Sequences and Mathematical Induction

CSE 215, Foundations of Computer Science

Stony Brook University

http://www.cs.stonybrook.edu/~cse215

Sequences

- A *sequence* is a function whose domain is
  - all the integers between two given integers

    $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$
  - all the integers greater than or equal to a given integer

    $a_m, a_{m+1}, a_{m+2}, \ldots$

  a_k is a *term* in the sequence

  k is the *subscript* or *index*

  m is the *subscript of the initial term*

  n is the *subscript of the last term* $(m \leq n)$
- An *explicit formula* or *general formula* for a sequence is a rule that shows how the values of a_k depend on k

Sequences

- Examples:

$a_k = 2^k$ is the sequence 2, 4, 8, 16, 32, 64, 128,…

Index	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Term	2	4	8	16	32	64	128	256

a_k = k/k + 1, for all integers k ≥ 1:

$$a_1 = \frac{1}{1+1} = \frac{1}{2}$$
$$a_2 = \frac{2}{2+1} = \frac{2}{3}$$
$$a_3 = \frac{3}{3+1} = \frac{3}{4}$$

b_i = i-1/i, for all integers i ≥ 2:

$$b_2 = \frac{2-1}{2} = \frac{1}{2}$$
$$b_3 = \frac{3-1}{3} = \frac{2}{3}$$
$$b_4 = \frac{4-1}{4} = \frac{3}{4}$$

- a_k for k ≥ 1 is the same sequence with b_i for i ≥ 2

Sequences

- **An *Alternating Sequence*:**

  $c_j = (-1)^j$ for all integers $j \geq 0$:

$$c_0 = (-1)^0 = 1$$
$$c_1 = (-1)^1 = -1$$
$$c_2 = (-1)^2 = 1$$
$$c_3 = (-1)^3 = -1$$
$$c_4 = (-1)^4 = 1$$
$$c_5 = (-1)^5 = -1$$
$$\dots$$

Find an explicit formula for a sequence

- The initial terms of a sequence are:

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}$$

  - a_k is the general term of the sequence, a_1 is the first element
  - observe that the denominator of each term is a perfect square

$$\begin{array}{cccccc} \frac{1}{1^2}, & \frac{(-1)}{2^2}, & \frac{1}{3^2}, & \frac{(-1)}{4^2}, & \frac{1}{5^2}, & \frac{(-1)}{6^2} \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \end{array}$$

  - observe that the numerator equals ± 1: $a_k = \frac{\pm 1}{k^2}$
  - alternating sequence with -1 when k is even:

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1$$

Find an explicit formula for a sequence

- Result sequence:

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for all integers } k \geq 1$$

- Alternative sequence:

$$a_k = \frac{(-1)^k}{(k+1)^2} \quad \text{for all integers } k \geq 0$$

Summation Notation

- If m and n are integers and m ≤ n, the summation from k equals m to n of a_k, $\sum_{k=m}^{n} a_k$, is the sum of all the terms a_m, a_{m+1}, a_{m+2},…, a_n

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

k is the index of the summation

m is the lower limit of the summation

n is the upper limit of the summation

Summation Notation

- Example:

$$a_1 = -2, \quad a_2 = -1, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 2$$

$$\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$$

$$\sum_{k=2}^{2} a_k = a_2 = -1$$

$$\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$$

Summation Notation

- Summation notation with formulas example:

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

- Changing from Summation Notation to Expanded Form:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

Summation Notation

- Changing from Expanded Form to Summation Notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}$$

- The general term of this summation can be expressed as $\frac{k+1}{n+k}$ for integers k from 0 to n

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{k=0}^{n} \frac{k+1}{n+k}$$

Sequences

- Evaluating expression for given limits:

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n\cdot(n+1)}$$

$$n=1 \qquad \frac{1}{1\cdot 2}=\frac{1}{2}$$

$$n=2 \qquad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$$

$$n=3 \qquad \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{3}{4}$$

Sequences

- Recursive definitions:

$$\sum_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \sum_{k=m}^{n} a_k = \sum_{k=m}^{n-1} a_k + a_n \quad \text{for all integers } n > m$$

- Use of recursion examples:

$$\sum_{i=1}^{n+1} \frac{1}{i^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

$$\sum_{k=0}^{n} 2^k + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

Sequences

- **Telescoping Sums:**

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1) - k}{k(k+1)} = \frac{1}{k(k+1)}$$

$$= 1 - \frac{1}{n+1}$$

Product Notation

- The product from k equals m to n of a_k , $\prod_{k=m}^{n} a_k$,for m and n integers and $m \leq n$, is the product of all the terms a_m, a_{m+1}, $a_{m+2}, \ldots, a_n$

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n$$

- Example: $\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5$

$$\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

Product Notation

- Recursive definition:

