Functions

CSE 215, Foundations of Computer Science

Stony Brook University

http://www.cs.stonybrook.edu/~cse215

Functions Defined on General Sets

- A function f from a set X to a set Y

$$f : X \rightarrow Y$$

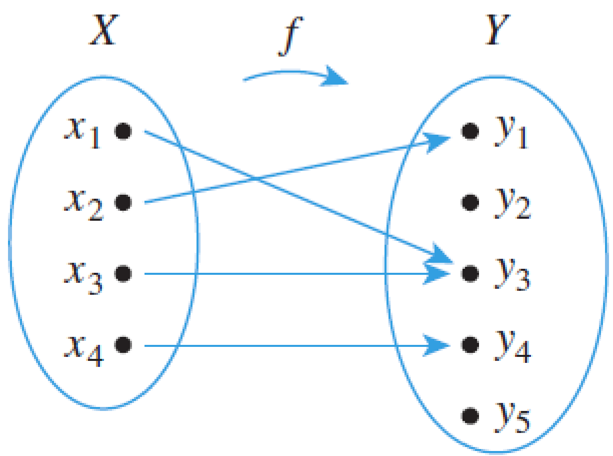
X is the domain

Y is the co-domain

1. every element in X is related to some element in Y
2. no element in X is related to more than one element in Y
   - For any element $x \in X$, there is a unique element $y \in Y$ such that $f(x)=y$
- Range of f (image of X under f) $= \{y \in Y \mid y = f(x), x \in X\}$
  - The inverse image of $y = \{x \in X \mid f(x) = y\}$

Arrow diagrams

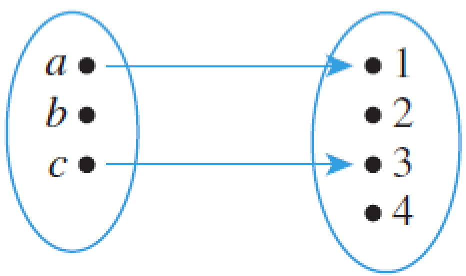
- An arrow diagram defines a function *iff*
  - Every element of X has an arrow coming out of it
  - No element of X has two arrows coming out of it that point to two different elements of Y



Arrow diagrams

- Example 1:

  X = {a, b, c},   Y = {1, 2, 3, 4}



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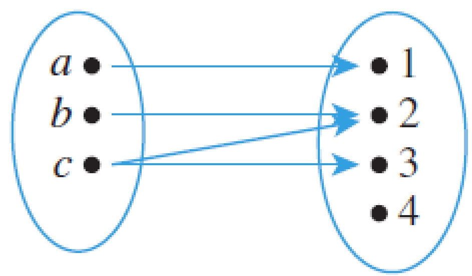

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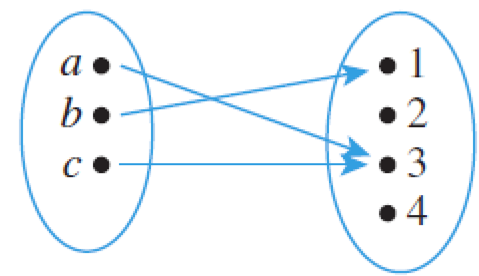
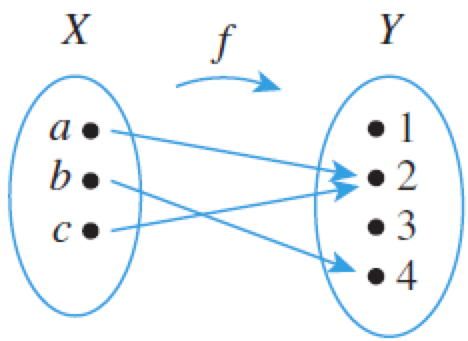


Yes

Arrow diagrams

- Example 2:

  $X = \{a, b, c\}, \quad Y = \{1, 2, 3, 4\}$



f (a) = 2
f (b) = 4
f (c) = 2

  - domain of f $= \{a, b, c\}$, co-domain of f $= \{1, 2, 3, 4\}$
  - range of f $= \{2, 4\}$
  - inverse image of 2 $= \{a, c\}$
  - inverse image of 4 $= \{b\}$
  - inverse image of 1 $= \emptyset$
  - function representation as a set of pairs $= \{(a,2),(b,4),(c,2)\}$

Function Equality

Def.: the set notation for a function: $F(x) = y \Leftrightarrow (x,y) \in F$

- If $F: X \rightarrow Y$ and $G: X \rightarrow Y$ are functions, then $F = G$ **if, and only if,** $F(x) = G(x)$ for all $x \in X$.

Proof:

$F \subseteq X \times Y$ $\qquad$ $G \subseteq X \times Y$

$F(x) = y \Leftrightarrow (x, y) \in F$ $\qquad$ $G(x) = y \Leftrightarrow (x, y) \in G$

F = G ➔ F(x) = G(x) for all x ∈ X. Then for all $x \in X$,

$$F(x) = y \Leftrightarrow (x, y) \in F \Leftrightarrow (x, y) \in G \Leftrightarrow G(x) = y$$

$$F(x) = y = G(x)$$

F(x) = G(x) for all x ∈ X ➔ F = G Then for any element x of X:

$$(x, y) \in F \Leftrightarrow y = F(x) \Leftrightarrow y = G(x) \Leftrightarrow (x, y) \in G$$

F and G consist of exactly the same elements and hence $F = G$.

