

# Conformal Modules via Geometric Complex Analysis

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# Surface Uniformization

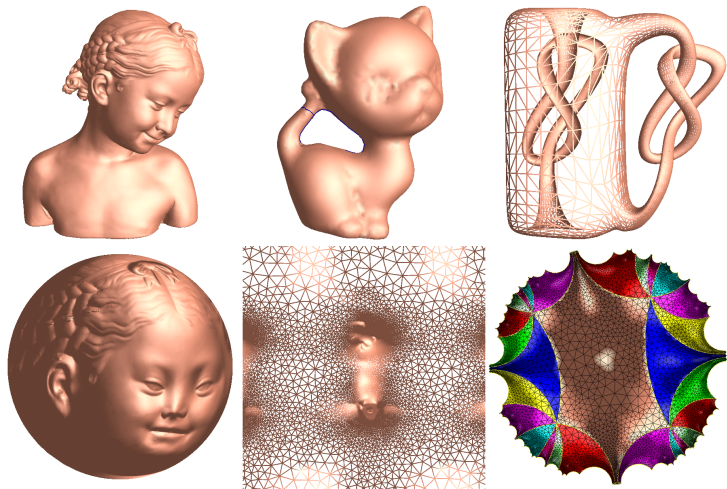


Figure: Closed surface uniformization.

# Surface Uniformization

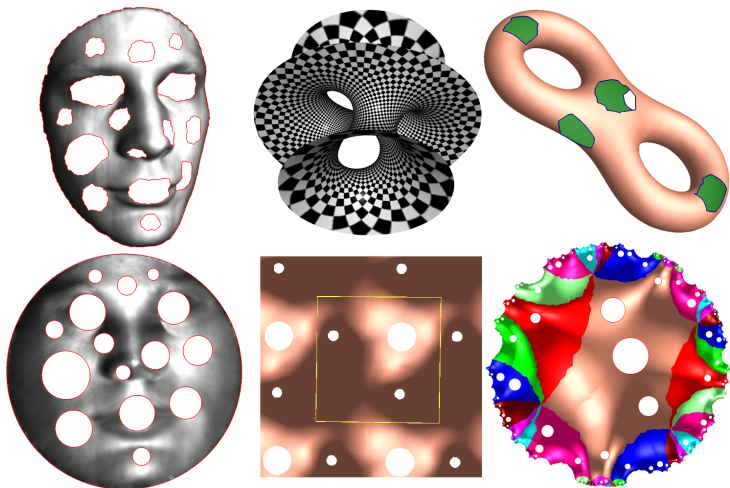


Figure: Open surface uniformization.

# Historic Open Question

## Problem

*Can we find a 3-5 quad-mesh on a torus ?*

## Strategy

- 1 Define a family of complex functions, with some constraints;
- 2 Show the family is a normal family;
- 3 **Estimate** some geometric or analytic bounds, such as distortions;
- 4 Maximize some coefficient of an item in the Laurent series;
- 5 Show the limit exists by normal family property;
- 6 Show the limit is the desired mapping.

Examples include Riemann mapping, slit mapping and so on.

# Basic Concepts in Geometric Complex Analysis

# Normal Family Definition

## Definition (Uniform Convergence)

Assume  $\{f_n : \Omega \rightarrow \mathbb{C}\}$  is a sequence of holomorphic functions defined on an open set  $\Omega$ . We say the functions uniformly converge to a function  $f : E \rightarrow \mathbb{C}$ , if for any  $\varepsilon > 0$ , there is a  $n_0$ , such that for any  $n > n_0$  and any  $z \in E$ , we have

$$|f_n(z) - f(z)| < \varepsilon.$$

## Definition (Normal Family)

Let  $\Omega \subset \mathbb{C}$  be an open set on  $\mathbb{C}$ ,  $\mathcal{F}$  is a normal family, if any subsequence  $\{f_n\}$  in  $\mathcal{F}$  uniformly converge on any compact subset in  $\Omega$ .

## Theorem (Weierstrass)

*Let  $\{f_n : \Omega \rightarrow \mathbb{C}\}$  be a sequence of holomorphic functions defined on an open set  $\Omega \subset \mathbb{C}$ , assume  $\{f_n\}$  uniformly converges to  $f : \Omega \rightarrow \mathbb{C}$  on compact subsets in  $\Omega$ , then  $f$  is holomorphic and  $\{f'_n : \Omega \rightarrow \mathbb{C}\}$  uniformly converges to  $f' : \Omega \rightarrow \mathbb{C}$ .*



# Normal Family Properties

## Definition (Univalent Map)

Let  $U \subset \mathbb{C}$  be open subset on  $\mathbb{C}$ , if holomorphic map  $f : U \rightarrow \mathbb{C}$  is injective, namely  $z_1 \neq z_2$  implies  $f(z_1) \neq f(z_2)$ , then  $f$  is called a univalent map or univalent function.

## Theorem (Hurwitz)

*Let  $\{f_n : \Omega \rightarrow \mathbb{C}\}$  be a family of holomorphic functions defined on an open set  $\Omega \subset \mathbb{C}$ , such that for any  $n$  and  $z \in \Omega$ ,  $f_n(z) \neq 0$ . If  $\{f_n\}$  uniformly converges to  $f : \Omega \rightarrow \mathbb{C}$  on compact sets of  $\Omega$ , then either  $f \equiv 0$  or for any  $z \in \Omega$ ,  $f(z) \neq 0$ .*

## Corollary

*Let  $\Omega$  be an open set in  $\mathbb{C}$ , let  $\{f_n : \Omega \rightarrow \mathbb{C}\}$  be a holomorphic function series, and uniformly converges to  $f : \Omega \rightarrow \mathbb{C}$  on compact sets. If each  $f_n$  on  $\Omega$  is univalent, then either  $f$  is constant, or  $f$  is univalent on  $\Omega$ .*

## Definition (Uniformly Bounded on Compact Sets)

Let  $\mathcal{F}$  be a family of holomorphic functions, if for any compact set  $E \subset \Omega$ , there exists a constant  $M$ , such that for any  $z \in E$  and any function  $f \in \mathcal{F}$ , we have  $|f(z)| \leq M$ , then we say  $\mathcal{F}$  is uniformly bounded on compact sets.

## Definition (equicontinuous)

Let  $\mathcal{F}$  be a family of holomorphic functions defined on open set  $\Omega \subset \mathbb{C}$ . We say  $\mathcal{F}$  is equicontinuous, if for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any distinct points  $z$  and  $z'$ ,  $|z - z'| \leq \delta$  implies  $|f(z) - f(z')| < \varepsilon$  for any  $f \in \mathcal{F}$ .

## Theorem (Montel)

Let  $\mathcal{F}$  be a family of holomorphic functions defined on an open set  $\Omega \subset \mathbb{C}$ , if  $\mathcal{F}$  is uniformly bounded on compact sets in  $\Omega$ , then

- 1  $\mathcal{F}$  is equicontinuous on each compact set in  $\Omega$ ;
- 2  $\mathcal{F}$  is a normal family.