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m$$

Sequences

- If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers:

$$\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$$

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$$

- Generalized distributive law: if c is any real number:

$$c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$$

Sequences

- **Using Properties of Summation and Product:**

$a_k = k + 1$ $b_k = k - 1$

$$\begin{aligned}
\sum_{k=m}^{n} a_k + 2\cdot\sum_{k=m}^{n} b_k &= \sum_{k=m}^{n}(k+1) + 2\cdot\sum_{k=m}^{n}(k-1) \\
&= \sum_{k=m}^{n}(k+1) + \sum_{k=m}^{n} 2\cdot(k-1) \\
&= \sum_{k=m}^{n}((k+1) + 2\cdot(k-1)) \\
&= \sum_{k=m}^{n}(3k-1)
\end{aligned}$$

Sequences

- **Using Properties of Summation and Product:**

$a_k = k + 1$ $\qquad\qquad$ $b_k = k - 1$

$$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n} (k+1)\right) \cdot \left(\prod_{k=m}^{n} (k-1)\right)$$

$$= \prod_{k=m}^{n} (k+1) \cdot (k-1)$$

$$= \prod_{k=m}^{n} (k^2 - 1)$$

Sequences

- Change of variable examples:

$$\sum_{j=2}^{4}(j-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2$$
$$= 1^2 + 2^2 + 3^2$$
$$= \sum_{k=1}^{3} k^2.$$

$$\sum_{k=0}^{6}\frac{1}{k+1} \quad \textit{change of variable: } j = k+1$$

$$\frac{1}{k+1} = \frac{1}{(j-1)+1} = \frac{1}{j}$$

$$k=0, \quad j = k+1 = 0+1 = 1$$
$$k=6, \quad j = k+1 = 6+1 = 7$$

$$\sum_{k=0}^{6}\frac{1}{k+1} = \sum_{j=1}^{7}\frac{1}{j}$$

Factorial Notation

- The quantity n factorial, n!, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$$

0! is defined to be 1: $0! = 1$

$$0! = 1 \qquad 1! = 1$$

$$2! = 2 \cdot 1 = 2 \qquad 3! = 3 \cdot 2 \cdot 1 = 6$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \qquad 5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$$

$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$$

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$$

Factorial Notation

- A recursive definition for factorial is:

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

- Computing with Factorials:

$$\frac{8!}{7!} = \frac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$$

$$\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \frac{5 \cdot 4}{2 \cdot 1} = 10$$

$$\frac{(n+1)!}{n!} = \frac{(n+1) \cdot \cancel{n!}}{\cancel{n!}} = n + 1$$

$$\frac{n!}{(n-3)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \cancel{(n-3)!}}{\cancel{(n-3)!}} = n \cdot (n-1) \cdot (n-2)$$

$$= n^3 - 3n^2 + 2n$$

n choose r

- n choose r, $\binom{n}{r}$, represents the number of subsets of size r that can be chosen from a set with n elements, for n and r integers with $0 \le r \le n$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

- Examples:

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8\cdot 7\cdot \cancel{6}\cdot \cancel{5}\cdot \cancel{4}\cdot \cancel{3\cdot 2\cdot 1}}{(\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)\cdot(\cancel{3\cdot 2\cdot 1})} = 56$$

$$\binom{n+1}{n} = \frac{(n+1)!}{n!((n+1)-n)!} = \frac{(n+1)!}{n!1!} = \frac{(n+1)\cdot \cancel{n!}}{\cancel{n!}} = n+1$$

n choose r

- *4* choose 2 = *4! / (2!*2!) = 6*
- Example: Let S = {1,2,3,4}
  - The 6 subsets of S with 2 elements are:

    {1,2}

    {1,3}

    {1,4}

    {2,3}

    {2,4}

    {3,4}

Sequences in Computer Programming

- Arrays: int[] a = new int[50];

$$
\begin{aligned}
&s := a[1] \\
&\textbf{for } k := 2 \textbf{ to } n \\
&\qquad s := s + a[k] \\
&\textbf{next } k
\end{aligned}
$$

Example Algorithm with Arrays

- Convert from Base 10 to Base 2:

$$38 = 19\cdot2 + 0$$
$$= (9\cdot2 + 1)\cdot2 + 0 = 9\cdot22 + 1\cdot2 + 0$$
$$= (4\cdot2 + 1)\cdot2^2 + 1\cdot2 + 0 = 4\cdot2^3 + 1\cdot2^2 + 1\cdot2 + 0$$
$$= (2\cdot2 + 0)\cdot2^3 + 1\cdot2^2 + 1\cdot2 + 0$$
$$= 2\cdot2^4 + 0\cdot2^3 + 1\cdot2^2 + 1\cdot2 + 0$$
$$= (1\cdot2 + 0)\cdot2^4 + 0\cdot2^3 + 1\cdot2^2 + 1\cdot2 + 0$$
$$= 1\cdot2^5 + 0\cdot2^4 + 0\cdot2^3 + 1\cdot2^2 + 1\cdot2 + 0$$

$$a = 2^k\cdot r[k] + 2^{k-1}\cdot r[k-1] + \cdots + 2^2\cdot r[2] + 2^1\cdot r[1] + 2^0\cdot r[0]$$

$$a_{10} = (r[k]r[k-1]\cdots r[2]r[1]r[0])_2$$

Convert from Base 10 to Base 2

Input: n [a nonnegative integer]

