Discrete Laplace-Beltrami Operator Determines Discrete Riemannian Metric

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20th Fall Workshop in Computational Geometry (FWCG)

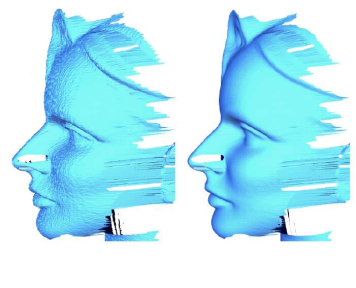
Thanks

Thanks for the invitation.

Motivation

In computational geometry and computer graphics, many recent applications based on Laplace-Beltrami operator.

Mesh Smoothing



[Desbrun et al 1999, etc]

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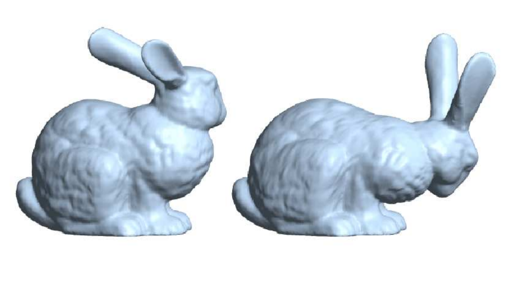
Mesh Editing

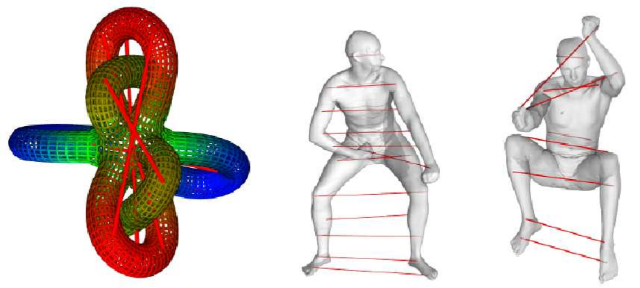


[Zhou et al 2005, Lipman et al 2005, etc]

Motivation

In computational geometry and computer graphics, many recent applications based on Laplace-Beltrami operator.

Shape Analysis

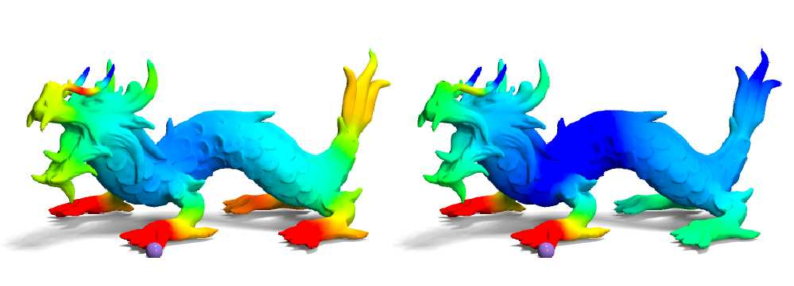


[Ovsjanikov, Sun and Guibas 2008, etc]

Motivation

In computational geometry and computer graphics, many recent applications based on Laplace-Beltrami operator.

Heat Kernel Signature



[Sun, Ovsjanikov, and Guibas 2008, etc]

Laplace-Beltrami operator

Suppose (M,g) is a complete Riemannian manifold, g is the Riemannian metric. $f,g : M \to \mathbb{R}$ are functions. The L^2 norm is given by

$$f,g = \int_M fgdv$$

Δ is the Laplace-Beltrami operator.

$$\Delta(f) = -div(grad(f)).$$

Laplace operator is elliptic, self-adjoint, positive definite.

Eigen values and eigen functions

The eigenvalues $\{\lambda_n\}$ and eigenfunctions $\{\phi_n\}$ of Δ are

$$\Delta\phi_n = -\lambda_n\phi_n,$$

where ϕ_n is normalized to be orthonormal in $L^2(M)$, which form the basis of $L^2(M)$. The collection of $\{\lambda_i\}$'s is called the spectrum of Δ.