Function Equality

- Example: $J_3 = \{0, 1, 2\}$

$$f : J_3 \rightarrow J_3 \quad \text{and} \quad g : J_3 \rightarrow J_3$$

$$f(x) = (x^2 + x + 1) \bmod 3$$

$$g(x) = (x + 2)^2 \bmod 3$$

x	$x^2 + x + 1$	$f(x) = (x^2 + x + 1)\ mod\ 3$	$(x + 2)^2$	$g(x) = (x + 2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
1	3	$3\ mod\ 3 = 0$	9	$9\ mod\ 3 = 0$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$

$$f(0) = g(0) = 1, \quad f(1) = g(1) = 0, \quad f(2) = g(2) = 1$$

$$f = g = \{(0,1), (1,0), (2,1)\}$$

Function Equality

- Example: F: $\mathbf{R} \to \mathbf{R}$ and G: $\mathbf{R} \to \mathbf{R}$

  F+G: $\mathbf{R} \to \mathbf{R}$ and G+F: $\mathbf{R} \to \mathbf{R}$

  $$(F + G)(x) = F(x) + G(x)$$

  $$(G + F)(x) = G(x) + F(x), \quad \text{for all } x \in \mathbf{R}$$

For all real numbers x:

$(F + G)(x) = F(x) + G(x)$ by definition of F + G

$= G(x) + F(x)$ by the commutative law for addition of real numbers

$= (G + F)(x)$ by definition of G + F

Hence F + G = G + F. ■

Functions

- **The Identity Function on a Set:**

Given a set X, $I_X: X \to X$ is an identity function *iff*

$$I_X(x) = x, \text{ for all } x \in X$$

- **The function for a sequence:**

$1, -1/2, 1/3, -1/4, 1/5, \ldots, (-1)^n/(n+1), \ldots$

$0 \to 1, \quad 1 \to -1/2, \quad 2 \to 1/3, \quad 3 \to -1/4, \quad 4 \to 1/5$

$$n \to (-1)^n/(n+1)$$

$f : \mathbf{N} \to \mathbf{R}$, for each integer $n \geq 0$, $f(n) = (-1)^n/(n+1)$

where ($\mathbf{N} = \mathbf{Z}^{\mathbf{nonneg}}$) OR

$g : \mathbf{Z}^+ \to \mathbf{R}$, for each integer $n \geq 1$, $g(n) = (-1)^{n+1}/n$

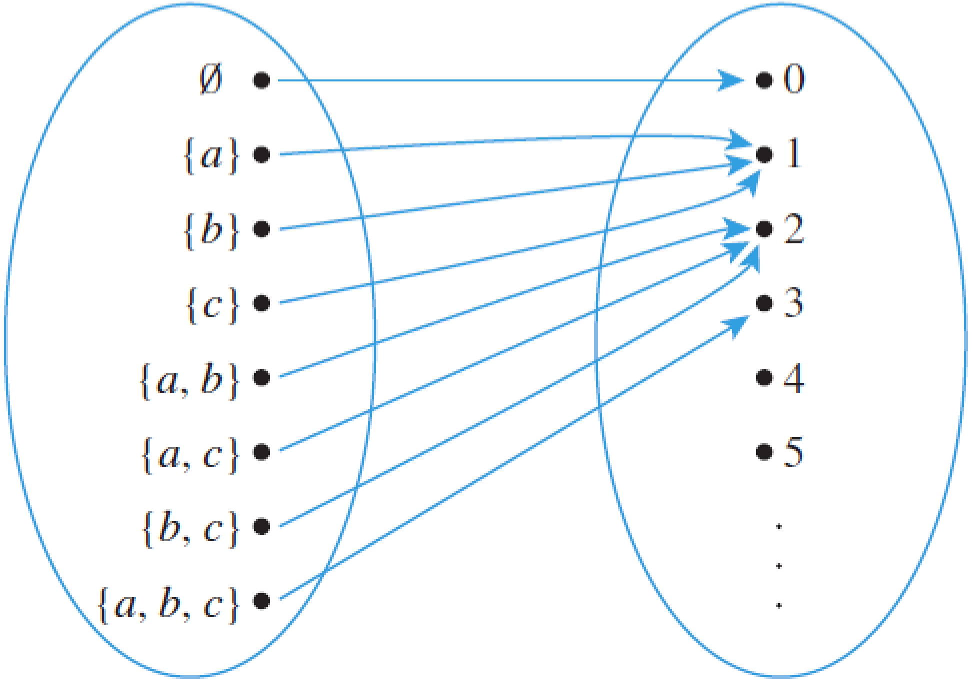
where ($\mathbf{Z}^+ = \mathbf{Z}^{\mathbf{nonneg}} - \{\mathbf{0}\}$)

Functions

- **Power set example:**

$$F : P(\{a, b, c\}) \rightarrow \mathbf{Z}^{\mathbf{nonneg}}$$

For each $X \in P(\{a, b, c\})$, $F(X)$ = the number of elements in X (i.e., the cardinality of X)



Functions

- **Cartesian product example:**

$$M : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \quad \text{and} \quad R : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$$

The multiplication function: M(a, b) = a*b

We omit parenthesis for tuples: M((a, b))=M(a,b)

M(1, 1) = 1, M(2, 2) = 4

The reflection function: R(a, b) = (-a, b)

R sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis

R(1, 1) = (-1, 1), R(2, 5) = (-2, 5), R(-2, 5) = (2, 5)