Algorithm Body:

```
q := n, i := 0
while (i = 0 or q = 0)
    r[i] := q mod 2
    q := q div 2
    i := i + 1
end while
```

Output: r[0], r[1], r[2], . . . , r[i − 1] [a sequence of integers]

Mathematical Induction

- **The Principle of Mathematical Induction:**

Let P(n) be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. P(a) is true.
2. For all integers k ≥ a, if P(k) is true then P(k + 1) is true.

Then the statement "for all integers n ≥ a, P(n)" is true.

That is:

$$P(a) \text{ is true.}$$

$$P(k) \rightarrow P(k+1),\ \forall\, k \geq a$$

$$\therefore P(n) \text{ is true, } \forall\, n \geq a$$

Mathematical Induction

- **The Method of Proof by Mathematical Induction:**

To prove a statement of the form:

"For all integers $n \geq a$, a property $P(n)$ is true."

Step 1 (base step): **Show that $P(a)$ is true.**

Step 2 (inductive step): **Show that for all integers $k \geq a$, if $P(k)$ is true then $P(k + 1)$ is true**:

- Inductive hypothesis: suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.
- Then show that $P(k + 1)$ is true.

Mathematical Induction

- Example: For all integers $n \geq 8$, n¢ can be obtained using 3¢ and 5¢ coins:

Base step: P(8) is true because 8¢ can = one 3¢ coin and one 5¢ coin

Inductive step: for all integers $k \geq 8$, if P(k) is true then P(k+1) is also true

Inductive hypothesis: suppose that k is any integer with $k \geq 8$:

P(k): k¢ can be obtained using 3¢ and 5¢ coins

We must show: P(k+1)is true:(k+1)¢ can be obtained using 3¢ and 5¢ coins

Case 1 (There is a 5¢ coin among those used to make up the k¢): replace the 5¢ coin by two 3¢ coins; the result will be (k + 1)¢.

Case 2 (There is not a 5¢ coin among those used to make up the k¢): because $k \geq 8$, at least three 3¢ coins must have been used. Remove three 3¢ coins (9¢) and replace them by two 5¢ coins (10¢); the result will be (k + 1)¢

Mathematical Induction

- Example: The Sum of the First n Integers:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \quad \text{for all integers } n \geq 1$$

Base step: P(1): $1 = \dfrac{1(1+1)}{2}$

Inductive step: P(k) is true, for a particular but arbitrarily chosen integer k ≥ 1:

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Prove P(k+1): $1 + 2 + \cdots + (k+1) = \dfrac{(k+1)(k+2)}{2}$

$$(1 + 2 + \cdots + k) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

The Sum of the First n Integers

- A formula in closed form represents a sum with a variable number of terms without an ellipsis or a summation symbol.
- Applying the Formula for the Sum of the First n Integers:

$$\begin{aligned} 2+4+6+\cdots+500 &= 2\cdot(1+2+3+\cdots+250) \\ &= 2\cdot\left(\frac{250\cdot 251}{2}\right) \\ &= 62{,}750. \end{aligned}$$

$$5+6+7+8+\cdots+50 = (1+2+3+\cdots+50) - (1+2+3+4)$$

Geometric sequence

- Each term is obtained from the preceding one by multiplying by a constant factor: if the first term is 1 and the constant factor is r: 1, r, r^2, r^3,…, r^n,…

$$1 + r + r^2 + \cdots + r^n = \sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}$$

Base step: Prove P(0): $\sum_{i=0}^{0} r^i = \frac{r^{0+1} - 1}{r - 1}$ ⇔ 1 = 1 (Proved)

Inductive hypothesis:Suppose P(k) is true:for $k \geq 0$: $\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1}$

Prove P(k + 1): $\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}$

Geometric sequence

$$
\begin{aligned}
\sum_{i=0}^{k+1} r^i &= 1 + r + r^2 + \cdots + r^k + r^{k+1} \\
&= \frac{r^{k+1} - 1}{r - 1} + r^{k+1} \\
&= \frac{r^{k+2} - 1}{r - 1}
\end{aligned}
$$

Geometric sequence

- Examples:

$$1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{(m-2)+1} - 1}{3 - 1}$$
$$= \frac{3^{m-1} - 1}{2}.$$

$$3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2}) \quad \text{by factoring out } 3^2$$
$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2}\right)$$

Mathematical Induction

- **Proving a Divisibility Property:**

  P(n): for all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3

Basic step P(0): $2^{2\cdot 0} - 1 = 0$ is divisible by 3

Induction hypothesis: Suppose P(k) is True: $2^{2k} - 1$ is divisible by 3

Prove: P(k+1): $2^{2(k+1)} - 1$ is divisible by 3 ?