- 1 Fix a point  $p \in \Omega$ , a family of univalent holomorphic functions  $\mathcal{F}$  is a normal family, if for any  $f \in \mathcal{F}$ ,  $|f(p)| < M$  and  $|f'(p)| < N$ .
- 2 A family of holomorphic functions  $\mathcal{F}$ , if there are three points  $\{z_1, z_2, z_3\}$ , such that for any  $f \in \mathcal{F}$ , the image of  $f$  doesn't include them, then  $\mathcal{F}$  is a normal family.
- 3 If  $\mathcal{F}$  is a normal family, then

$$\mathcal{F}^{-1} = \{f^{-1} | f \in \mathcal{F}\}$$

is also a normal family.

## Definition ( $\mathcal{S}$ Family)

All univalent holomorphic functions defined on the unit disk, with normalization condition form a normal family:

$$\mathcal{S} = \{f : \mathbb{D} \rightarrow \mathbb{C} : f \text{ univalent, } f(0) = 0, f'(0) = 1\}$$

any  $f \in \mathcal{F}$  has Taylor expansion in a neighborhood of 0,

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots,$$

Taylor series converge in the unit disk  $|z| < 1$ .

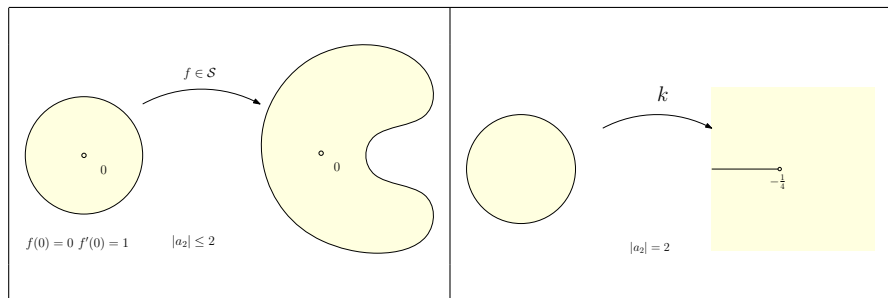
# Geometric Distortion Estimate

## Definition (Koebe Function)

The holomorphic function  $k(z) \in \mathcal{S}$ ,

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \dots$$

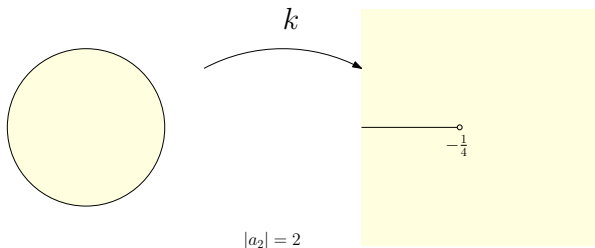
is called the Koebe function, which maps  $\mathbb{D}$  to  $\mathcal{C} \setminus (-\infty, -1/4]$ .



# Geometric Distortion Estimate

## Theorem (Bieberbach $a_2$ of $\mathcal{S}$ )

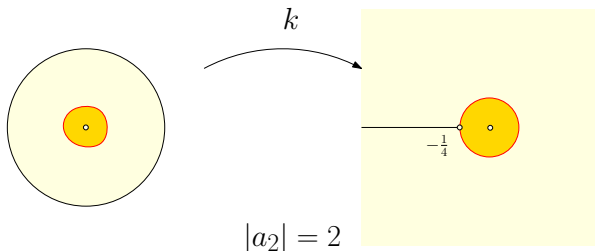
If  $f \in \mathcal{S}$ , then  $|a_2| \leq 2$ , equality holds if and only if  $f$  is a rotation of the Koebe function.



# Geometric Distortion Estimate

## Theorem (Koebe 1/4)

For any  $f \in \mathcal{S}$ ,  $f(\mathbb{D})$  includes an open disk  $|w| < 1/4$ . If there exists a  $|w| = 1/4$  and  $w \notin f(\mathbb{D})$ , then  $f$  is a rotation of Koebe function.



# Geometric Distortion Estimate

## Proof.

Let  $f(z) = z + a_2z^2 + a_3z^3 \dots$  be a function of  $\mathcal{S}$ ,  $w \notin f(\mathbb{D})$ . Construct a holomorphic function

$$h(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right)z^2 + \dots$$

then  $h(z)$  is in  $\mathcal{S}$ , by Bieberbach theorem,

$$\left|a_2 + \frac{1}{w}\right| \leq 2 \quad (1)$$

and  $|a_2| \leq 2$ , therefore  $|1/w| \leq 4$ ,  $|w| \geq 1/4$ . Equality holds if and only if  $f$  is a rotation of Koebe function.  $\square$



## Definition ( $\Sigma$ Family)

All holomorphic functions defined on  $\Delta = \{|w| > 1\}$  with normalization condition form a normal family,

$$\Sigma = \{g : \Delta \rightarrow \mathbb{C} : g \text{ univalent, } g(\infty) = \infty, g'(\infty) = 1\},$$

for any  $g \in \Sigma$ , it has Laurent power series in a neighborhood of  $\infty$ ,

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

the series converges in  $\Delta$ .

# Geometric Distortion Estimate

## Definition (Full Mapping Family)

The family of holomorphic functions

$$\tilde{\Sigma} := \{f : \Delta \rightarrow \mathbb{C} : f \in \Sigma, \mathbb{C} \setminus f(\Delta) \text{ has zero Lebesgue Measure}\}$$

## Theorem (Gronwall Area)

Suppose  $g \in \Sigma$ , and

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$$

then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

equality holds if and only if  $g$  is a full mapping,  $g \in \tilde{\Sigma}$ .

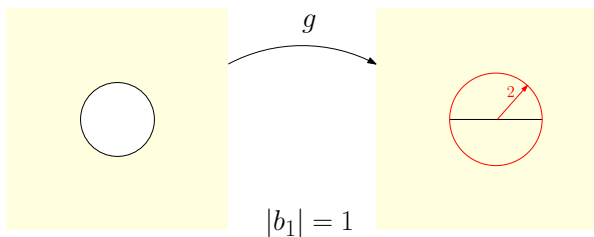
# Geometric Distortion Estimate

## Corollary ( $b_1$ of $\Sigma$ )

If  $g \in \Sigma$ , then  $|b_1| \leq 1$ , equality holds if and only if

$$g(z) = z + b_0 + \frac{b_1}{z}, |b_1| = 1 \quad (2)$$

$g$  maps  $\Delta$  to the complement of a length segment with length 4.