Functions

- **Logarithms and Logarithmic Functions:**
  - The base of a logarithm b is a positive real number with $b \neq 1$
  - The logarithm with base b of x: $\log_b x = y \Leftrightarrow b^y = x$
  - The **logarithmic function with base b**:

$$\log_b x : \mathbf{R}^+ \rightarrow \mathbf{R}$$

Examples:

$\log_3 9 = 2$ because $3^2 = 9$

$\log_{10}(1) = 0$ because $10^0 = 1$

$\log_2 ½ = -1$ because $2^{-1} = ½$

$\log_2 (2^m) = m$

Functions

- **Example: Encoding and Decoding Functions**

For each string $s \in A$,

E(s) = the string obtained from s by replacing each bit of s by the same bit written three times

For each string $t \in T$,

D(t) = the string obtained from t by replacing each consecutive triple of three identical bits of t by a single copy of that bit

$E(s) = t$, for all $t \in T$ and $D(t) = s$

Functions

- **The Hamming Distance Function**

Let S_n be the set of all strings of 0's and 1's of length n.

$$H: S_n \times S_n \rightarrow Z^{nonneg}$$

For each pair of strings $(s, t) \in S_n \times S_n$

H(s, t)=the number of positions in which s and t differ

For n = 5, H(11111, 00000) = 5

H(10101, 00000) = 3

H(01010, 00000) = 2

Functions

- **Boolean functions:**

$$f : \{0, 1\}^n \longrightarrow \{0, 1\}$$

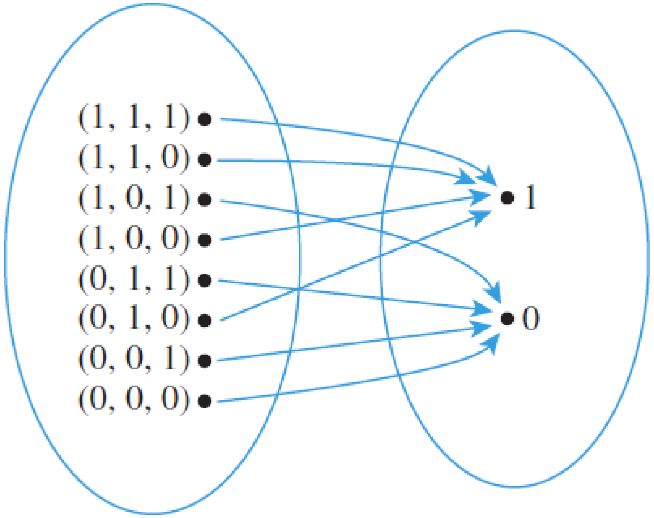
(n-place) Boolean function

the domain = the set of all ordered n-tuples of 0's and 1's

the co-domain = the set {0, 1}

Input			Output
P	*Q*	*R*	*S*
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

(P∧Q∧R) ∨ (P∧Q∧~R) ∨ (P∧~Q∧~R) ∨ (~P∧Q∧~R)



Functions

- **Boolean functions example:**

$$f : \{0, 1\}^3 \longrightarrow \{0, 1\}$$

$$f(x_1, x_2, x_3) = (x_1 + x_2 + x_3) \bmod 2$$

$$f(0, 0, 0) = (0 + 0 + 0) \bmod 2 = 0 \bmod 2 = 0$$

$$f(0, 0, 1) = (0 + 0 + 1) \bmod 2 = 1 \bmod 2 = 1$$

$$f(0, 1, 0) = (0 + 1 + 0) \bmod 2 = 1 \bmod 2 = 1$$

$$f(0, 1, 1) = (0 + 1 + 1) \bmod 2 = 2 \bmod 2 = 0$$

$$f(1, 0, 0) = (1 + 0 + 0) \bmod 2 = 1 \bmod 2 = 1$$

$$f(1, 0, 1) = (1 + 0 + 1) \bmod 2 = 2 \bmod 2 = 0$$

$$f(1, 1, 0) = (1 + 1 + 0) \bmod 2 = 2 \bmod 2 = 0$$

$$f(1, 1, 1) = (1 + 1 + 1) \bmod 2 = 3 \bmod 2 = 1$$

Functions

- **Checking Whether a Function Is Well Defined:**

  A function f is **"not well defined"** if:

(1) there is no element in the co-domain y that satisfies f(x)=y for some element x in the domain ***OR***

(2) there are two different values of y that satisfy f(x)=y

- Example 1:

  f : **R** → **R**, f (x) is the real number y such that $x^2 + y^2 = 1$

  f is "not well defined":

(1) x = 2, there is no real number y such that $2^2 + y^2 = 1$

OR

(2) x = 0, there are 2 real numbers y=1 and y=-1 such that $0^2 + y^2 = 1$

Functions

- Example 2 (Not Well Defined):

  $f : \mathbf{Q} \to \mathbf{Z}$

  $f(m/n) = m$, for all integers m and n with $n \neq 0$

  $1/2 = 2/4 \rightarrow f(1/2) = f(2/4)$!

  BUT

  $f(1/2) = 1 \quad \neq \quad 2 = f(2/4)$

Condition *(2):"there are two different values of y that satisfy f(x)=y"* is True.