Heat Kernel

Definition (Heat Kernel)

There is a heat kernel $K(x,y,t) \in C^\infty(M \times M \times \mathbb{R}^+)$, such that

$$K(x,y,t) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

Heat Kernel

Heat kernel $K(x,y,t)$ means, if we set a unit heat source at point x at time 0, the temperature at y at time t. The heat equation is

$$\frac{\partial}{\partial t}(f_t) + \Delta(f_t) = 0.$$

with initial condition $f_0(x)$. The solution is given by

$$f_t(x) = \int_M K(x,y,t) f_0(y) dy.$$

Heat Kernel

Heat kernel reflects all the information of the Riemannian metric **g**.

Theorem

Let $\Phi : (M_1, g_1) \to (M_2, g_2)$ be a diffeomorphism between two Riemannian manifolds. If f is an isometry, then

$$K_1(x,y,t) = K_2(\Phi(x),\Phi(y),t), \forall x,y \in M, t > 0. \quad (1)$$

Conversely, if f is a surjective map, and Eqn. (1) holds, then f is an isometry.

Main Problem

Prof. Leo Guibas raised the following question in SPM 2009.

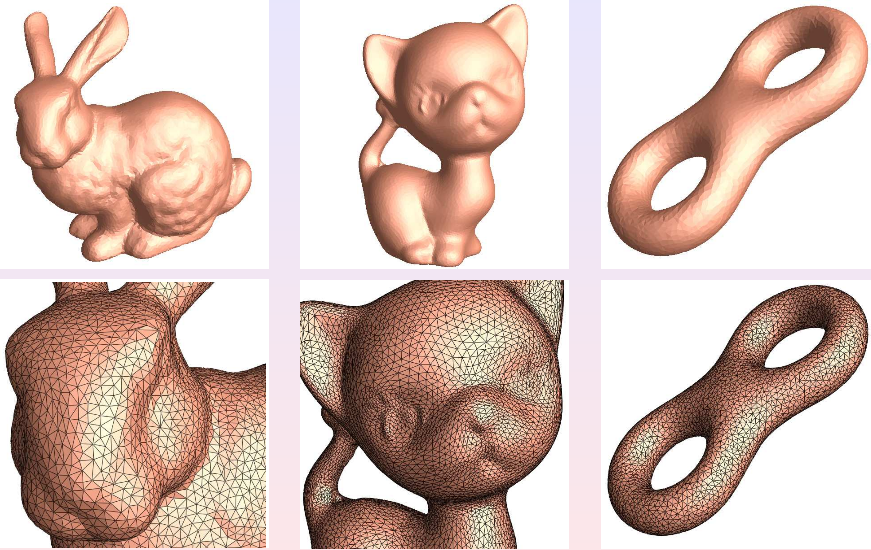
Central Problem

In discrete case, does heat kernel determine the Riemannian metric ?

Definition (Polyhedral Surface)

An Euclidean polyhedral surface is a triple (S,T,d) where S is a closed surface, T is a triangulation of S and d is a metric on S whose restriction to each triangle is isometric to an Euclidean triangle.

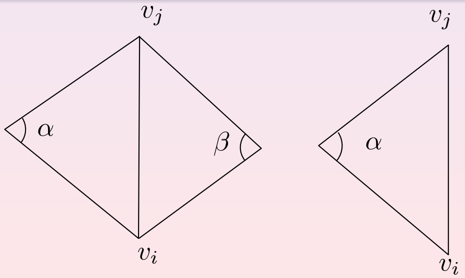
Polyhedral Surface



Euclidean polyhedral surfaces used in the experiments.

Definition (**Cotangent Edge Weight**)

Suppose $[v_i, v_j]$ is a boundary edge of M, $[v_i, v_j] \in \partial M$, then $[v_i, v_j]$ is associated with one triangle $[v_i, v_j, v_k]$, the angle against $[v_i, v_j]$ at the vertex v_k is α, then the weight of $[v_i, v_j]$ is given by $w_{ij} = \frac{1}{2}\cot\alpha$. Otherwise, if $[v_i, v_j]$ is an interior edge, the two angles against it are α, β, then the weight is $w_{ij} = \frac{1}{2}(\cot\alpha + \cot\beta)$.