# Geometric Distortion Estimate

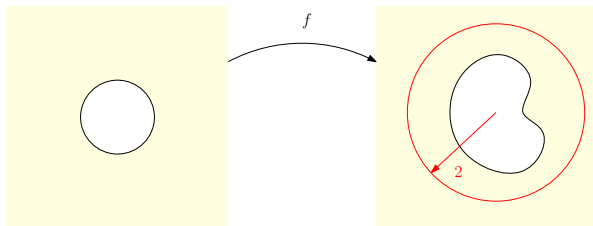
## Corollary

For any  $f \in \Sigma$ ,  $f : \{|z| > 1\} \rightarrow \mathbb{C}$ ,  $f(\infty) = \infty$ ,  $f'(\infty) = 1$ ,

$$f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots,$$

we have

$$\partial f(|z| > 1) = f(|z| = 1) \subset \{|w - b_0| \leq 2\}. \quad (3)$$



# Geometric Distortion Estimate

## Proof.

If  $f(z) \in \Sigma$ , then  $f(z^{-1})^{-1} \in \mathcal{S}$ ,

$$f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \dots$$

therefore

$$\begin{aligned} f(z^{-1}) &= \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \dots \\ f(z^{-1})^{-1} &= z(1 + b_0 z + b_1 z^2 + \dots)^{-1} \\ &= z(1 - (b_0 z + b_1 z^2 + \dots) + \dots) \\ &= z - b_0 z^2 - b_1 z^3 + \dots \end{aligned}$$

let  $g(z) = f(z^{-1})^{-1}$ , then  $g(0) = 0$ ,  $g'(0) = 1$ , hence  $g \in \mathcal{S}$ . □

## Continued.

Given any point  $\zeta \in \partial\mathbb{D}$ ,  $|\zeta| = 1$ , then  $w = g(\zeta) \notin g(\mathbb{D})$ , by Bieberbach inequality (1),

$$\left| -b_0 + \frac{1}{w} \right| \leq 2,$$

by  $w = g(\zeta) = 1/f(\zeta^{-1})$ , we obtain  $1/w = f(1/\zeta)$ . Set  $\zeta' = 1/\zeta \in \partial\mathbb{D}$ , we obtain

$$| -b_0 + f(\zeta') | \leq 2.$$

# Riemann Mapping

## Theorem (Riemann)

*Given a non-empty, simply connected, open subset  $\Omega \subset \mathbb{C}$ ,  $\Omega$  is not the entire complex plane  $\mathbb{C}$ , for any point  $z_0 \in \Omega$ , there exists a unique biholomorphic mapping from  $\Omega$  to the unit disk  $\mathbb{D}$ ,  $f : \Omega \rightarrow \mathbb{D}$ , such that  $f(z_0) = 0$  and  $f'(z_0) > 0$ .*



## Uniqueness

If we don't require  $f(z_0) = 0$  and  $f'(z_0) > 0$ , then conformal mapping is not unique. All such kind of mappings differ by a Möbius transformation,  $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ ,

$$\varphi(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad z_0 \in \mathbb{D}, \theta \in [0, 2\pi)$$

## Extendibility

If  $\Omega$  is a Jordan domain, the boundary  $\partial\Omega$  is a piecewise analytical curves, then the conformal mapping  $\varphi$  can be extended to the boundary

$$\varphi : \partial\Omega \rightarrow \partial\mathbb{D}.$$

# Riemann Mapping

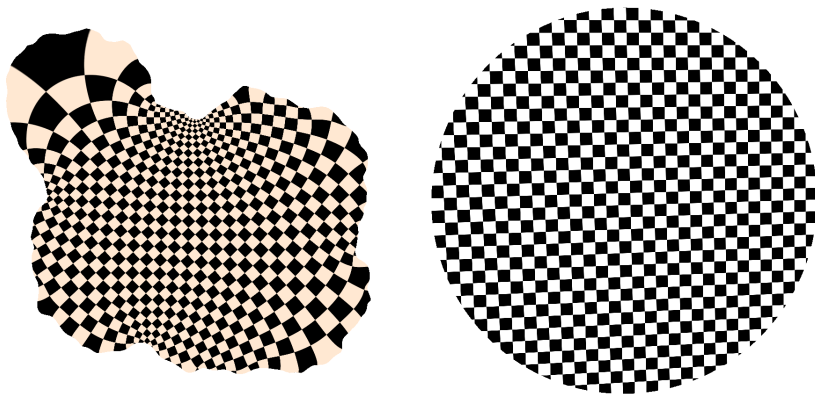


Figure: Riemann Mapping

# Riemann Mapping

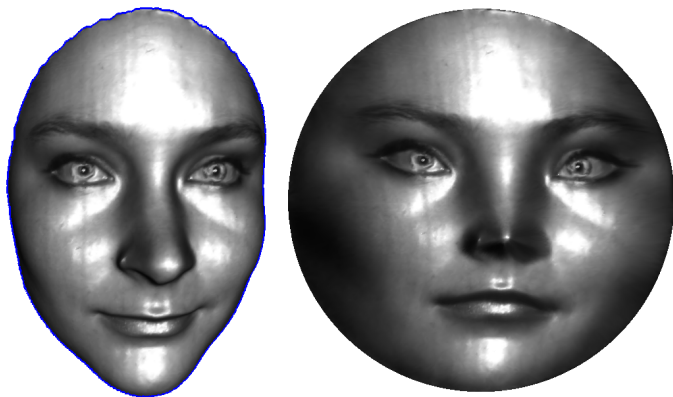


Figure: Riemann Mapping

# Riemann Mapping

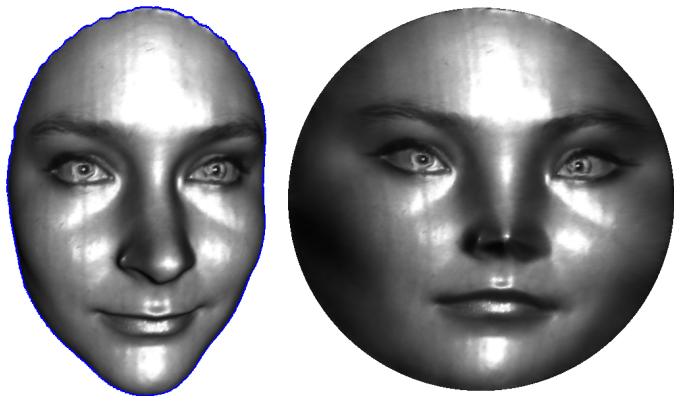


Figure: Riemann Mapping

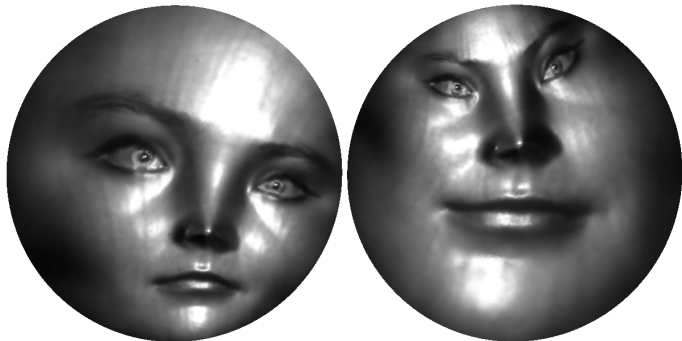


Figure: Möbius Transformation.

