Surface Quasi-Conformal Mapping by Solving Beltrami Equations

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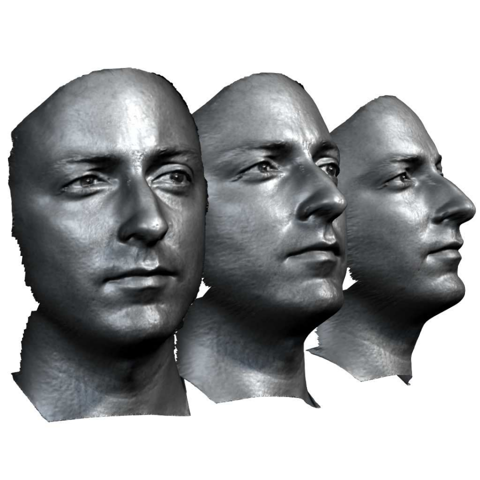
Thanks

Thank for the organizer and reviewers.

Motivation

Motivation

3D geometric data acquisition technology becomes mature.

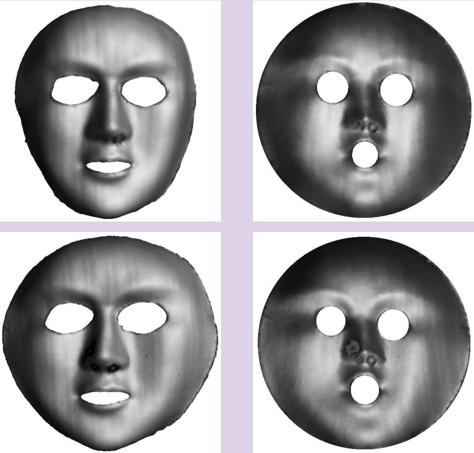


Motivation

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

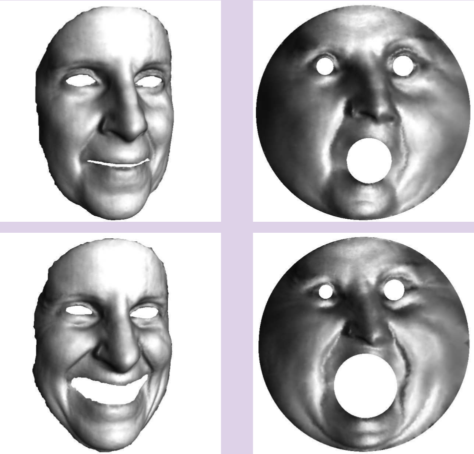
Motivation

Conformal geometry has been applied for studying surface deformation. The following image shows an isometric deformation, which is conformal.



Motivation

In reality, most deformations are quasi-conformal.Same face with different expressions have different conformal moduli.



Motivation

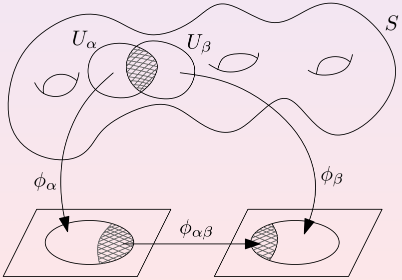
Generalize computational algorithms for conformal mappings to quasi-conformal mappings.

Theoretic Background

Theoretic Background

Definition (Conformal Atlas)

suppose S is a surface covered by a collection of open sets $\mathscr{A} = \{U_\alpha\}$, $S \subset \bigcup U_\alpha$. A chart is (U_α, ϕ_α), where $\phi_\alpha : U_\alpha \to \mathbb{C}$ is a homeomorphism. The chart transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$, $\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$. If all transition functions are holomorphic, then the atlas $\mathscr{A}$ is a conformal atlas.