Mathematical Induction

- **Proving a Divisibility Property:**

$$
\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\
&= 2^{2k} \cdot 2^2 - 1 && \text{by the laws of exponents} \\
&= 2^{2k} \cdot 4 - 1 \\
&= 2^{2k}(3 + 1) - 1 \\
&= 2^{2k} \cdot 3 + (2^{2k} - 1) && \text{by the laws of algebra} \\
&= 2^{2k} \cdot 3 + 3r && \text{by inductive hypothesis} \\
&= 3(2^{2k} + r) && \text{by factoring out the 3.}
\end{aligned}
$$

$2^{2k} + r$ is an integer because integers are closed under multiplication and summation

so, **$2^{2(k+1)} - 1$ is divisible by 3** ■

Mathematical Induction

- **Proving an Inequality:**

  $P(n)$: for all integers $n \geq 3$, $2n + 1 < 2^n$

Base step: Prove $P(3)$: $2 \cdot 3 + 1 < 2^3$

$7 < 8$ (True)

Inductive step: Suppose that for $k \geq 3$, $P(k)$ is True: $2k + 1 < 2^k$

Show: $P(k+1)$: $2(k+1) + 1 < 2^{k+1}$

That is: $2k + 3 < 2^{k+1}$

$$2k + 3 = (2k + 1) + 2 < 2^k + 2^k = 2^{k+1}$$

because $2k + 1 < 2^k$ by the inductive hypothesis

and because $2 < 2^k$ for all integers $k \geq 3$

Mathematical Induction

- A sequence: $a_1 = 2$ and $a_k = 5a_{k-1}$ for all integers $k \geq 2$
- Prove: $a_n = 2 \cdot 5^{n-1}$

Proof by induction: P(n): $a_n = 2 \cdot 5^{n-1}$ for all integers $n \geq 1$

Base step: P(1): $a_1 = 2 \cdot 5^{1-1} = 2 \cdot 5^0 = 2 \cdot 1 = 2$

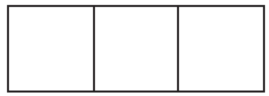
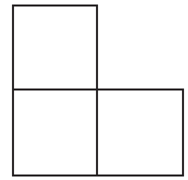
Inductive hypothesis: assume P(k) is true: $a_k = 2 \cdot 5^{k-1}$

Show: P(k+1): $a_{k+1} = 2 \cdot 5^{(k+1)-1} = 2 \cdot 5^k$?

$a_{k+1} = 5a_{(k+1)-1}$	by definition of $a_1, a_2, a_3, \ldots$
$= 5 \cdot a_k$	since $(k + 1) - 1 = k$
$= 5 \cdot 2 \cdot 5^{k-1}$	by inductive hypothesis
$= 2 \cdot (5 \cdot 5^{k-1})$	by regrouping
$= 2 \cdot 5^k$	by the laws of exponents

Mathematical Induction

- **A Problem with Trominoes (Tetris):**

straight  and L-shaped 

For any integer $n \geq 1$, if one square is removed from a $2^n \times 2^n$ checkerboard, the remaining squares can be completely covered by L-shaped trominoes

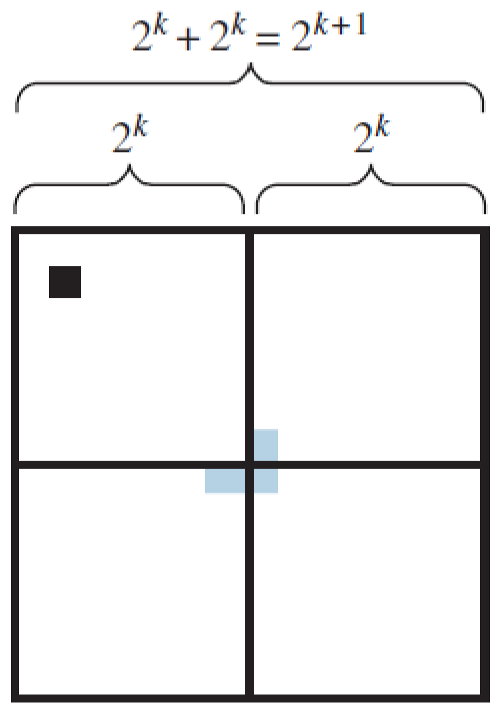
Base case: a 2×2 checkerboard can be covered by 1 L-shaped tromino

Mathematical Induction

Inductive hypothesis for $k \geq 1$: P(k): if one square is removed from a $2^k \times 2^k$ checkerboard, the remaining squares can be completely covered by L-shaped trominoes

P(k+1):

if one square is removed from a $2^{k+1} \times 2^{k+1}$ checkerboard, the remaining squares can be completely covered by L-shaped trominoes



Strong Mathematical Induction

- The Principle of Strong Mathematical Induction (or the principle of complete induction):

P(n) is a property that is defined for integers n, and a and b are fixed integers with $a \leq b$.