# Riemann Mapping

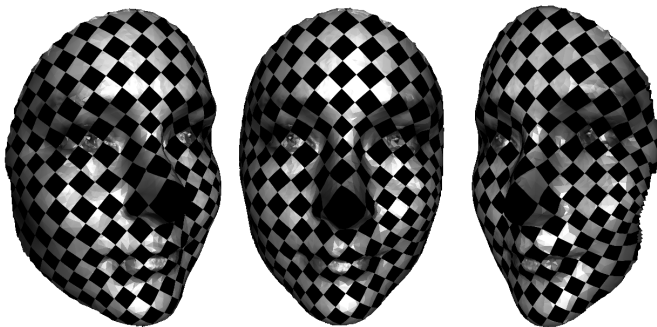


Figure: Texture mapping.

## Lemma (Schwartz)

Assume  $f(z)$  is analytic on  $\mathbb{D} = \{|z| < 1\}$ , satisfying  $|f(z)| \leq 1$ , and  $f(0) = 0$ , then  $|f'(0)| \leq 1$  and for  $\forall z \in \mathbb{D}$ ,

$$|f(z)| \leq |z|.$$

If  $|f'(0)| = 1$ , or  $\exists 0 \neq z_0 \in \mathbb{D}$ , such that  $|f(z_0)| = |z_0|$ , then  $f$  is a rotation,

$$f(z) = e^{i\theta} z.$$

# Schwartz Lemma

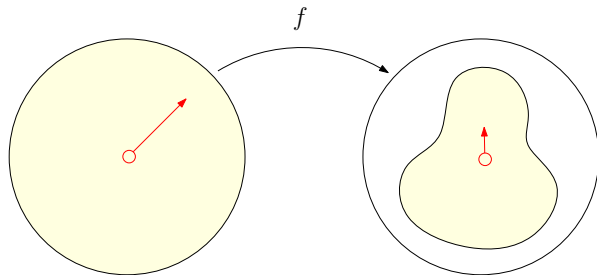


Figure: Schwarz lemma.



## Proof.

Since  $f$  is holomorphic, it can be represented as power series in a neighborhood of 0,

$$f(z) = a_0 + a_1z + a_2z^2 + \cdots$$

Because  $f(0) = 0$ ,  $a_0 = 0$ , hence

$$f(z) = a_1z + a_2z^2 + \cdots = z(a_1 + a_2z + a_3z^2 + \cdots),$$

the power series in the parenthesis converge. Construct auxiliary holomorphic function,

$$g(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$



## Proof.

here the auxiliary function has power series  $g(z) = a_1 + a_2z + a_3z^3 + \dots$  converges in  $\mathbb{D}$ , where  $g(0) = a_1 = f'(0)$ . On every circle  $|z| = r < 1$ ,  $|f(z)| < 1$ , the norm of the function

$$|g(z)| = \frac{|f(z)|}{|z|} < 1/r.$$

By maximal value principle, on the entire disk  $|z| < r$ ,  $|g(z)| < 1/r$ , let  $r \rightarrow 1$ , we obtain on the unit disk  $\mathbb{D}$ ,

$$|g(z)| \leq 1,$$

namely  $|f(z)| \leq |z|$ . If at some interior point  $z_0$ ,  $|g(z_0)| = 1$ , by maximal value principle,  $g(z)$  must be a constant  $a$ . By  $|a| = 1$ , we get  $a = e^{i\theta}$ ,

$$f(z) = e^{i\theta} z.$$

# Uniqueness of Riemann Mapping

## Lemma

Assume  $f : \mathbb{D} \rightarrow \mathbb{D}$  is a conformal automorphism of the unit disk, then  $f(z)$  must be a Möbius transformation.

## Proof.

We construct a Möbius transformation

$$\varphi(z) = \frac{z - f(0)}{1 - \overline{f(0)}z},$$

then  $g = \varphi \circ f$  is a conformal automorphism of  $\mathbb{D}$ , and  $g(0) = 0$ . By Schwarz lemma, for all  $z \in \mathbb{D}$ ,  $|g(z)| \leq |z|$ . Similarly,  $w = g(z)$ , then  $|g^{-1}(w)| \leq |w|$ , therefore, for all  $z \in \mathbb{D}$ ,  $|g(z)| = |z|$ . By Schwarz lemma, we get  $g(z) = e^{i\theta}z$ . Hence  $f(z) = \varphi^{-1}(z) \circ g(z)$  is a Möbius transformation. □

# Existence Proof

Consider the functions family  $\mathcal{F}$ , consisting all functions  $g(z) : \Omega \rightarrow \mathbb{D}$  satisfying the following 3 conditions:

- 1  $g(z)$  is analytic and univalent on  $\Omega$ ;
- 2  $\forall z \in \Omega, |g(z)| < 1$ ;
- 3  $g(z_0) = 0$  and  $g'(z_0) > 0$ .

The whole proof has three steps:

- 1 the function family  $\mathcal{F}$  is non-empty,  $\mathcal{F} \neq \emptyset$ ;
- 2 there exists a function  $f \in \mathcal{F}$ , such that  $f'(z_0)$  is maximized;
- 3 this function  $f$  is the desired conformal mapping.

## Step 1 $\mathcal{F} \neq \emptyset$

There is a point  $a \neq \infty$ ,  $a \notin \Omega$ . Since  $\Omega$  is simply connected, we can define a single-valued branch of  $\sqrt{z-a}$ , denoted as  $h(z)$ .  $h(z)$  won't take the same value twice, or take the opposite value: if  $w \in h(\Omega)$ , then  $-w \notin h(\Omega)$ . Choose a small disk  $|w - h(z_0)| < \rho$  inside  $h(\Omega)$ , then  $|w + h(z_0)| < \rho$  has no intersection point with  $h(\Omega)$ . Therefore for any  $z \in \Omega$ ,  $|h(z) + h(z_0)| > \rho$ ,

$$h_0(z) := \frac{\rho}{h(z) + h(z_0)}$$

is univalent on  $\Omega$ , and  $\forall z \in \Omega$ ,  $|h_0(z)| < 1$ . Choose  $\theta_0 \in [0, 2\pi)$ , such that  $h_0'(z_0) > 0$ , where

$$h_1(z) := e^{i\theta_0} \frac{h_0(z) - h_0(z_0)}{1 - \overline{h_0(z_0)} h_0(z)}, \quad h_1 \in \mathcal{F}.$$

$$\mathcal{F} \neq \emptyset$$

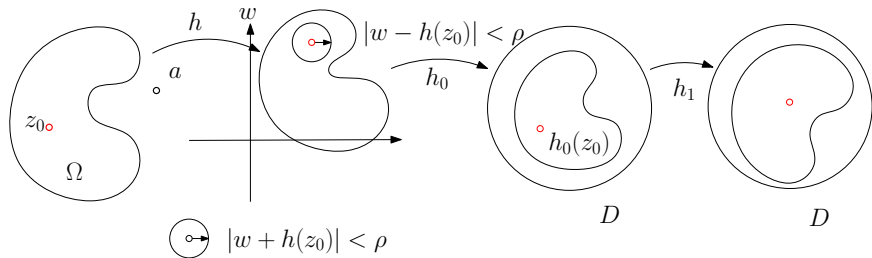


Figure:  $\mathcal{F}$  is non-empty.