Theoretic Background

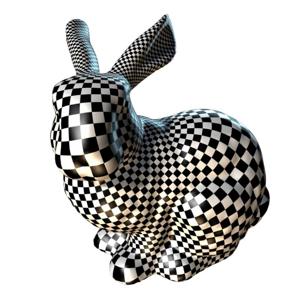
Theorem (Riemann Surface)

Any metric surfaces can be covered by conformal atlas, such that each local coordinate system is isothermal.

Isothermal Coordinates

A surface Σ with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$

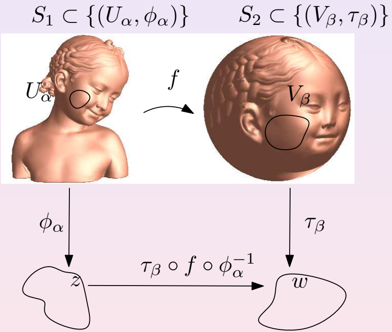


Definition (Conformal Map)

suppose S_1 and S_2 are two Riemann surfaces with conformal atlas $\{(U_\alpha, \phi_\alpha\}$ and $\{(V_\beta, \tau_\beta\}$ respectively, $f : S_1 \to S_2$ is a homeomorphism. If the local representation of f

$$\tau_\beta \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \to \tau_\beta(V_\beta)$$

is holomorphic, then f is a conformal map.



Definition (Quasi-Conformal Map)

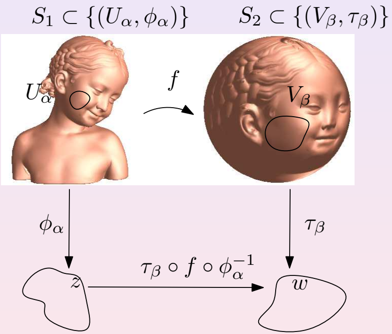
If f is a diffeomorphism, locally

$$df = f_z dz + f_{\bar{z}} d\bar{z}.$$

Let Beltrami coefficient $\mu(z)$ be

$$\mu(z) = \frac{f_{\bar{z}}}{f_z}.$$

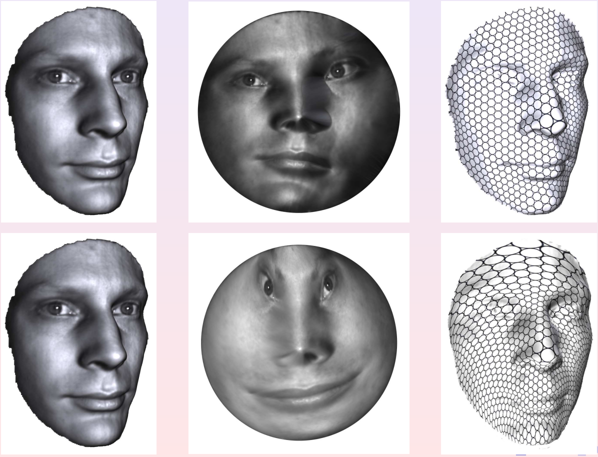
If $|\mu(z)| < \infty$, then f is a quasi-conformal map. $\mu(z) \equiv 0$ iff f is conformal.



Theoretic Background

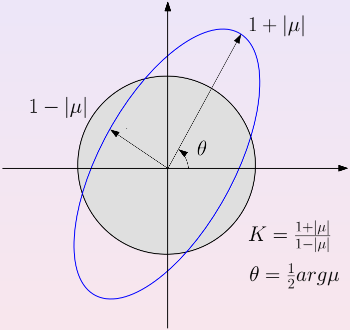
Property

A Conformal mapping maps infinitesimal circles to infinitesimal circles; a quasi-conformal mapping maps circles to ellipses.



Geometric Meaning

The argument of $\mu(z)$ is determined by the orientation of the ellipse; the norm of $\mu(z)$ is determined by the eccentric rate of the ellipse.



Theoretic Background

Beltrami Equation

Suppose $\Omega \subset \mathbb{C}$, given a complex valued measurable function $\mu(z) : \Omega \rightarrow \mathbb{C}$, such that $\| \mu(z) \|_\infty < 1$ almost everywhere on Ω, then the Beltrami equation is

$$\bar{\partial} f(z) = \mu(z) \partial f(z),$$

where $f : \Omega \rightarrow \mathbb{C}$.

