assume the total energy is conserved

$$\int_{\Omega} f(x)\,dx = \int_{\Omega^*} g(y)\,dy > 0, \tag{39}$$

where dx*,* dy *represent the Hausdorff measure on* $\mathbb{S}^{n-1}$*. Then for* $\kappa < 1$*, there exists a week solution* Γ_ρ*, all such solutions* Γ_ρ*'s differ by a scaling.* □

证明.

By the (DP) theorem in optimal transportation, there are a pair of functions (ϕ, ψ), unique up to a constant, maximizing the following energy

$$\sup\{I(u,v) : (u,v) \in K\},$$

where
$$I(u,v) = \int_{\Omega} f(x)u(x)\,dx + \int_{\Omega^*} v(y)g(y)\,dy,$$

$$K = \left\{(u,v) \in (C(\overline{\Omega}), C(\overline{\Omega^*})) : u(x) + v(y) \le c(c,y), \forall x \in \Omega, y \in \Omega^*\right\},$$

$$c(x,y) = -\log(1 - \kappa\langle x, y\rangle),$$

where $\langle x, y\rangle$ is the inner product in $\mathbb{R}^n$, such that $\rho = e^{\phi}$ is the solution to the spherical Monge-Ampère equation with the natural boundary condition. □

定理

Suppose Ω and Ω^* are domains in $\mathbb{S}^{n-1}$, the illumination intensity of the emanating ray lights is represented by a positive bounded function $f(x)$ defined on Ω, the illumination intensity of the refracted rays is represented by a positive bounded function $g(y)$ on $\overline{\Omega^*}$. Suppose $|\partial\Omega| = 0$ and satisfies the physical constraint

$$\inf_{x\in\Omega, y\in\Omega^*} \langle x, y\rangle \geq \frac{1}{\kappa}. \tag{40}$$

furthermore, assume the total energy is conserved

$$\int_\Omega f(x)\,dx = \int_{\Omega^*} g(y)\,dy > 0, \tag{41}$$

where dx, dy represent the Hausdorff measure on $\mathbb{S}^{n-1}$. Then for $\kappa > 1$, there exists a week solution Γ_ρ, all such solutions Γ_ρ's differ by a scaling. □

Proof for Existence and Uniqueness

The proof is similar to the proof for the case of $\kappa < 1$, but the cost function is modified as

$$c(x, y) = -\log(\kappa\langle x, y\rangle - 1). \tag{42}$$

Simulation Result

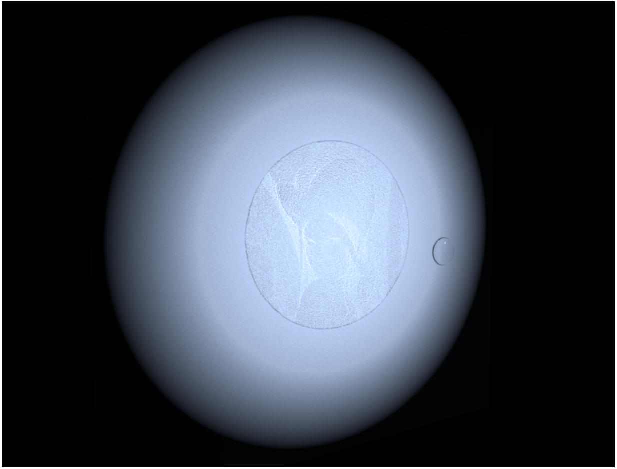


图: Reflector Design

1. Area-preserving Parameterization;
2. Minkowski Problem I;
3. Reflector Design;
4. Refractor Design $\kappa < 1$;
5. Refractor Design $\kappa > 1$;

Summary

Source measure (Ω, μ), target measure (Ω^*, ν), cost function $c(x, y)$, Kantorovich potential function (φ, ψ), density function $d\mu(x) = f(x)\,dx$, $d\nu(y) = g(y)\,dy$,

$$\sup\left\{\int_\Omega \varphi f + \int_{\Omega^*} \psi g : \varphi \oplus \psi \leq c\right\}$$

$$\psi(y) = \varphi^c, \quad \varphi(x) = \psi^{\bar{c}}$$

	cost $c(x, y)$	support $c(x, y) - \psi(y)$	potential $\varphi = \inf_y c(x, y) - \psi(y)$
1	$\langle x, y\rangle$	$\langle x, y\rangle - \psi(y)$	$\varphi(x) = \sup_y \langle x, y\rangle - \psi(y)$
2	$-\log\langle x, y\rangle$	$\frac{e^{-\psi(y)}}{\langle x,y\rangle}$	$\rho(x) = e^{\varphi(x)} = \inf_y \frac{e^{-\psi(y)}}{\langle x,y\rangle}$
3	$-\log(1 - \langle x, y\rangle)$	$\frac{e^{-\psi(y)}}{1-\langle x,y\rangle}$	$\rho(x) = e^{\varphi(x)} = \inf_y \frac{e^{-\psi(y)}}{1-\langle x,y\rangle}$
4	$-\log(1 - \kappa\langle x, y\rangle)$	$\frac{e^{-\psi(y)}}{1-\kappa\langle x,y\rangle}$	$\rho(x) = e^{\varphi(x)} = \inf_y \frac{e^{-\psi(y)}}{1-\kappa\langle x,y\rangle}$
5	$-\log(\kappa\langle x, y\rangle - 1)$	$\frac{e^{-\psi(y)}}{\kappa\langle x,y\rangle-1}$	$\rho(x) = e^{\varphi(x)} = \inf_y \frac{e^{-\psi(y)}}{\kappa\langle x,y\rangle-1}$

Summary

	map $\nabla_x c(x, T(x)) = \nabla\varphi(x)$	support $c(x, y) - \psi(y)$	Legendre Dual $\psi(y) = \inf_x c(x, y) - \varphi(x)$
1	$T(x) = \nabla\varphi(x)$	plane	$\psi(y) = \sup_x \langle x, y\rangle - \varphi(x)$
2	$T(x) = n(x)$	plane	$\eta(y) = e^{\psi(y)} = \inf_x \frac{e^{-\varphi(x)}}{\langle x,y\rangle}$
3	$T(x) = x - 2\langle x, n\rangle n$	paraboloid	$\eta(y) = e^{\psi(y)} = \inf_x \frac{e^{-\varphi(x)}}{1-\langle x,y\rangle}$
4	$n(x) = \frac{x-\kappa T(x)}{\lvert x-\kappa T(x)\rvert}$	ellipsoid	$\eta(y) = e^{\psi(y)} = \inf_x \frac{e^{-\varphi(x)}}{1-\kappa\langle x,y\rangle}$
5	$n(x) = \frac{x-\kappa T(x)}{\lvert x-\kappa T(x)\rvert}$	hyperboloid	$\eta(y) = e^{\psi(y)} = \inf_x \frac{e^{-\varphi(x)}}{\kappa\langle x,y\rangle-1}$

For more information, please contact gu@cs.stonybrook.edu

Thank You !