## Step 2

Define supreme

$$\beta = \sup_{g \in \mathcal{F}} g'(z_0),$$

there is a sequence  $\{g_n\} \subset \mathcal{F}$ , such that

$$\lim_{n \rightarrow \infty} g_n'(z_0) = \beta.$$

Based on Montel theorem,  $\mathcal{F}$  is a normal function family, hence there is a subsequence  $\{g_{n_k}\} \subset \{g_n\}$ , which converges to an analytic function  $f$  on  $\Omega$ , and uniformly converges on any compact subset on  $\Omega$ . By Weierstrass theorem,  $\beta = f'(z_0)$ . Because  $\beta$  is finite, and  $\beta > 0$ , we obtain  $f$  is not constant.

## Step 3

Because  $\{g_n\}$  on  $\Omega$  is univalent, by Hurwitz the limit function  $f$  is also univalent.  $f$  is analytic, therefore conformal. We need to show  $f$  is surjective. Because  $f$  is bijective,  $\Omega$  is simply connected, hence  $f(\Omega)$  is simply connected. Assume there is an interior point  $w_0 \in \mathbb{D}$ , such that  $w_0 \notin f(\Omega)$ . Define a function  $f_2 : \Omega \rightarrow \mathbb{D}$ ,

$$f_2(z) := \sqrt{\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}}$$

has an analytic branch, restricted on the image set  $f_2 : \Omega \rightarrow f_2(\Omega)$  is bijective,  $f_2(\Omega) \subset \mathbb{D}$ . Let

$$F(z) = \frac{f_2(z) - f_2(z_0)}{1 - \overline{f_2(z_0)} f_2(z)},$$

then  $F : \Omega \rightarrow \mathbb{D}$  is injective.



## Step 3

By  $f(z_0) = 0$ , we obtain  $|f_2(z_0)| = \sqrt{|w_0|}$ ,

$$\begin{aligned} |F'(z)| &= \left| \frac{1 - \overline{f_2(z_0)}f_2(z_0)}{[1 - \overline{f_2(z_0)}f_2(z)]^2} \right| |f_2'(z)| \\ &= \left| \frac{1 - \overline{f_2(z_0)}f_2(z_0)}{[1 - \overline{f_2(z_0)}f_2(z)]^2} \right| \frac{1}{2\sqrt{\frac{f(z)-w_0}{1-\overline{w_0}f(z)}}} \left| \frac{1 - \overline{w_0}w_0}{[1 - \overline{w_0}f(z)]^2} \right| |f'(z)| \end{aligned} \quad (4)$$

## Step 3

plug in  $z = z_0$

$$\begin{aligned} |F'(z_0)| &= \left| \frac{1 - |f_2(z_0)|^2}{[1 - |f_2(z_0)|^2]^2} \right| \frac{1}{2\sqrt{\left| \frac{f(z_0) - w_0}{1 - \bar{w}_0 f(z_0)} \right|}} \left| \frac{1 - |w_0|^2}{[1 - \bar{w}_0 f(z_0)]^2} \right| |\beta| \\ &= \frac{1}{1 - |w_0|} \frac{1}{2\sqrt{|w_0|}} |1 - |w_0|^2| \cdot |\beta| \\ &= \frac{1 + |w_0|}{2\sqrt{|w_0|}} |\beta| > |\beta| \end{aligned} \tag{5}$$

Construct the function

$$g(z) = \frac{|F'(z_0)|}{F'(z_0)} F(z),$$

then  $g \in \mathcal{F}$  and  $g'(z_0) > \beta$ . Contradiction. Hence  $f : \Omega \rightarrow \mathbb{D}$  is surjective.

# Topological Annulus

# Conformal Mapping for Annulus



(a) Topological annulus



(b) Conformal module

Figure: Canonical conformal mapping for topological annulus.

## Theorem

*Suppose  $\Omega$  is a doubly connected domain on  $\mathbb{C}$ , then  $\Omega$  is conformally equivalent to a canonical annulus.*

# Conformal Module for Topological Annulus

## Step 1.

Assume  $\partial\Omega = \gamma_1 - \gamma_2$ , both  $\gamma_1$  and  $\gamma_2$  include more than 1 point, and  $\gamma_1$  is finite. Suppose the complementary of  $\gamma_1$  has two connected components, the one containing  $\Omega$  is denoted as  $\Omega_1$ . By Riemann mapping theorem, we can conformally map  $\Omega_1$  onto the unit planar disk  $|z'| < 1$ ,  $\Omega$  is mapped to  $\Omega'$ ,  $\gamma_2$  to  $\gamma'_2$  inside the unit disk.

## Step 2.

The complementary of  $\gamma'_2$  has two connected components, the one containing  $\Omega'$  is denoted as  $\Omega'_2$ . We conformally map  $\Omega'_2$  onto the exterior to the unit disk  $|z''| > 1$ , mapping  $z' = \infty$  to  $z'' = \infty$ .  $\gamma'_1 \mapsto \gamma''_1$ ,  $\Omega' \mapsto \Omega''$ ,  $\infty \notin \Omega''$ ,  $\partial\Omega'' = \gamma''_1 - \gamma''_2$ .

# Conformal Module for Topological Annulus

## Step 3.

Use the map  $t = \log z''$ , map  $\Omega''$  to  $B_1$ ,  $B_1$  is included in the right half plane  $\{t | \Re t > 0\}$ . The mapping is not one-to-one,  $B_1$  is a infinite stripe,  $B_1$  is periodic, for any  $t \in B_1$ ,  $t + 2k\pi i$ ,  $k \in \mathbb{Z}$  is also in  $B_1$ .

## Step 4.

By Riemann mapping theorem, there is a map  $\omega = f(t)$ , which maps  $B_1$  to the vertical stripe region

$$B_2 := \{\omega | 0 < \Re \omega < h\},$$

the mapping  $f : B_1 \rightarrow B_2$  maps

$$f : \{-\sqrt{-1}\infty, 0 + \sqrt{-1}\infty\} \mapsto \{-\sqrt{-1}\infty, 0 + \sqrt{-1}\infty\}.$$

Because both  $B_1$  and  $B_2$  are simply connected, the boundaries are with more than one point, since they are conformal equivalent.