Cotangent edge weight.

Discrete Laplace Matrix

The discrete Laplace-Beltrami operator is constructed from the cotangent edge weight.

$$\Delta f(v_i) = \sum_{[v_i,v_j]\in E} w_{ij}(f(v_i) - f(v_j)).$$

Definition (Discrete Laplace Matrix)

The discrete Laplace matrix $L = (L_{ij})$ for an Euclidean polyhedral surface is given by

$$L_{ij} = \begin{cases} -w_{ij} & i \neq j \\ \sum_k w_{ik} & i = j \end{cases}$$

Discrete Heat Kernel

Because L is symmetric, it can be decomposed as

$$L = \Phi\Lambda\Phi^T \tag{2}$$

where $\Lambda = diag(\lambda_0, \lambda_1, \cdots, \lambda_n)$, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of L, and $\Phi = (\phi_0|\phi_1|\phi_2|\cdots|\phi_n)$, $L\phi_i = \lambda_i\phi_i$ are the orthonormal eigenvectors, such that $\phi_i^T\phi_j = \delta_{ij}$.

Definition (Discrete Heat Kernel)

The discrete heat kernel is defined as follows:

$$K(t) = \Phi exp(-\Lambda t)\Phi^T. \tag{3}$$

Theorem (Global Rigidity)

Suppose two Euclidean polyhedral surfaces $(S, T, \mathbf{d_1})$ *and* $(S, T, \mathbf{d_2})$ *are given,*

$$L_1 = L_2,$$

if and only if $\mathbf{d_1}$ *and* $\mathbf{d_2}$ *differ by a scaling.*

Corollary

Suppose two Euclidean polyhedral surfaces $(S, T, \mathbf{d_1})$ *and* $(S, T, \mathbf{d_2})$ *are given,*

$$K_1(t) = K_2(t), \forall t > 0,$$

if and only if $\mathbf{d_1}$ *and* $\mathbf{d_2}$ *differ by a scaling.*

Proof.

Note that,

$$\frac{dK(t)}{dt}|_{t=0} = -L.$$

Therefore, the discrete Laplace matrix and the discrete heat kernel mutually determine each other. □

Proof Overview

Connectivity

Fix the connectivity of the polyhedral surface (S,T). Suppose the edge set of (S,T) is sorted as $E = \{e_1, e_2, \cdots, e_m\}$, where $m = |E|$ number of edges, the face set is denoted as F. A triangle $[v_i, v_j, v_k] \in F$ is also denoted as $\{i,j,k\} \in F$.

Metric

An Euclidean polyhedral metric on (S,T) is given by its edge length function $d : E \to \mathbb{R}^+$, denoted as $d = (d_1, d_2, \cdots, d_m)$, where $d_i = d(e_i)$ is the length of edge e_i, such that on each triangle $[v_i, v_j, v_k]$

$$\{(d_1, d_2, d_3) | d_i + d_j > d_k\}$$

Proof Overview-Admissible Metric Space

Definition (Admissible Metric Space)

Given a triangulated surface (S, K), the admissible metric space is defined as

$$\Omega_u = \{(u_1, u_2, u_3 \cdots, u_m) | \sum_{k=1}^{m} u_k = m, (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_d(2), \forall \{i, j, k\} \in$$

where

$$E_d(2) = \{(d_1, d_2, d_3) | d_i + d_j > d_k\}$$

We show that Ω_u is a convex domain in $\mathbb{R}^m$.