Functions Acting on Sets

- If $f : X \rightarrow Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

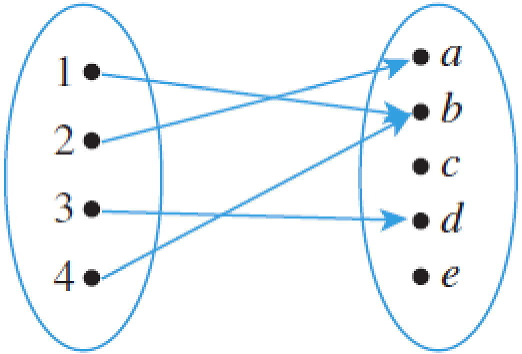
$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

$f(A)$ is the image of A

$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}$$

$f^{-1}(C)$ is the **inverse** image of C

Example: $X = \{1, 2, 3, 4\}$, $Y = \{a, b, c, d, e\}$, $f : X \rightarrow Y$



$f(\{1,4\}) = \{b\}$ $f^{-1}(\{a,b\}) = \{1, 2, 4\}$

$f(X) = \{a, b, d\}$ $f^{-1}(\{c,e\}) = \emptyset$

Functions Acting on Sets

- Let X and Y be sets, let $F : X \to Y$ be a function and $A \subseteq X$ and $B \subseteq X$, then $F(A \cup B) \subseteq F(A) \cup F(B)$

Proof:

Suppose $y \in F(A \cup B)$.

By definition of function, $y = F(x)$ for some $x \in A \cup B$.

By definition of union, $x \in A$ or $x \in B$.

Case 1, $x \in A$: $F(x) = y$, so $y \in F(A)$.

By definition of union: $y \in F(A) \cup F(B)$

Case 2, $x \in B$: $F(x) = y$, so $y \in F(B)$.

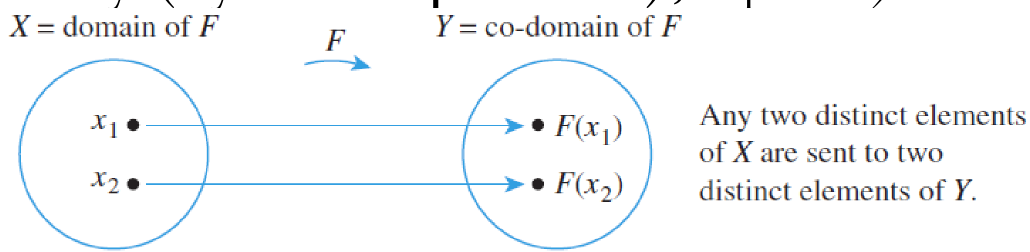
By definition of union: $y \in F(A) \cup F(B)$ ■

Functions

- ***One-to-One Functions (injective)*:**
  - A function F : X ⟶ Y is ***one-to-one (injective)*** ⇔

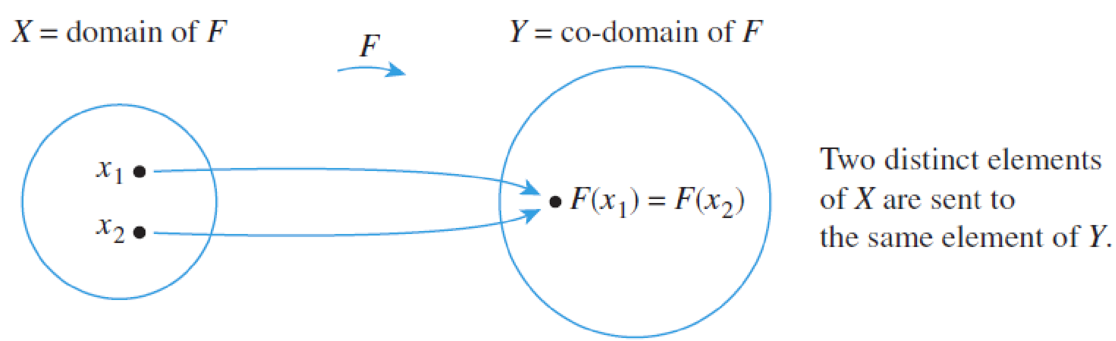
for all elements $x_1 \in X$ and $x_2 \in X$, $F(x_1) = F(x_2) \rightarrow x_1 = x_2$

or, equivalently (by contraposition), $x_1 \neq x_2 \rightarrow F(x_1) \neq F(x_2)$



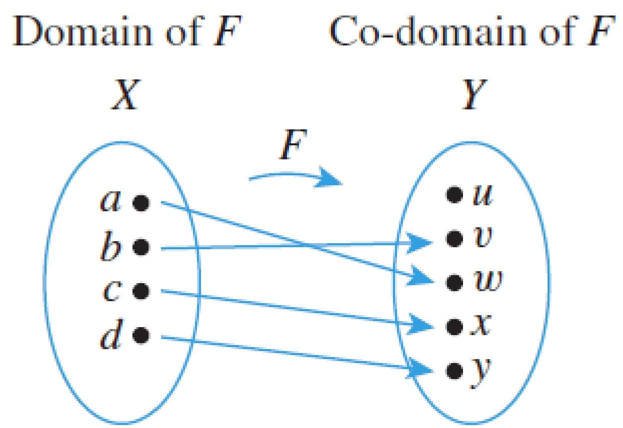
  - A function F : X ⟶ Y is **NOT** ***one-to-one (injective)*** ⇔

∃ elements $x_1 \in X$ and $x_2 \in X$, such that $x_1 \neq x_2$ and $F(x_1) = F(x_2)$.