# Conformal Module for Topological Annulus

## Step 4. Continued

Assume  $f(2\pi i) = \omega_0$ , by scaling map, we can assume  $f(2\pi i) = 2\pi i$ . We prove the mapping has the property:

$$f(t + 2\pi i) = f(t) + 2\pi i.$$

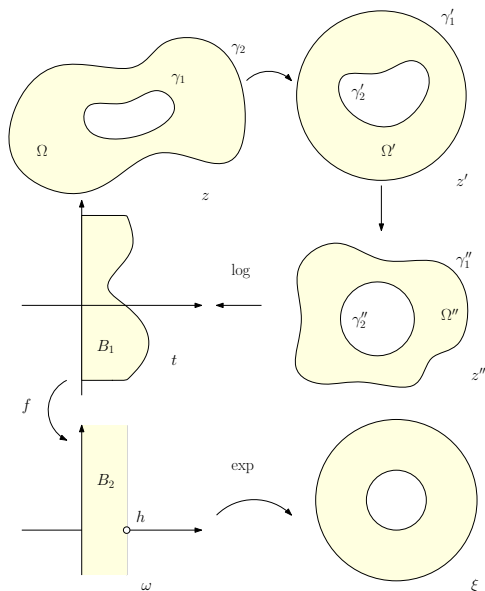
Since both two conformal mappings  $f(t + 2\pi i) - 2\pi i$  and  $f(t)$  map  $B_1$  to  $B_2$ , and maps  $-\infty i, 0, +\infty i$  to  $-\infty i, 0, +\infty i$ , therefore by the uniqueness of Riemann mapping  $f(t + 2\pi i) - 2\pi i = f(t)$ .

## Step 5.

The map  $\xi = \exp(\omega)$  maps  $B_2$  to the canonical annulus  $1 < |\xi| < e^h$ , the composition  $\xi = \exp(f(\log z''))$  maps  $\Omega''$  to the annulus  $1 < |\xi| < e^h$ , which is conformal injective. So the composition of all the mappings together is the conformal mapping between  $\Omega$  to  $1 < |\xi| < e^h$ .



# Conformal Module for Topological Annulus



# Slit Map Topological Poly-annulus

# Slit Map

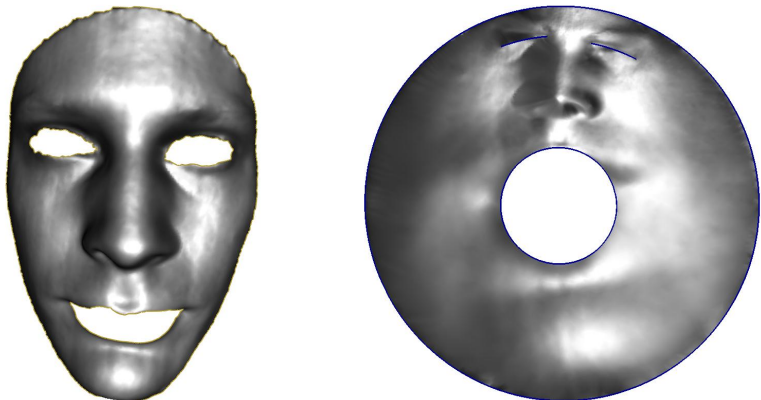


Figure: Slit map.

## Definition (slit domain)

A connected open set (domain)  $\Omega \subset \mathbb{C}$  is called a slit domain, if every connected component of its boundary  $\partial\Omega$  is either a point or a horizontal closed interval.

## Theorem (Hilbert)

*Given any domain  $\Omega \subset \mathbb{C}$ , its boundary has finite number of connected components, then  $\Omega$  is conformal equivalent to a slit domain.*

# Hilbert Theorem

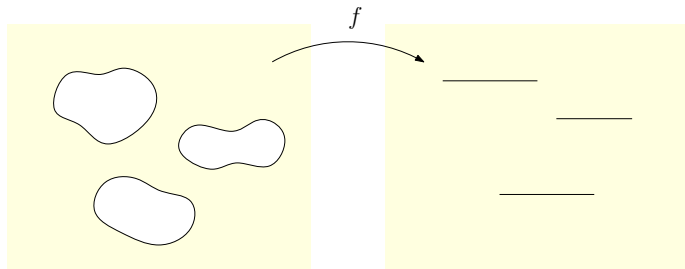


Figure: Hilbert theorem.

# Hilbert Theorem

## Lemma

*In a neighborhood of  $\infty$ , given analytic functions*

$$\alpha(z) = z + \frac{k_1}{z} + \cdots, \quad \beta(z) = z + \frac{h_1}{z} + \cdots,$$

*then*

$$\beta \circ \alpha(z) = z + \frac{k_1 + h_1}{z} + \cdots \quad (6)$$

## Proof.

By direct computation. □

## Proof.

Given a planar domain  $\Omega \subset \hat{\mathbb{C}}$ , by a Möbius transformation, we can assume  $\infty \in \Omega$  and  $\Omega \subset \{|z| > 1\}$ , let univalent holomorphic mapping family be

$$\Sigma = \left\{ f : \Omega \rightarrow \hat{\mathbb{C}} \mid f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, |z| > 1 \right\},$$

if  $f \in \Sigma$ , then  $f(\infty) = \infty$  and  $f'(\infty) = 1$ . Let  $f(z) = z$ , then  $f \in \Sigma$ ,  $\Sigma \neq \emptyset$ .

Consider the function family  $\Sigma^{-1} = \{f^{-1} \mid f \in \Sigma\}$ , by Corollary (3), we have

$$\{|z| < 1\} \subset [f^{-1}(|w - b_0| > 2)]^c,$$

hence  $f^{-1}(|w - b_0| > 2)$  excludes three points  $\{-1 + \varepsilon, 0, 1 - \varepsilon\}$ , therefore  $\Sigma^{-1}$  is a normal function family, hence  $\Sigma$  is a normal functional family.  $\square$

# Normal family

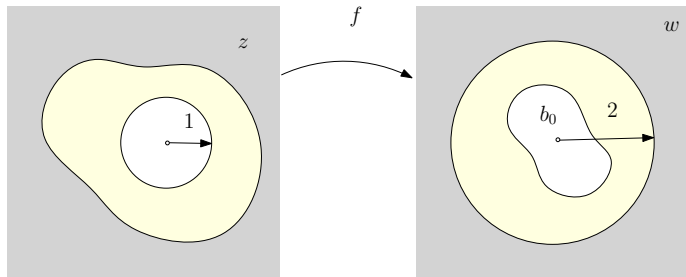


Figure: Estimate of image.