Proof Overview-Energy

Definition (Energy)

An energy $E:\Omega_u \to \mathbb{R}$ is defined as:

$$E(u_1,u_2\cdots,u_m)=\int_{(1,1,\cdots,1)}^{(u_1,u_2\cdots,u_m)}\sum_{k=1}^{m} w_k(\mu)d\mu_k, \tag{4}$$

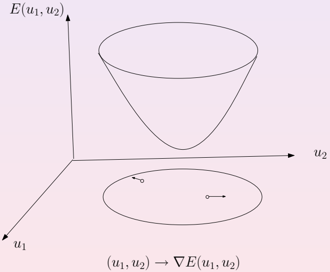
where $w_k(\mu)$ is the cotangent weight on the edge e_k determined by the metric μ.

We show that the energy is convex.

Proof Overview - Rigidity Lemman

Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is an open convex domain in $\mathbb{R}^n$, $E : \Omega \to \mathbb{R}$ is a strictly convex function with positive definite Hessian matrix, then $\nabla E : \Omega \to \mathbb{R}^n$ is a smooth embedding.



Rigidity lemma.

Proof of Rigidity lemma

Proof.

If $\mathbf{p} \neq \mathbf{q}$ in Ω, let $\gamma(t) = (1-t)\mathbf{p} + t\mathbf{q} \in \Omega$ for all $t \in [0,1]$. Then $f(t) = E(\gamma(t)) : [0,1] \to \mathbb{R}$ is a strictly convex function, so that

$$\frac{df(t)}{dt} = \nabla E|_{\gamma(t)} \cdot (\mathbf{q} - \mathbf{p}).$$

Because

$$\frac{d^2 f(t)}{dt^2} = (\mathbf{q}-\mathbf{p})^T H|_{\gamma(t)} (\mathbf{q}-\mathbf{p}) > 0,$$

$\frac{df(0)}{dt} \neq \frac{df(1)}{dt}$, therefore

$$\nabla E(\mathbf{p}) \cdot (\mathbf{q}-\mathbf{p}) \neq \nabla E(\mathbf{q}) \cdot (\mathbf{q}-\mathbf{p}).$$

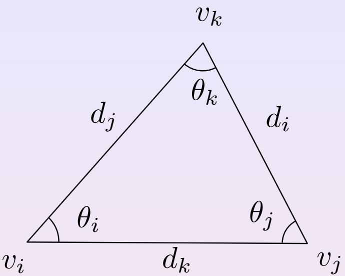
This means $\nabla E(\mathbf{p}) \neq \nabla E(\mathbf{q})$, therefore ∇E is injective.
On the other hand, the Jacobi matrix of ∇E is the Hessian matrix of E, which is positive definite. It follows that $\nabla E : \Omega \to \mathbb{R}^n$ is a smooth embedding.

Proof of main theorem

Proof.

Because the energy $E:\Omega_u \to \mathbb{R}$ is strictly convex on Ω_u, and Ω_u is convex, therefore $\nabla E:\Omega_u \to \mathbb{R}^m$ is an embedding. $\nabla E = (w_1, w_2, \cdots, w_m)$ are the edge weights. Namely, the map $(u_1, u_2, \cdots, u_m) \to (w_1, w_2, \cdots, w_m)$ is one-to-one, the metric is determined by the wedge weight unique up to scaling. □

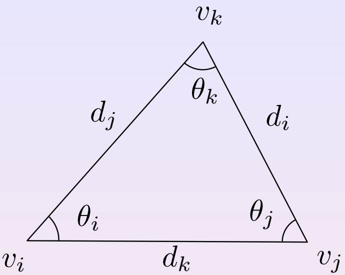
Simple Case - One Euclidean Triangle



An Euclidean triangle. By direct computation,

Lemma

Suppose an Euclidean triangle is with angles $\{\theta_i, \theta_j, \theta_k\}$ and edge lengths $\{d_i, d_j, d_k\}$, angles are treated as the functions of the edge lengths, $\theta_i(d_i, d_j, d_k)$ then $\frac{\partial \theta_i}{\partial d_i} = \frac{d_i}{2A}$ and $\frac{\partial \theta_i}{\partial d_j} = -\frac{d_i}{2A} \cos \theta_k$, where A is the area of the triangle.