- **Base step:** P(a), P(a + 1), . . . , and P(b) are all true
- **Inductive step:** For any integer $k \geq b$, if P(i) is true for all integers i from a through k (**inductive hypothesis**), then P(k + 1) is true

Then the statement for all integers $n \geq a$, P(n) is true.

That is: P(a), P(a+1),…, P(b-1), P(b) are true.

$\forall k \geq b, (\forall a \leq i \leq k, P(i)) \rightarrow P(k+1)$

$\therefore P(n)$ is true, $\forall n \geq a$

Strong Mathematical Induction

- Any statement that can be proved with ordinary mathematical induction can be proved with strong mathematical induction (and vice versa).

Strong Mathematical Induction

- **Divisibility by a Prime:**

Any integer greater than 1 is divisible by a prime number

P(n): n is divisible by a prime number

Base case: P(2): 2 is divisible by a prime number

2 is divisible by 2 and 2 is a prime number

Inductive hypothesis: Let k be any integer with $k \geq 2$

P(i): i is divisible by a prime number for all integers

P(i) is true for all integers i from 2 through k

Show: P(k + 1): k + 1 is divisible by a prime number

Strong Mathematical Induction

- Case 1 ($k + 1$ is prime): In this case $k + 1$ is divisible by itself (a prime number): $k+1 = 1*(k+1)$

- Case 2 ($k + 1$ is not prime): $k + 1 = a*b$

where a and b are integers with $1<a<k+1$ and $1<b<k+1$.

From $k + 1 = a*b$, $k + 1$ is divisible by a

By inductive hypothesis, a is divisible by a prime number p

By transitivity of divisibility, $k + 1$ is divisible by the prime number p.

Therefore, $k+1$ is divisible by a prime number p. ■

Strong Mathematical Induction

- A sequence $s_0, s_1, s_2, \ldots$

$s_0=0$, $s_1=4$, $s_k=6s_{k-1}-5s_{k-2}$ for all integers $k \geq 2$

$s_2 = 6s_1 - 5s_0 = 6\cdot4 - 5\cdot0 = 24,$

$s_3 = 6s_2 - 5s_1 = 6\cdot24 - 5\cdot4 = 144 - 20 = 124$

Prove: $s_n=5^n-1$

Base step P(0) and P(1) are true:

P(0): $s_0=5^0-1 = 1 - 1 = 0$

P(1): $s_1= 5^1-1 = 5 - 1 = 4$

Inductive step: Let k be any integer with $k \geq 1$,

$s_i=5^i-1$ for all integers i with $0 \leq i \leq k$

Strong Mathematical Induction

- **We must show** P(k + 1) is true: $s_{k+1} = 5^{k+1} - 1$

$s_{k+1} = 6s_k - 5s_{k-1}$	by definition of $s_0, s_1, s_2, \ldots$
$= 6(5^k - 1) - 5(5^{k-1} - 1)$	by definition hypothesis
$= 6 \cdot 5^k - 6 - 5^k + 5$	by multiplying out and applying a law of exponents
$= (6 - 1)5^k - 1$	by factoring out 6 and arithmetic
$= 5 \cdot 5^k - 1$	by arithmetic
$= 5^{k+1} - 1$	by applying a law of exponents

■

Strong Mathematical Induction

- **The Number of Multiplications Needed to Multiply n Numbers is (n-1)**
  - P(n): If $x_1, x_2, \ldots, x_n$ are n numbers, then no matter how parentheses are inserted into their product, the number of multiplications used to compute the product is $n - 1$.

➢ Base case P(1): The number of multiplications needed to compute the product of x_1 is $1 - 1 = 0$

➢ Inductive hypothesis: Let k by any integer with $k \geq 1$ and for all integers i from 1 through k, if $x_1, x_2, \ldots, x_i$ are numbers, then no matter how parentheses are inserted into their product, the number of multiplications used to compute the product is $i - 1$.

Strong Mathematical Induction

- We must show: $P(k + 1)$: If $x_1, x_2,\ldots, x_{k+1}$ are $k + 1$ numbers, then no matter how parentheses are inserted into their product, the number of multiplications used to compute the product is $(k + 1) - 1 = k$

When parentheses are inserted in order to compute the product $x_1 x_2\ldots x_{k+1}$, some multiplication is the final one: let L be the product of the left-hand l factors and R be the product of the right-hand r factors: $l + r = k + 1$

By inductive hypothesis, evaluating L takes $l - 1$ multiplications and evaluating R takes $r - 1$ multiplications

$(l - 1) + (r - 1) + 1 = (l + r) - 1 = (k + 1) - 1 = k$ ■

Strong Mathematical Induction

- **Existence and Uniqueness of Binary Integer Representations:** any positive integer n has a unique representation in the form

$$n = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

where r is a nonnegative integer, $c_r = 1$, and $c_j = 0$ or 1 for $j = 0, \ldots, r-1$

Proof of Existence:

Base step: P(1): $1 = c_0 \cdot 2^0$ where $c_0 = 1$, $r = 0$.