## Proof.

By the compactness of normal function family, there exists a limit  $f \in \Sigma$ , such that

$$\Re_f(b_1) = \max_{g \in \Sigma} \Re_g(b_1),$$

we will show  $f(\Omega)$  is a slit domain. Otherwise, there is a connected component  $\Gamma$  of  $\partial f(\Omega)$ ,  $\Gamma$  is neither a point or a horizontal line segment. Construct a map

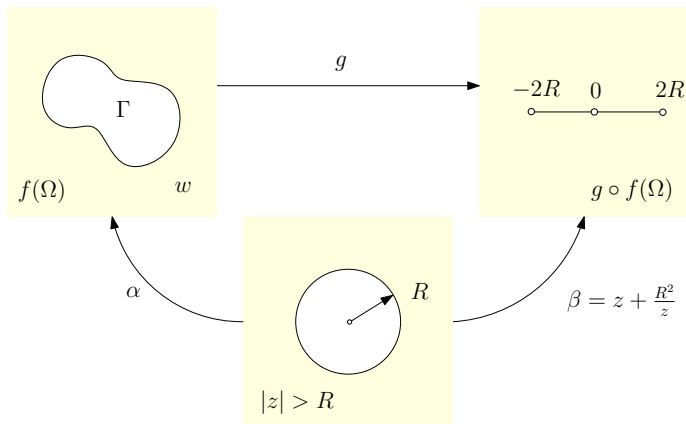
$$g : \hat{\mathbb{C}} \setminus \Gamma \rightarrow \hat{\mathbb{C}} \setminus [-2R, 2R]$$

as follows: first construct the inverse map of a Riemann mapping  $\alpha : \{|z| > R\} \rightarrow \hat{\mathbb{C}} \setminus \Gamma$ ,

$$\alpha(z) = z + \frac{\varepsilon}{z} + \dots$$

and slit map  $\beta : \{|z| > R\} \rightarrow \hat{\mathbb{C}} \setminus [-2R, 2R]$ ,  $\beta(z) = z + \frac{R^2}{z}$ , □

# Slit Map



continued.

The the composition map  $g : \hat{\mathbb{C}} \setminus \Gamma \rightarrow \hat{\mathbb{C}} \setminus [-2R, 2R]$ ,

$$g(w) = \beta \circ \alpha^{-1}(w) = w + \frac{\lambda}{w} + \dots,$$

by the corollary of Gronwall theorem (2), compare  $\alpha$  and  $\beta$ , they maps the complement of the disk to planar domains, the real part of  $b_1$  of the slit map reaches the maximum, hence

$$R^2 = \Re_{\beta}(b_1) > \Re_{\alpha}(b_1) = \varepsilon.$$

By Eqn. (6),  $\beta(z) = g \circ \alpha(z)$ , we obtain

$$R^2 = \Re_{\beta}(b_1) = \Re_{g \circ \alpha}(b_1) = \Re_g(b_1) + \Re_{\alpha}(b_1) = \lambda + \varepsilon > \varepsilon.$$

Therefore  $\Re_g(b_1) = \lambda > 0$ .

continued.

By Eqn. (6), on  $\{|z| > 1\}$ , composition map

$$g \circ f(z) = z + \frac{\Re_f(b_1) + \lambda}{z} + \dots,$$

by  $\lambda > 0$ , we obtain  $\Re_{g \circ f}(b_1) > \Re_f(b_1)$ , this contradicts to the choice of  $f$ . Hence the assumption is incorrect, the claim holds.

# Slit Map Algorithm



Figure: Exact harmonic forms.

# Slit Map Algorithm



Figure: Closed, non-exact harmonic forms.

# Slit Map Algorithm

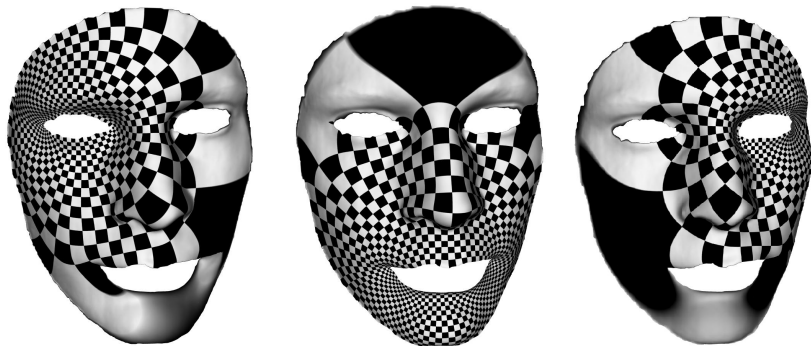


Figure: Holomorphic forms.

# Slit Map Algorithm

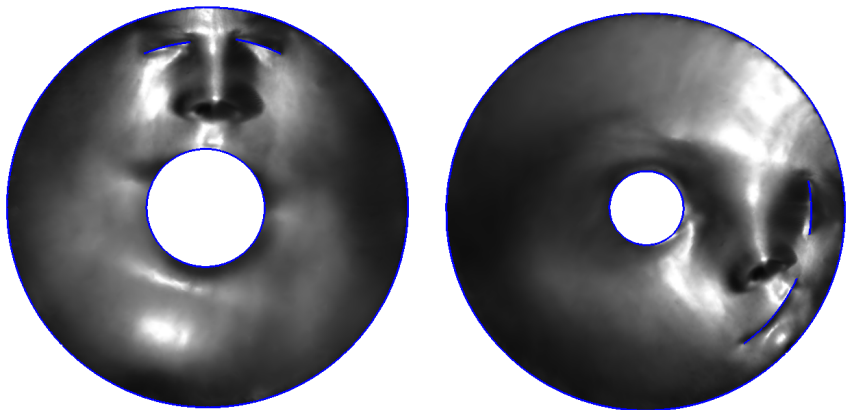


Figure: Slit maps.



# Slit Map Algorithm

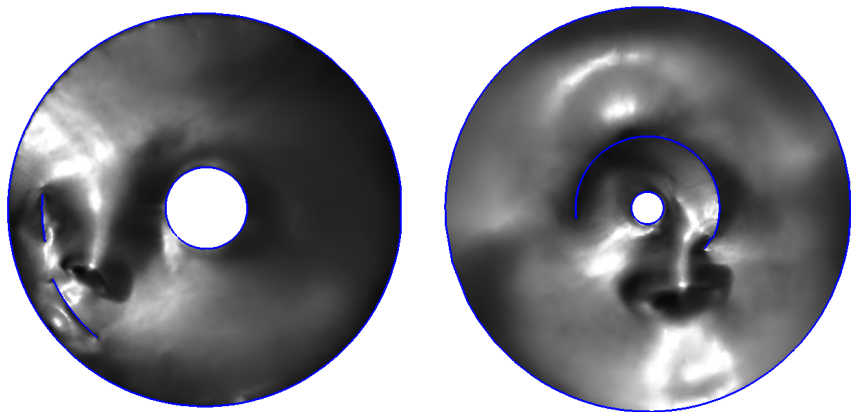


Figure: Slit maps.