An Euclidean triangle. By direct computation,

Lemma

In an Euclidean triangle, let $u_i = \frac{1}{2}d_i^2$ *and* $u_j = \frac{1}{2}d_j^2$ *then*

$$\frac{\partial \cot \theta_i}{\partial u_j} = \frac{\partial \cot \theta_j}{\partial u_i} \tag{5}$$

Corollary

Definition (Admissible Metric Space)

Let $u_i = \frac{1}{2}d_i^2$, the admissible metric space is defined as

$$\Omega_u := \{(u_i, u_j, u_k) | (\sqrt{u_i}, \sqrt{u_j}, \sqrt{u_k}) \in E_d(2), u_i + u_j + u_k = 3\}$$

Lemma

The admissible metric space Ω_u is a convex domain in $\mathbb{R}^3$.

By direct argument.

Edge Weight Space

Definition (Edge Weight Space)

The edge weights of an Euclidean triangle form the edge weight space

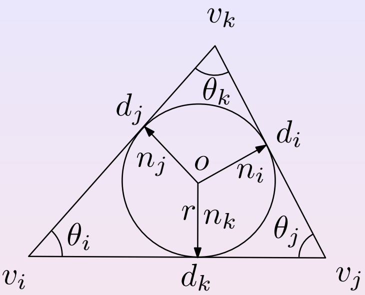
$$\Omega_\theta = \{(\cot\theta_i, \cot\theta_j, \cot\theta_k) | 0 < \theta_i, \theta_j, \theta_k < \pi, \theta_i + \theta_j + \theta_k = \pi\}$$

Lemma

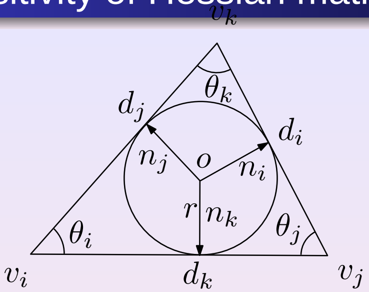
The energy $E : \Omega_u \to \mathbb{R}$

$$E(u_i, u_j, u_k) = \int_{(1,1,1)}^{(u_i,u_j,u_k)} \cot\theta_i d\tau_i + \cot\theta_j d\tau_j + \cot\theta_k d\tau_k \qquad (7)$$

is well defined on the admissible metric space Ω_u *and is convex.*



The geometric interpretation of the Hessian matrix. The incircle of the triangle is centered at O, with radius r. The perpendiculars n_i, n_j and n_k are from the incenter of the triangle and orthogonal to the edge e_i, e_j and e_k respectively.



By direct computation, we show the Hessian matrix

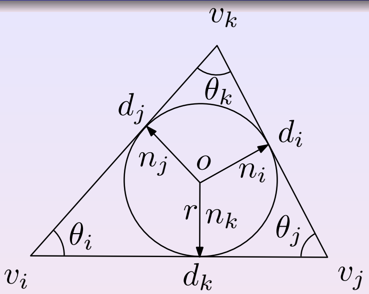
$$H = -\frac{2R^2}{A}\begin{bmatrix} (\eta_i,\eta_i) & (\eta_i,\eta_j) & (\eta_i,\eta_k) \\ (\eta_j,\eta_i) & (\eta_j,\eta_j) & (\eta_j,\eta_k) \\ (\eta_k,\eta_i) & (\eta_k,\eta_j) & (\eta_k,\eta_k) \end{bmatrix}$$

As shown in Figure 30, $d_i\mathbf{n}_i + d_j\mathbf{n}_j + d_k\mathbf{n}_k = 0$,

$$\eta_i = \frac{\mathbf{n}_i}{rd_i}, \eta_j = \frac{\mathbf{n}_j}{rd_j}, \eta_k = \frac{\mathbf{n}_k}{rd_k},$$

where r is the radius of the incircle of the triangle.