Theorem

Let μ be a measurable function in $\Omega \subset \mathbb{C}$ and suppose $\| \mu \|_\infty < 1$. Then there is a homeomorphic solution $g : \Omega \rightarrow \mathbb{C}$ to the Beltrami equation.

Central Result

Suppose S is a topological surface, $\mathscr{A}_1$ and $\mathscr{A}_2$ are two conformal structures on S. Suppose the local complex parameters of $(S,\mathscr{A}_1)$ and $(S,\mathscr{A}_2)$ are z and w. The identity map is a quasi-conformal map, with Beltrami coefficient $\mu(z)$,

$$\frac{\partial w}{\partial \bar{z}} = \mu(z)\frac{\partial w}{\partial z}.$$

Then we construct another conformal atlas $\tilde{\mathscr{A}}_1$. For each chart $(U_\alpha, z_\alpha) \in \mathscr{A}_1$, convert it to $(U_\alpha, \tilde{z}_\alpha)$, such that

$$d\tilde{z}_\alpha = dz_\alpha + \mu(z) d\bar{z}_\alpha.$$

Theorem

The atlas $\tilde{\mathscr{A}}_1$ is a conformal atlas. The Riemann surface $(S,\tilde{\mathscr{A}}_1)$ is conformal equivalent to $(S,\mathscr{A}_2)$.

Converting quasi-conformal mappings to conformal mappings

Given a surface $(S,\mathbf{g})$ and Beltrami coefficient μ, $\|\mu\|_\infty < 1$, find a map $f: S \to \mathbb{C}$, such that $\bar{\partial} f = \mu \partial f$.

1. Compute a conformal mapping $\phi: (S,\mathbf{g}) \to \mathbb{C}$, such that
$$\mathbf{g} = e^{2\lambda(z)} dz d\bar{z}.$$
2. Construct a new metric
$$\tilde{\mathbf{g}} = |dz + \mu(z) d\bar{z}|^2$$
3. Find a conformal map $f: (S,\tilde{\mathbf{g}}) \to \mathbb{C}$, then f is the solution.

Solveing Beltrami Equation

Given metric surfaces $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$, let z, w be isothermal coordinates of S_1, S_2, $w = \phi(z)$.

$$\mathbf{g}_1 = e^{2u_1} dz d\bar{z} \quad (1)$$

$$\mathbf{g}_2 = e^{2u_2} dw d\bar{w}, \quad (2)$$

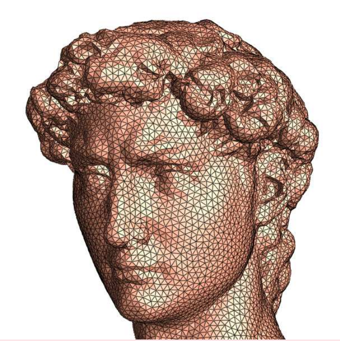
Then

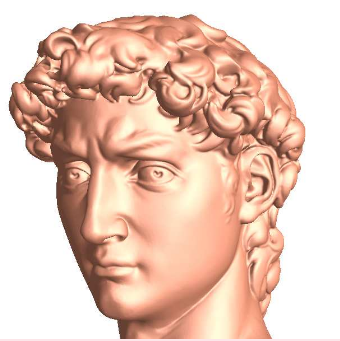
- $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$, quasi-conformal with Beltrami coefficient μ.
- $\phi : (S_1, \phi^* \mathbf{g}_2) \to (S_2, \mathbf{g}_2)$ is isometric
- $\phi^* \mathbf{g}_2 = e^{u_2} |dw|^2 = e^{u_2} |dz + \mu d\bar{z}|^2$.
- $\phi : (S_1, |dz + \mu d\bar{z}|^2) \to (S_2, \mathbf{g}_2)$ is conformal.