One-to-One Functions Defined on Finite Sets

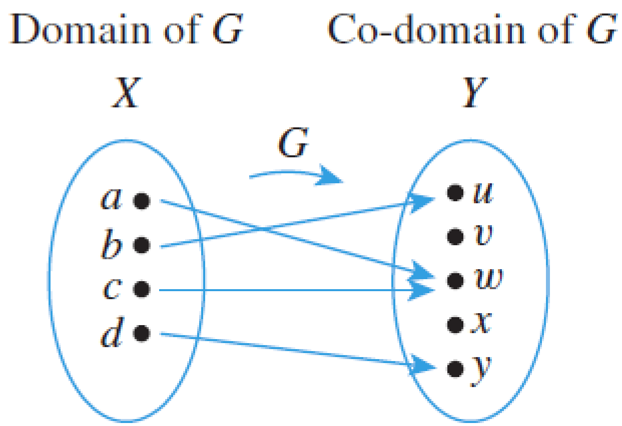
- Example 1: F: {a,b,c,d} $\rightarrow$ {u,v,w,x,y} defined by the following arrow diagram is **one-to-one**:



$$\forall\, x_1 \in X \text{ and } x_2 \in X, \quad x_1 \neq x_2 \Rightarrow F(x_1) \neq F(x_2)$$

One-to-One Functions Defined on Finite Sets

- Example 2: G: {a,b,c,d} → {u,v,w,x,y} defined by the following arrow diagram is **NOT one-to-one**:



G(a) = G(c) = w

∃ elements $x_1 \in X$ and $x_2 \in X$, such that $x_1 \neq x_2$ and $G(x_1) = G(x_2)$

I.e., a ∈ X and c ∈ X, such that a ≠ c and G(a) = G(c)

One-to-One Functions Defined on Finite Sets

- Example 3: $H: \{1, 2, 3\} \rightarrow \{a, b, c, d\}$

$H(1) = c$, $H(2) = a$, and $H(3) = d$

H is **one-to-one:**

$\forall\ x_1 \in X$ and $x_2 \in X$, $x_1 \neq x_2 \Rightarrow H(x_1) \neq H(x_2)$

- Example 4: $K: \{1, 2, 3\} \rightarrow \{a, b, c, d\}$

$K(1) = d$, $K(2) = b$, and $K(3) = d$

K is **NOT one-to-one:**

$$K(1) = K(3) = d$$

$\exists$ elements $x_1 \in X$ and $x_2 \in X$, such that $x_1 \neq x_2$ and $K(x_1) = K(x_2)$

One-to-One Functions on Infinite Sets

- f is one-to-one ⇔ $\forall x_1, x_2 \in X$, if $f(x_1)=f(x_2)$ then $x_1=x_2$
- To show f is one-to-one, we will generally use the method of direct proof:
  - **suppose** x_1 and x_2 are elements of X such that $f(x_1)=f(x_2)$
  - **show** that $x_1 = x_2$.
- To show f is **not** one-to-one, we will try to use the method of direct proof and detect that we cannot (similar to counterexample method):
  - **find** elements x_1 and x_2 in X so that $f(x_1)=f(x_2)$ but $x_1 \neq x_2$.

One-to-One Functions on Infinite Sets

- Example: $f : \mathbf{R} \rightarrow \mathbf{R}$,

  $f(x) = 4x - 1$ for all $x \in \mathbf{R}$ is f one-to-one?

f is one-to-one ⇔ $\forall x_1, x_2 \in X$, if $f(x_1)=f(x_2)$ then $x_1=x_2$

suppose x_1 and x_2 are any real numbers such that $4x_1-1=4x_2-1$

Adding 1 to both sides and and dividing by 4 both sides gives

$\mathbf{x_1=x_2}$ Yes! → **f is one-to-one** ■

- Example: $g : \mathbf{Z} \rightarrow \mathbf{Z}$,

  $g(n) = n^2$ for all $n \in \mathbf{Z}$ is g one-to-one?

Start by try to show that g is one-to-one:

suppose n_1 and n_2 are integers such that $n_1^2=n_2^2$ and **try to show** that $n_1=n_2$ **No!** $1^2=(-1)^2=1$ → **g is not one-to-one** ■

Hash Functions

- Hash Functions are functions defined from larger to smaller sets of integers used in ***signing*** documents.
- Example: Hash:SSN $\rightarrow$ {0, 1, 2, 3, 4, 5, 6}

SSN = the set of all social security numbers (ignoring hyphens)

Hash(n) = n mod 7   for all social security numbers n.

Hash(328343419) = 328343419 − (7·46906202) = 5

- Hash is not one-to one: called a **collision** for hash functions.

Hash(328343412) = 328343412 − (7· 46906201) = 5

- Collision resolution methods: if position Hash(n) in the hash array is already occupied, then start from that position and search downward to place the record in the first empty position.

Onto Functions

- $F: X \rightarrow Y$ is ***onto*** (***surjective***) $\Leftrightarrow$

$$\forall y \in Y, \quad \exists x \in X \text{ such that } F(x) = y.$$

For arrow diagrams, a function is onto if each element of the co-domain has an arrow pointing to it from some element of the domain.