Inductive hypothesis: $k \geq 1$ is an integer and for all integers i from 1 through k: P(i): $i = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$

We must show that k + 1 can be written in the required form.

Strong Mathematical Induction

- Case 1 (k + 1 is even): (k + 1)/2 is an integer

By inductive hypothesis:

$$(k + 1)/2 = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

$$k + 1 = c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2$$

$$= c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 0 \cdot 2^0$$

- Case 2 (k + 1 is odd): k is even, so k/2 is an integer

By inductive hypothesis:

$$k/2 = c_r \cdot 2^r + c_{r-1} \cdot 2^{r-1} + \cdots + c_2 \cdot 2^2 + c_1 \cdot 2 + c_0$$

$$k = c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2$$

$$k + 1 = c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2 + 1$$

$$= c_r \cdot 2^{r+1} + c_{r-1} \cdot 2^r + \cdots + c_2 \cdot 2^3 + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 1 \cdot 2^0$$

Strong Mathematical Induction

- **Proof of Uniqueness:**

Proof by contradiction: Suppose that there is an integer n with two different representations as a sum of nonnegative integer powers of 2:

$$2^r + c_{r-1}\cdot 2^{r-1} + \cdots + c_1\cdot 2 + c_0 = 2^s + d_{s-1}\cdot 2^{s-1} + \cdots + d_1\cdot 2 + d_0$$

r and s are nonnegative integers, and each c_i and each d_i equal 0 or 1

Assume: $r < s$

By geometric sequence:

$$2^r + c_{r-1}\cdot 2^{r-1} + \cdots + c_1\cdot 2 + c_0 \leq 2^r + 2^{r-1} + \cdots + 2 + 1 = 2^{r+1} - 1 < 2^s$$

$$2^r + c_{r-1}\cdot 2^{r-1} + \cdots + c_1\cdot 2 + c_0 < 2^s + d_{s-1}\cdot 2^{s-1} + \cdots + d_1\cdot 2 + d_0$$

Contradiction

Recursion

- A sequence can be defined in 3 ways:
  - enumeration: -2,3,-4,5,...
  - general pattern: $a_n=(-1)^n(n+1)$, for all integers $n \geq 1$
  - recursion: $a_1=-2$ and $a_n=(-1)^{n-1}\ a_{n-1} +(-1)^n$
    - **define one or more initial values for the sequence AND**
    - **define each later term in the sequence by reference to earlier terms**
- A **recurrence relation** for a sequence a_0, a_1, a_2,... is a formula that relates each term a_k to certain of its predecessors a_{k-1}, a_{k-2},..., a_{k-i}, where i is an integer with $k-i \geq 0$
- The **initial conditions** for a recurrence relation specify the values of a_0, a_1, a_2,..., a_{i-1}, if i is a fixed integer, OR

  a_0, a_1,..., a_m, where m is an integer with $m \geq 0$, if i depends on k.

Recursion

- **Computing Terms of a Recursively Defined Sequence:**
  - Example:

Initial conditions: $c_0 = 1$ and $c_1 = 2$

Recurrence relation: $c_k = c_{k-1} + k * c_{k-2} + 1$, for all integers $k \geq 2$

$c_2 = c_1 + 2\,c_0 + 1$ by substituting $k = 2$ into the recurrence relation

$= 2 + 2 \cdot 1 + 1$ since $c_1 = 2$ and $c_0 = 1$ by the initial conditions

$= 5$

$c_3 = c_2 + 3\,c_1 + 1$ by substituting $k = 3$ into the recurrence relation

$= 5 + 3 \cdot 2 + 1$ since $c_2 = 5$ and $c_1 = 2$

$= 12$

$c_4 = c_3 + 4\,c_2 + 1$ by substituting $k = 4$ into the recurrence relation

$= 12 + 4 \cdot 5 + 1$ since $c_3 = 12$ and $c_2 = 5$

$= 33$

Recursion

- **Writing a Recurrence Relation in More Than One Way:**
  - Example:

Initial condition: $s_0 = 1$

Recurrence relation 1: $s_k = 3s_{k-1} - 1$, for all integers $k \geq 1$

Recurrence relation 2: $s_{k+1} = 3s_k - 1$, for all integers $k \geq 0$

Recursion

- **Sequences That Satisfy the Same Recurrence Relation:**
  - Example:

    Initial conditions: $a_1 = 2$ and $b_1 = 1$

    Recurrence relations: $a_k = 3a_{k-1}$ and $b_k = 3b_{k-1}$ for all integers $k \geq 2$