Discrete Algorithm

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





Discrete Forms

Definition (Chain Space)

linear combination of splices

$$C_k = \{\sum_i \lambda_i \sigma_i^k | \lambda_i \in \mathbb{Z}\}$$

Definition (Boundary Operator on a simplex)

$$\partial_n [v_0, v_1, \cdots, v_n] = \sum_k (-1)^k [v_0, \cdots, v_{k-1}, v_{k+1}, \cdots, v_n]$$

Definition (Boundary Operator on a k-chain)

$\partial_k : C_k \to C_{k-1}$

$$\partial_k \sum_i \lambda_i \sigma_i^k = \sum_i \lambda_i \partial_k \sigma_i^k .$$

Homology Group

Definition (Homology Group)

linear combination of splices

$$H_k = \frac{Ker\,\partial_k}{Img\,\partial_{k+1}}$$

Cochain Space

Definition (Cochain)

A k-form ω is a linear functional on C_k

$$\omega : C_k \to \mathbb{R}.$$

Definition (Cochain Space)

A k-cochain space C^k is the dual space of C_k

$$C^k = \{\omega | \omega \ k - form\}$$

Definition (Exterior Differentiation)

$d_k : C^k \to C^{k+1}$ linear operator

$$(d_k \omega)(\sigma) = \omega(\partial_k \sigma), \omega \in C^k, \sigma \in C_{k+1}.$$

Cohomology Group

Definition (Cohomology Group)

$$H^k = \frac{Ker d_k}{Img d_{k-1}}$$

Laplace Beltrami Operator

Definition (Edge Weight)

Given an edge $[v_i, v_j]$ adjacent to two faces $[v_i, v_j, v_k]$ and $[v_j, v_i, v_l]$, then the edge weight is defined as

$$w_{ij} = \cot\theta_{ij}^k + \cot\theta_{ji}^l.$$

Given an 1-form ω

$$\Delta\omega(v_i) = \sum_j w_{ij}\,\omega([v_i, v_j]).$$

Definition (Harmonic 1-form)

Let ω be a 1-form, ω is harmonic if

$$\Delta\omega = 0.$$

Hodge Theory

Theorem (Hodge)

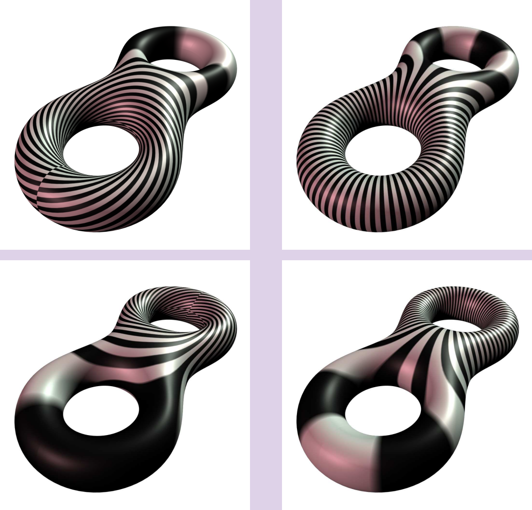
All the harmonic 1-forms form a group, which is isomorphic to $H_1(M)$.

Harmonic 1-form

Each cohomologous class has a unique harmonic 1-form, which represents a vortex free, source-sink free flow field.