# Slit Map Algorithm

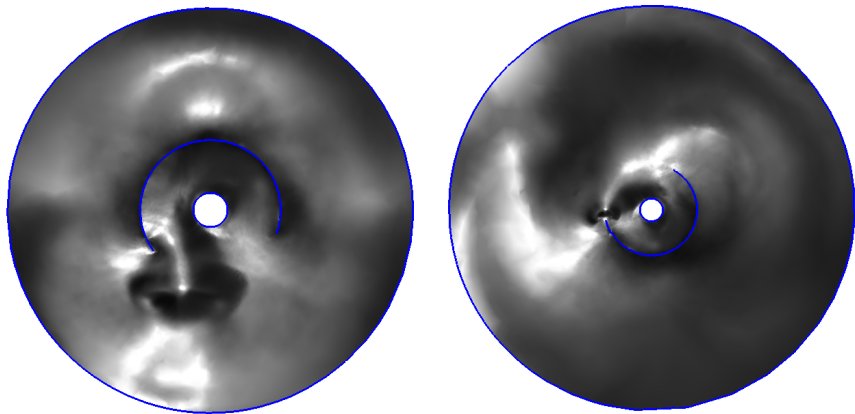


Figure: Slit maps.

# Slit Map Algorithm

Input: A genus zero mesh with  $n + 1$  boundary components  $M$ ,  
 $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$ ;

Output: A slit map  $: M \rightarrow D$ ,  $D$  is a circular slit domain.

- 1 Compute exact harmonic 1-forms  $\omega_1, \omega_2, \dots, \omega_n$ ;
- 2 Compute closed, non-exact harmonic 1-forms  $h_1, h_2, \dots, h_n$ ;
- 3 Compute conjugate harmonic 1-forms  ${}^*\omega_1, {}^*\omega_2, \dots, {}^*\omega_n$ ;
- 4 Find special holomorphic 1-form  $\varphi$

$$\Im \int_{\gamma_0} \varphi = 2\pi, \Im \int_{\gamma_1} \varphi = -2\pi, \Im \int_{\gamma_k} \varphi = 0, k = 2, 3, \dots, n.$$

- 5 Slit map  $f : M \rightarrow D$ , choose a fixed based point  $p \in M$ ,

$$f(q) := \exp \int_p^q \varphi$$

the integration path can be chosen arbitrarily.

## Exact Harmonic Forms

Construct  $n$  harmonic functions  $f_1, f_2, \dots, f_n$  with Dirichlet boundary condition, for each  $1 \leq k \leq n$ ,

$$\begin{cases} \Delta f_k(v_i) = 0 & v_i \notin \partial M \\ f_k(v_j) = -1 & v_j \in \gamma_k \\ f_k(v_l) = 0 & v_l \in \partial M \setminus \gamma_k \end{cases}$$

Exact harmonic 1-form group basis are given by

$$\omega_k = df_k, \quad 1 \leq k \leq n.$$

## Random Harmonic Forms

Generate a random 1-form  $\omega$ , according to Hodge decomposition theorem,

$$\omega = df + \delta\eta + h,$$

where  $f$  is a 0-form,  $\eta$  a 2-form,

$$\delta\omega = \delta df, \quad d\omega = d\delta\eta,$$

the harmonic form is given by

$$h = \omega - df - \delta\eta.$$

## Gram–Schmidt Orthonormalization

- 1 for  $k = 1, 2, \dots, n$ ,
  - 1 Generate a random harmonic form  $h_k$ ,
  - 2 Decompose  $h_k$  with respect to the orthonormal frame  $\{h_1, h_2, \dots, h_{k-1}\}$ ,

$$h_k \leftarrow h_k - \sum_{i=1}^{k-1} (h_i, h_k) h_i, \quad (h_i, h_j) := \int_M h_i \wedge^* h_j,$$

- 3 if  $\|h_k\|^2 = (h_k, h_k) < \varepsilon$ , then regenerate  $h_k$  and re-decompose  $h_k$ , until  $\|h_k\|^2 > \varepsilon$
- 4 normalize  $h_k$

$$h_k \leftarrow \frac{h_k}{\sqrt{(h_k, h_k)}}.$$

# Slit Map Algorithm

## Hodge Star Operator

Given an exact harmonic 1-form  $\omega_k$ , then

$$*\omega_k = \lambda_{k1}h_1 + \lambda_{k2}h_2 + \cdots + \lambda_{kn}h_n,$$

$$\begin{pmatrix} h_1 \wedge *\omega_k \\ h_2 \wedge *\omega_k \\ \vdots \\ h_n \wedge *\omega_k \end{pmatrix} = \begin{pmatrix} h_1 \wedge h_1 & h_1 \wedge h_2 & \cdots & h_1 \wedge h_n \\ h_2 \wedge h_1 & h_2 \wedge h_2 & \cdots & h_2 \wedge h_n \\ \vdots & \vdots & & \vdots \\ h_n \wedge h_1 & h_n \wedge h_2 & \cdots & h_n \wedge h_n \end{pmatrix} \begin{pmatrix} \lambda_{k1} \\ \lambda_{k2} \\ \vdots \\ \lambda_{kn} \end{pmatrix}$$

Taking integration on  $M$  for every element, and solve the linear system.

# Slit Map Algorithm

## Special Holomorphic 1-form

Suppose

$$\partial M = \gamma_0 - \gamma_1 - \gamma_2 \cdots - \gamma_n,$$

choose a holomorphic 1-form

$$\varphi = \sum_{i=1}^n \mu_i \left( \omega_i + \sqrt{-1}^* \omega_i \right),$$

$$\begin{pmatrix} 2\pi \\ -2\pi \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \mathfrak{F} \begin{pmatrix} \int_{\gamma_0}^* \omega_1 & \int_{\gamma_0}^* \omega_2 & \cdots & \int_{\gamma_0}^* \omega_n \\ \int_{\gamma_1}^* \omega_1 & \int_{\gamma_1}^* \omega_2 & \cdots & \int_{\gamma_1}^* \omega_n \\ \int_{\gamma_2}^* \omega_1 & \int_{\gamma_2}^* \omega_2 & \cdots & \int_{\gamma_2}^* \omega_n \\ \vdots & \vdots & & \vdots \\ \int_{\gamma_n}^* \omega_1 & \int_{\gamma_n}^* \omega_2 & \cdots & \int_{\gamma_n}^* \omega_n \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_n \end{pmatrix}$$



## Integration

Choose a vertex  $p \in M$ , use width first search to access all the vertices on  $M$ , and for each vertex  $q \in M$ , we obtain a path  $\gamma_q$  from  $p$  to  $q$ , the circular slit mapping is given by

$$f(q) := \exp \left( \int_{\gamma_q} \varphi \right),$$

where

$$\exp(a + \sqrt{-1}b) = e^a(\cos b + \sqrt{-1}\sin b).$$