- $F: X \rightarrow Y$ is **NOT** ***onto*** (***surjective***) $\Leftrightarrow$

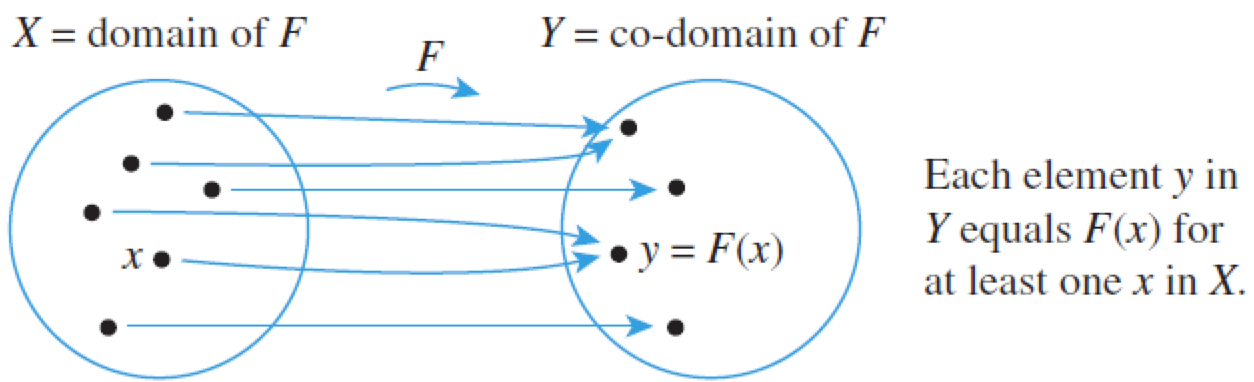
$$\exists y \in Y \text{ such that } \forall x \in X, F(x) \neq y.$$

There is some element in Y that is not the image of any element in X.

For arrow diagrams, a function is not onto if at least one element in its co-domain does not have an arrow pointing to it.

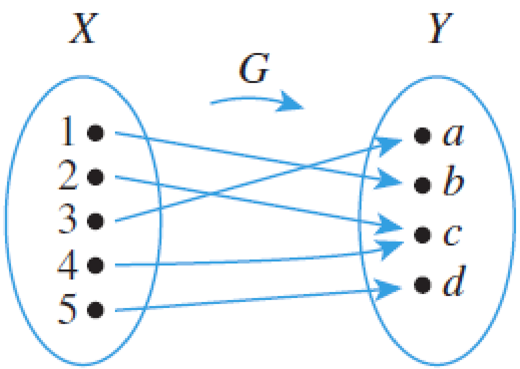
Onto Functions with Arrow Diagrams

- F is onto:



Onto Functions with Arrow Diagrams

- Example: G: {1,2,3,4,5} → {a,b,c,d}

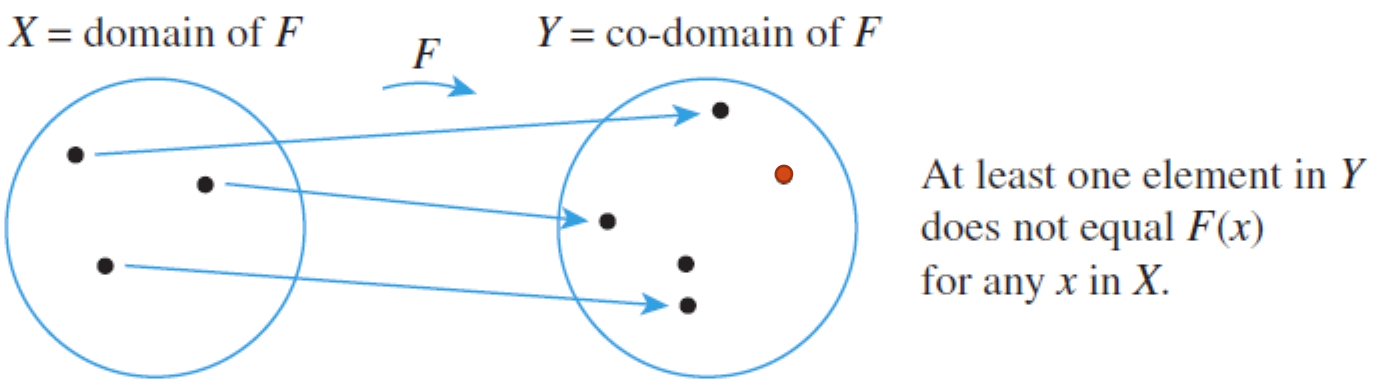


G is onto

$\forall y \in Y, \ \exists x \in X$ such that $G(x) = y$

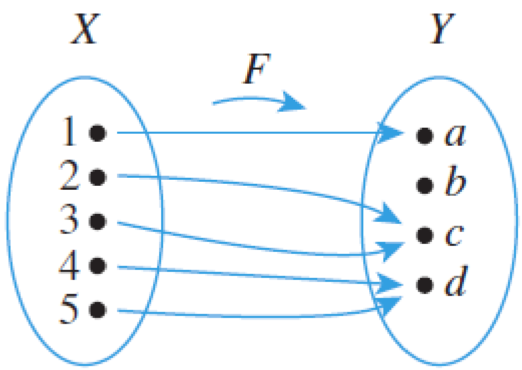
Onto Functions with Arrow Diagrams

- F is not onto



Onto Functions with Arrow Diagrams

- Example: F: {1,2,3,4,5} ⟶ {a,b,c,d}



F is **not onto** because b ≠ F(x) for any x in X

∃y ∈ Y such that ∀x ∈ X, F(x) ≠ y

Onto Functions

- Example: H: $\{1,2,3,4\} \to \{a,b,c\}$

$H(1) = c$, $H(2) = a$, $H(3) = c$, and $H(4) = b$

H is onto because $\forall y \in Y$, $\exists x \in X$ such that $H(x) = y$:

$a = H(2)$

$b = H(4)$

$c = H(1) = H(3)$

- Example: K: $\{1,2,3,4\} \to \{a,b,c\}$

$K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$

H is not onto because $a \neq K(x)$ for any $x \in \{1, 2, 3, 4\}$.