Harmonic 1-form

Harmonic 1-form Basis

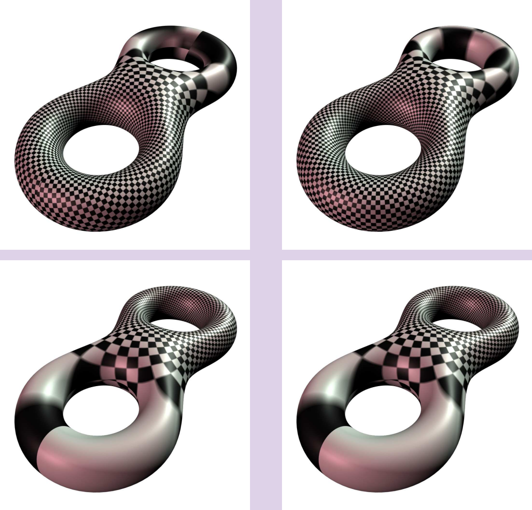


Harmonic 1-form

Conjugate Harmonic 1-form

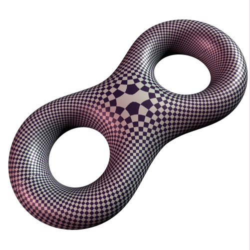
Holomorphic 1-form

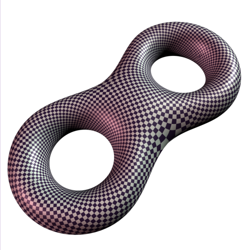
Holomorphic 1-form Basis



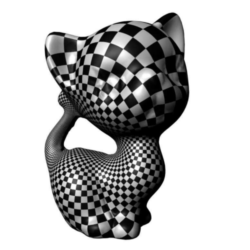
Holomorphic 1-form

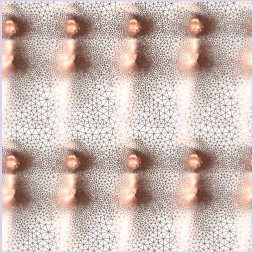
Holomorphic 1-form Basis



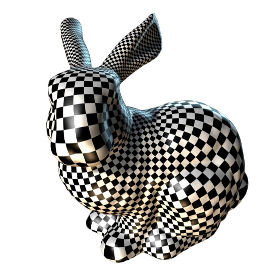


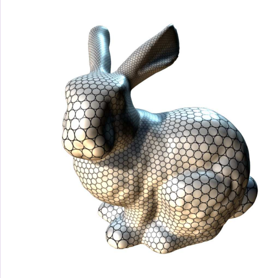
Holomorphic 1-form - Global Conformal Parameterization





Holomorphic 1-form - Global Conformal Parameterization





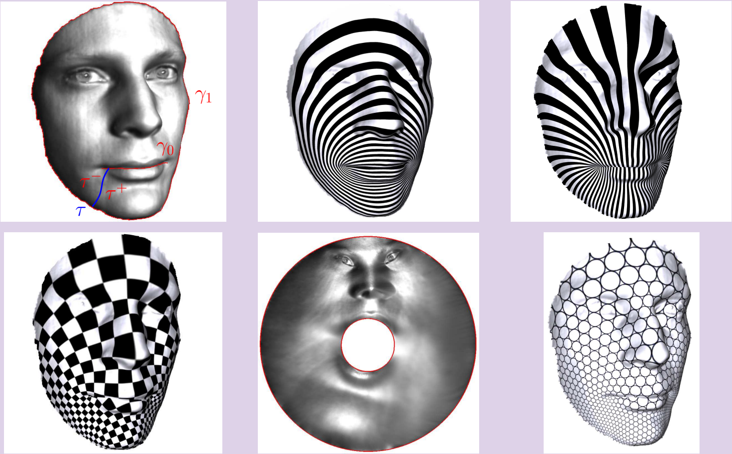
Topological Annulus

1. Compute an exact harmonic 1-form, and its conjugate harmonic 1-form.
2. Combine the two harmonic 1-forms to a holomorphic 1-form ω on the annulus, such that $Im(\int_{\gamma_1} \omega = 2\pi$.
3. Choose a base point p. For any point $q \in S$, map it to

$$\phi(q) = exp(\int_p^q \omega).$$

This mapping maps the whole surface to a planar annulus conformally.