    $a_2 = 3a_1 = 3 \cdot 2 = 6$

    $a_3 = 3a_2 = 3 \cdot 6 = 18$

    $a_4 = 3a_3 = 3 \cdot 18 = 54$

    $b_2 = 3b_1 = 3 \cdot 1 = 3$

    $b_3 = 3b_2 = 3 \cdot 3 = 9$

    $b_4 = 3b_3 = 3 \cdot 9 = 27$

Recursion

- Fibonacci numbers:
  1. We have one pair of rabbits (male and female) at the beginning of a year.
  2. Rabbit pairs are not fertile during their first month of life but thereafter give birth to one new male&female pair at the end of every month.

$$\begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k\end{bmatrix} = \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-1\end{bmatrix} + \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs born}\\ \text{at the end}\\ \text{of month } k\end{bmatrix}$$

$$= \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-1\end{bmatrix} + \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-2\end{bmatrix}$$

Recursion

- Fibonacci numbers:

The initial number of rabbit pairs: **$F_0 = 1$**

F_n : the number of rabbit pairs at the end of month n, for each integer n ≥ 1

$F_n = F_{n-1} + F_{n-2}$, for all integers k ≥ 2

$F_1 = 1$, because the first pair of rabbits is not fertile until the second month

How many rabbit pairs are at the end of one year?

January 1st: $F_0 = 1$

February 1st: $F_1 = 1$

March 1st : $F_2 = F_1 + F_0 = 1+1= 2$

April 1st : $F_3 = F_2 + F_1 = 2+1= 3$

May 1st : $F_4 = F_3 + F_2 = 3+2= 5$

June 1st : $F_5 = F_4 + F_3 = 5+3= 8$

July 1st : $F_6 = F_5 + F_4 = 8+5= 13$

August 1st : $F_7 = F_6 + F_5 = 13+8= 21$

September 1st : $F_8=F_7+F_6=21+13=34$

October 1st : $F_9=F_8+F_7=34+21=55$

November 1st : $F_{10}=F_9+F_8=55+34=89$

December 1st : $F_{11}=F_{10}+F_9=89+55=144$

January 1st : $F_{12}=F_{11}+F_{10}=144+89=$**233**

Recursion

- **Compound Interest:**
  - A deposit of \$100,000 in a bank account earning 4% interest compounded annually:

the amount in the account at the end of any particular year =
the amount in the account at the end of the previous year +
the interest earned on the account during the year

= the amount in the account at the end of the previous year +
0.04 · the amount in the account at the end of the previous year

$A_0 = \$100{,}000$

$\mathbf{A_k = A_{k-1} + (0.04) \cdot A_{k-1}} = 1.04 \cdot A_{k-1}$, for each integer $k \geq 1$

$A_1 = 1.04 \cdot A_0 = \$104{,}000$

$A_2 = 1.04 \cdot A_1 = 1.04 \cdot \$104{,}000 = \$108{,}160$

…

Recursion

- **Compound Interest with Compounding Several Times a Year:**
  - An annual interest rate of i is compounded m times per year:

    the interest rate paid per each period is i/m

    P_k is the sum of the the amount at the end of the $(k - 1)$ period

    + the interest earned during k-th period

    $P_k = P_{k-1} + P_{k-1} \cdot i/m = P_{k-1} \cdot (1+ i/m)$
  - If 3% annual interest is compounded quarterly, then the interest rate paid per quarter is $0.03/4 = 0.0075$

Compound Interest

- Example: deposit of \$10,000 at 3% compounded quarterly

For each integer $n \geq 1$, P_n = the amount on deposit after n consecutive quarters.

$P_k = 1.0075 \cdot P_{k-1}$

$P_0 = \$10{,}000$

$P_1 = 1.0075 \cdot P_0 = 1.0075 \cdot \$10{,}000 = \$10{,}075.00$

$P_2 = 1.0075 \cdot P_1 = (1.0075) \cdot \$10{,}075.00 = \$10{,}150.56$

$P_3 = 1.0075 \cdot P_2 \sim (1.0075) \cdot \$10{,}150.56 = \$10{,}226.69$

$P_4 = 1.0075 \cdot P_3 \sim (1.0075) \cdot \$10{,}226.69 = \$10{,}303.39$

The annual percentage rate (APR) is the percentage increase in the value of the account over a one-year period:

$APR = (10303.39 - 10000)/\ 10000 = 0.03034 = 3.034\%$

Recursive Definitions of Sum and Product

- The summation from i=1 to n of a sequence is defined using recursion:

$$\sum_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n, \quad \text{if } n > 1.$$

- The product from i=1 to n of a sequence is defined using recursion:

$$\prod_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n, \quad \text{if } n > 1.$$

Sum of Sums

- For any positive integer n, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