Onto Functions on Infinite Sets

- F is onto ⇔ $\forall y \in Y, \exists x \in X$ such that $F(x) = y$.
- We prove F is **onto** using the method of generalizing from the generic particular:
  - **suppose** that y is any element of Y,
  - **show** that there is an element x of X with $F(x)=y$.
- Prove F is **not onto:**
  - **find** an element y of Y such that $y \neq F(x)$ for any x in X.

Onto Functions on Infinite Sets

- Example: $f : \mathbf{R} \rightarrow \mathbf{R}$ Prove f is onto or give counterexample.

  $f(x) = 4x - 1$ for all $x \in \mathbf{R}$

  suppose $y \in \mathbf{R}$

  show that there exists a real number x such that $y = 4x - 1$.

  $4x - 1 = y \Leftrightarrow x = (y + 1)/4 \in \mathbf{R}$ by adding 1 and dividing by 4

  $\rightarrow$ **f is onto** $\blacksquare$

- Example: $h : \mathbf{Z} \rightarrow \mathbf{Z}$ Prove h is onto or give counterexample.

  $h(n) = 4n - 1$ for all $n \in \mathbf{Z}$

  $0 \in \mathbf{Z}$, if $h(n) = 0$, then $4n - 1 = 0 \Leftrightarrow n = 1/4 \notin \mathbf{Z}$

  $h(n) \neq 0$ for any integer n $\rightarrow$ h **is not onto** $\blacksquare$

Exponential Functions

- The exponential function with base b: $\exp_b : \mathbf{R} \rightarrow \mathbf{R}^+$

$$\exp_b(x) = b^x$$

$\exp_b(0) = b^0 = 1$

$\exp_b(-x) = b^{-x} = 1/b^x$

- The exponential function is one-to-one and onto

  For any positive real number $b \neq 1$, $b^v = b^u \rightarrow u = v, \forall u,v \in \mathbf{R}$

- Laws of Exponents: $\forall$ b, c $\in \mathbf{R}^+$ and u,v $\in \mathbf{R}$

$$b^u b^v = b^{u+v}$$

$$(b^u)^v = b^{uv}$$

$$b^u / b^v = b^{u-v}$$

$$(bc)^u = b^u c^u$$

Logarithmic Functions

- The logarithmic function with base b: $\log_b : \mathbf{R}^+ \rightarrow \mathbf{R}$

$$\log_b(x) = y \Leftrightarrow b^y = x$$

- The logarithmic function is one-to-one and onto.

For any positive real number $b \neq 1$,

$$\log_b u = \log_b v \rightarrow u = v, \forall u, v \in \mathbf{R}^+$$

- Properties of Logarithms: $\forall$ b, c, x $\in \mathbf{R}^+$, with $b \neq 1$ and $c \neq 1$

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b(x^a) = a \log_b x$$

$$\log_c x = \log_b x \,/\, \log_b c$$

Exponential and Logarithmic Functions

- $\forall$ b, c, x $\in$ $\mathbf{R}^+$, with b$\neq$1 and c$\neq$1: $\log_c x = \log_b x \,/\, \log_b c$

Proof: Suppose positive real numbers b, c, and x are given, s.t. (1) $\log_b c = u$ (2) $\log_c x = v$ (3) $\log_b x = w$

By definition of logarithm: $c = b^u$, $x = c^v$ and $x = b^w$

$x = c^v = (b^u)^v = b^{uv}$, by laws of exponents

But $x = b^w = b^{uv}$, so $uv = w$ (by one-one exponent) ➔

By (1), (2) and (3): $(\log_b c)(\log_c x) = \log_b x$

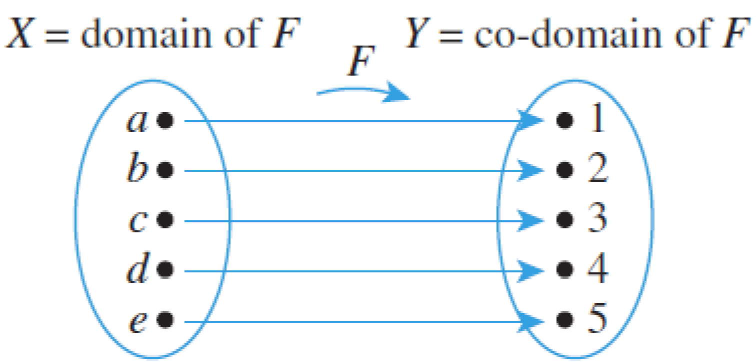
By dividing both sides by $\log_b c$: $\log_c x = \log_b x \,/\, \log_b c$ ■

Exponential and Logarithmic Functions

- Notations:
  - Logarithms with base 10 are called **common logarithms** and are denoted by simply log.
  - Logarithms with base *e* are called **natural logarithms** and are denoted by ln.
- Example: $\log_2 5 = \log 5 \;/\; \log 2 = \ln 5 \;/\; \ln 2$

One-to-One Correspondences

- A ***one-to-one correspondence*** (or ***bijection***) from a set X to a set Y is a function F: X ⟶ Y that is **both one-to-one** and **onto**.
- Example:



One-to-One Correspondences

- Example: A Function from a Power Set to a Set of Strings

$$h : P(\{a, b\}) \rightarrow \{00, 01, 10, 11\}$$

If a is in A, write a 1 in the 1^{st} position of the string h(A).