Topological Annulus



Algorithms

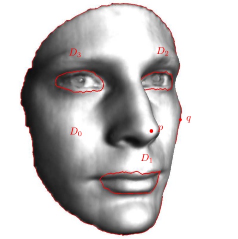
Simply Connected Domains

By removing one point to convert it to a topological annulus.

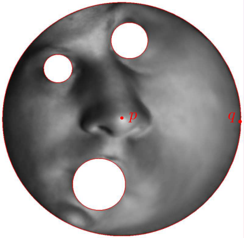
Algorithms

Multiply Connected Domains

Fill all the holes except two, map it to an annulus. Then we fill these two circles, and open another two holes. Iterate this procedure.



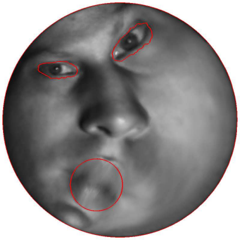
(a) Original surface



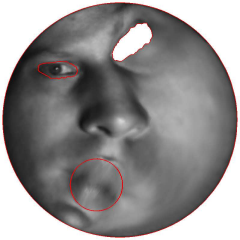
(b) Planar domain

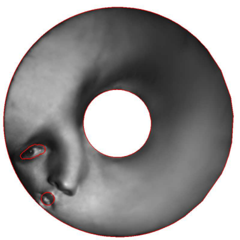
(c) Texture mapping

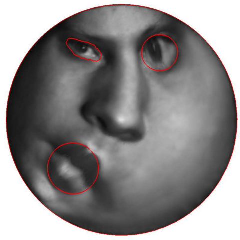
(1). D_1 is removed. (2). Conformal mapping for $S - D_1$. (3). D_1 is glued back.



(1). D_2 is removed. (2). Conformal mapping for $S - D_2$. (3). D_2 is glued back followed by a Möbius transformation.

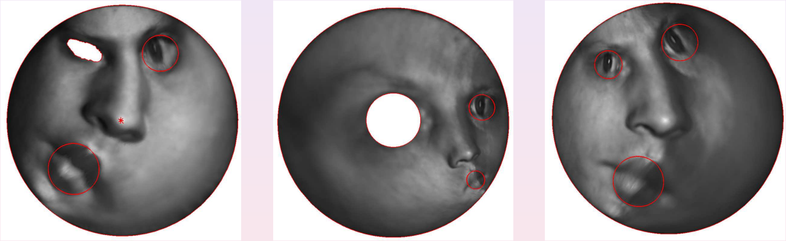






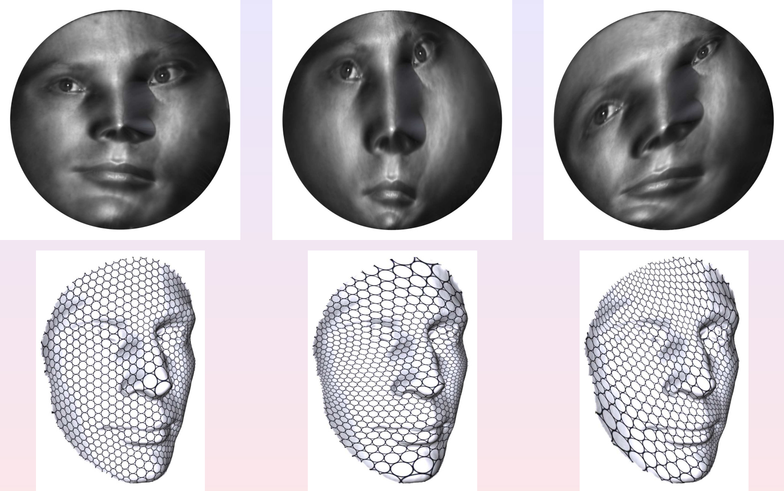
Algorithms

(1). D_3 is removed. (2). Conformal mapping for $S - D_3$. (3). D_3 is glued back followed by a Möbius transformation.