- Proof by induction

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \qquad \leftarrow P(n)$$

  - base step: $$\sum_{i=1}^{1}(a_i + b_i) = a_1 + b_1 = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i$$
  - inductive hypothesis: $$\sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i. \qquad \leftarrow P(k)$$

Sum of Sums

- Cont.: We must show that:

$$\sum_{i=1}^{k+1}(a_i + b_i) = \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i. \qquad \leftarrow P(k+1)$$

$$\begin{aligned}
\sum_{i=1}^{k+1}(a_i + b_i) &= \sum_{i=1}^{k}(a_i + b_i) + (a_{k+1} + b_{k+1}) && \text{by definition of } \Sigma \\
&= \left(\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i\right) + (a_{k+1} + b_{k+1}) && \text{by inductive hypothesis} \\
&= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) + \left(\sum_{i=1}^{k} b_i + b_{k+1}\right) && \text{by the associative and cummutative laws of algebra} \\
&= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i && \text{by definition of } \Sigma
\end{aligned}$$

Q.E.D.

Recursion

- **Arithmetic sequence**: there is a constant d such that

$$a_k = a_{k-1} + d \text{ for all integers } k \geq 1$$

It follows that, $a_n = a_0 + d*n$ for all integers $n \geq 0$.

- **Geometric sequence**: there is a constant r such that

$$a_k = r * a_{k-1} \text{ for all integers } k \geq 1$$

It follows that, $a_n = r^n * a_0$ for all integers $n \geq 0$.

Recursion

- A **second-order linear homogeneous recurrence relation** with constant coefficients is a recurrence relation of the form:

$$a_k = A * a_{k-1} + B * a_{k-2}$$

for all integers $k \geq$ some fixed integer

where A and B are fixed real numbers with $B = 0$.

Supplemental material on Sequences: Correctness of Algorithms

- A program is correct if it produces the output specified in its documentation for each set of inputs
  - initial state (inputs): pre-condition for the algorithm
  - final state (outputs): post-condition for the algorithm

- Example:
  - Algorithm to compute a product of nonnegative integers

Pre-condition: The input variables m and n are nonnegative integers

Post-condition: The output variable p equals m*n

Correctness of Algorithms

- The steps of an algorithm are divided into sections with assertions about the current state of algorithm

  [Assertion 1: pre-condition for the algorithm]

  {Algorithm statements}

  [Assertion 2]

  {Algorithm statements}

  …

  [Assertion $k - 1$]

  {Algorithm statements}

  [Assertion k: post-condition for the algorithm]

Correctness of Algorithms

- **Loop Invariants:** used to prove correctness of a loop with respect to pre- and post-conditions

  [Pre-condition for the loop]

  while (G)

  [Statements in the body of the loop]

  end while

  [Post-condition for the loop]

A loop is correct with respect to its pre- and post-conditions if, and only if, whenever the algorithm variables satisfy the pre-condition for the loop and the loop terminates after a finite number of steps, the algorithm variables satisfy the post-condition for the loop

Loop Invariant

- A **loop invariant I(n)** is a predicate with domain a set of integers, which for each iteration of the loop, **(induction)** if the predicate is true before the iteration, the it is true after the iteration

If **the loop invariant I(0) is true before the first iteration of the loop** AND

After a finite number of iterations of the loop, the guard G becomes false **AND**

The truth of **the loop invariant ensures the truth of the post-condition of the loop**

then the loop will be correct with respect to it pre- and post-conditions

Loop Invariant

- **Correctness of a Loop to Compute a Product:**

A loop to compute the product m*x for a nonnegative integer m and a real number x, without using multiplication

[Pre-condition: m is a nonnegative integer, x is a real number, i = 0, and product = 0]

while $(i \neq m)$

    product := product + x

    i := i + 1

end while

[Post-condition: product = mx]

Loop invariant I(n): **[i = n and product = n*x]**

Guard G: $i \neq m$

Base Property: I (0) is "i = 0 and product = 0· x = 0"

Inductive Property: [If G ∧ I (k) is true before a loop iteration (where k ≥ 0), then I (k+1) is true after the loop iteration.]

Let k is a nonnegative integer such that G ∧ I (k) is true:

$$i \neq m \wedge \ i = n \wedge \ \text{product} = n*x$$

Since i ≠ m, the guard is passed and

product = product + x = k*x + x = (k + 1)*x

i = i + 1 = k + 1

I(k + 1): (i = k + 1 and product = (k + 1)*x) is true

Eventual Falsity of Guard: [After a finite number of iterations of the loop, G becomes false]

After m iterations of the loop: i = m and G becomes false

Correctness of the Post-Condition: [If N is the least number of iterations after which G is false and I (N) is true, then the value of the algorithm variables will be as specified in the post-condition of the loop.]

I(N) is true at the end of the loop: i = N and product = N*x

G becomes false after N iterations, i = m, so m = i = N

The post-condition: the value of product after execution of the loop should be mx is true.