Positivity of Hessian matrix



$(x_i, x_j, x_k) \in \mathbb{R}^3$ is a vector in $\mathbb{R}^3$,

then

$$[x_i, x_j, x_k] \begin{bmatrix} (\eta_i, \eta_i) & (\eta_i, \eta_j) & (\eta_i, \eta_k) \\ (\eta_j, \eta_i) & (\eta_j, \eta_j) & (\eta_j, \eta_k) \\ (\eta_k, \eta_i) & (\eta_k, \eta_j) & (\eta_k, \eta_k) \end{bmatrix} \begin{bmatrix} x_i \\ x_j \\ x_k \end{bmatrix} = \|x_i \eta_i + x_j \eta_j + x_k \eta_k\|^2$$

If the result is zero, then $(x_i, x_j, x_k) = \lambda(u_i, u_j, u_k), \lambda \in \mathbb{R}$. That is the null space of the Hessian matrix. In the admissible metric space Ω_u, $u_i + u_j + u_k = C (C = 3)$.

Lemma

The admissible metric space Ω_u is convex.

Proof.

For a triangle $\{i,j,k\} \in F$, define

$$\Omega_u^{ijk} := \{(u_i,u_j,u_k)|(\sqrt{u_i},\sqrt{u_j},\sqrt{u_k}) \in E_d(2)\}.$$

Similar to the proof of Lemma 16, Ω_u^{ijk} is convex. The admissible metric space for the mesh is

$$\Omega_u = \bigcap_{\{i,j,k\}\in F} \Omega_u^{ijk} \bigcap \{(u_1,u_2,\cdots,u_m)|\sum_{k=1}^{m} u_k = m\},$$

the intersection Ω_u is still convex. □

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the intersection Ω_u is still convex. □

Definition (Differential Form)

The differential form ω defined on Ω_u is the summation of the differential form on each face,

$$\omega = \sum_{\{i,j,k\}\in F} \omega_{ijk} = \sum_{i=1}^{m} 2w_i du_i,$$

where ω_{ijk} is given in Eqn. (6) in Corollary 14. w_i is the edge weight on e_i.

Lemma

The differential form ω is a closed 1-form.

Proof.

According to Corollary 14,

$$d\omega = \sum d\omega_{ijk} = 0.$$

Energy function

Lemma

The energy function

$$E(u_1,u_2,\cdots,u_n) = \sum_{\{i,j,k\}\in F} E_{ijk}(u_1,u_2,\cdots,u_n) = \int_{(1,1,\cdots,1)}^{(u_1,u_2,\cdots,u_n)} \sum_{i=1}^{n} w_i du_i$$

is well defined and convex on Ω_u, where E_{ijk} is the energy on the face, defined in Eqn. (7).

Main Theorem

Theorem

The mapping on a closed Euclidean polyhedral surface $\nabla E : \Omega_u \rightarrow \mathbb{R}^m, (u_1, u_2, \cdots, u_n) \rightarrow (w_1, w_2, \cdots, w_n)$ is a smooth embedding.

Proof.

The admissible metric space Ω_u is convex as shown in Lemma 20, the total energy is convex as shown in Lemma 23. According to Lemma 11, ∇E is a smooth embedding. □

Surface with boundaries

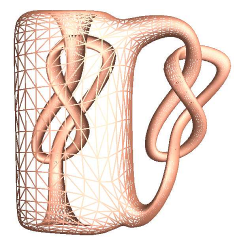
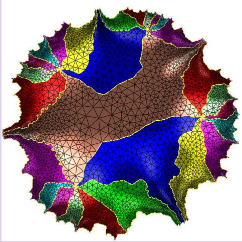
By using double covering technique, we convert a Euclidean polyhedral surface with boundary to a Euclidean polyhedral closed surface.

Future Direction

Generalize the theorem to higher dimesnional Euclidean polyhedral manifolds.

Thanks

For more information, please email to gu@cs.sunysb.edu.

Thank you!