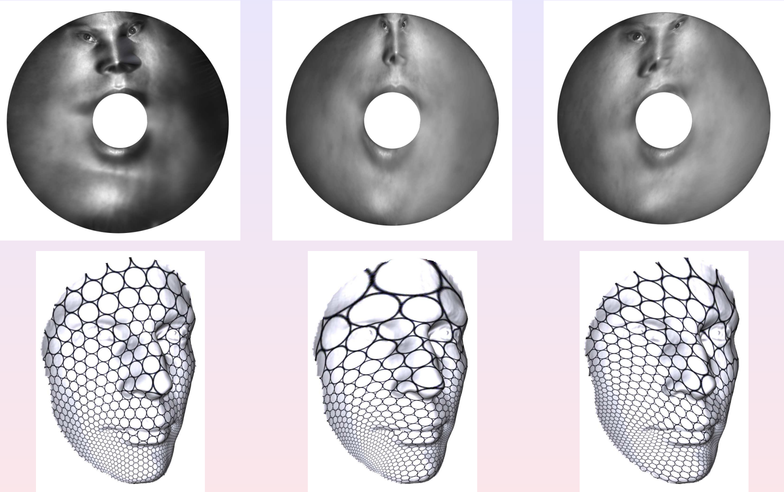
Experimental Results

Experimental Results



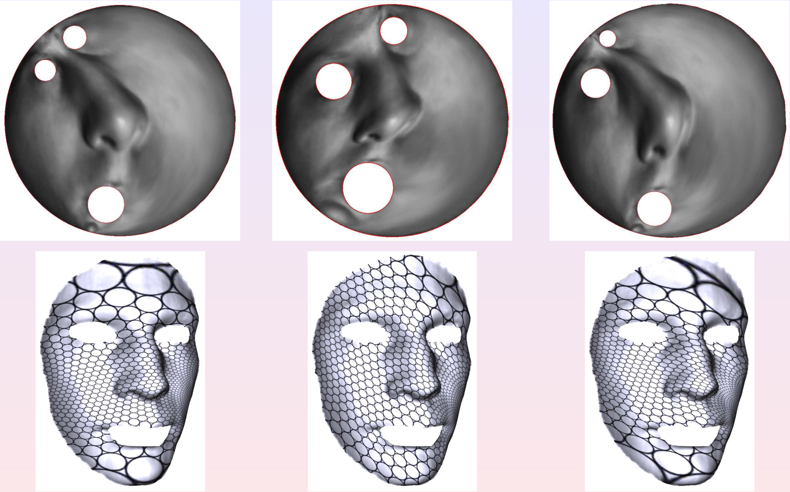
(a) $\mu = 0.0$ (b) $\mu = 0.25$ (c) $\mu = 0.25i$

Experimental Results



(a) $\mu = 0.0$ (b) $\mu = 0.25$ (c) $\mu = 0.25i$

Experimental Results



(a) $\mu = 0.25$ (b) $\mu = 0.25i$ (c) $\mu = 0.25 + 0.25i$

Table: Computational Time.

Figure	#Vertex	#Face	#Bnd	Iter	Time (sec)
Fig. 1(e)	80593	160054	1	1	102
Fig. 6(a)	80593	160054	1	1	73
Fig. 6(b)	80593	160054	1	1	110
Fig. 6(c)	80593	160054	1	1	105
Fig. 7(a)	80724	160054	2	1	78
Fig. 7(b)	80724	160054	2	1	110
Fig. 7(c)	80724	160054	2	1	112
Fig. 9(c)	15160	29974	4	2	156
Fig. 9(a)	15160	29974	4	2	160
Fig. 9(b)	15160	29974	4	2	156
Fig. 9(c)	15160	29974	4	2	157

Summary

- Introduce a generic method for computing quasi-conformal mappings by solving Beltrami equations.
- Give algorithmic details for genus zero surfaces based on holomorphic 1-forms
- The method can be directly applied for surfaces with arbitrary topologies

For more information, please email to gu@cs.sunysb.edu.

Thank you!