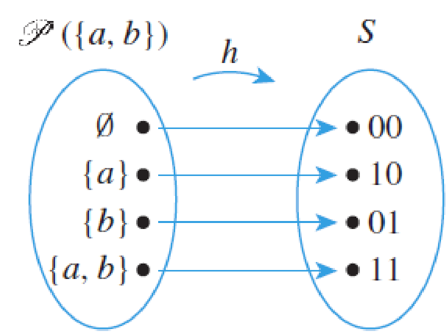
If a is not in A, write a 0 in the 1^{st} position of the string h(A).

If b is in A, write a 1 in the 2^{nd} position of the string h(A).

If b is not in A, write a 0 in the 2^{nd} position of the string h(A).

h

Subset of $\{a, b\}$	Status of a	Status of b	String in S
∅	not in	not in	00
$\{a\}$	in	not in	10
$\{b\}$	not in	in	01
$\{a, b\}$	in	in	11



One-to-One Correspondences

- Example: F: $\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R} \times \mathbf{R}$

$$F(x, y) = (x + y, x - y), \text{ for all } (x, y) \in \mathbf{R} \times \mathbf{R}$$

Part 1: Proof that F is one-to-one:

Suppose that (x_1,y_1) and (x_2,y_2) are any ordered pairs in $\mathbf{R} \times \mathbf{R}$ such that $\mathbf{F(x_1,y_1) = F(x_2,y_2)}$.

$\Leftrightarrow (x_1 + y_1, x_1 - y_1) = (x_2 + y_2, x_2 - y_2)$

$\Leftrightarrow$ (1)$x_1 + y_1 = x_2 + y_2$ and (2) $x_1 - y_1 = x_2 - y_2$

(1)+(2) $\rightarrow 2x_1 = 2x_2 \rightarrow$ (3) $x_1 = x_2$

Substituting (3) in (2) $\rightarrow x_2 + y_1 = x_2 + y_2 \rightarrow y_1 = y_2$

$\rightarrow \mathbf{(x_1, y_1) = (x_2, y_2)}$

Yes, F is one-to-one.

One-to-One Correspondences

- Example: F: $\mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$

$$F(x, y) = (x + y, x - y), \text{ for all } (x, y) \in \mathbf{R} \times \mathbf{R}$$

Part2: Proof that F is onto:

Let (u,v) be any ordered pair in $\mathbf{R} \times \mathbf{R}$

Suppose that we found $(r,s) \in \mathbf{R} \times \mathbf{R}$ such that $F(r,s) = (u, v)$.

$\Leftrightarrow (r + s, r - s) = (u, v) \Leftrightarrow r + s = u$ and $r - s = v$

$\Leftrightarrow 2r = u + v$ (by sum of 2 eqs) and $2s = u - v$ (by diff eqs)

$\Leftrightarrow r = (u + v)/2$ and $s = (u - v)/2$

We found $(r, s) \in \mathbf{R} \times \mathbf{R}$

Yes, F is onto.

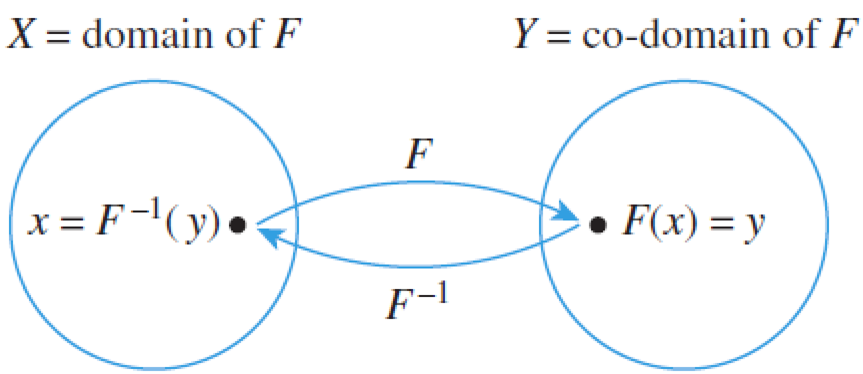
So, F is a One-to-One correspondence. ■

Inverse Functions

- If $F: X \rightarrow Y$ is a one-to-one correspondence, then there is an ***inverse function for*** F, $F^{-1}: Y \rightarrow X$, s.t.for any element $y \in Y$

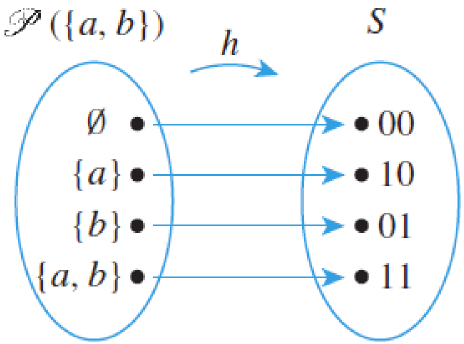
$$F^{-1}(y) = \text{that unique element } x \in X \text{ such that } F(x)=y$$

$$F^{-1}(y) = x \Leftrightarrow F(x) = y$$

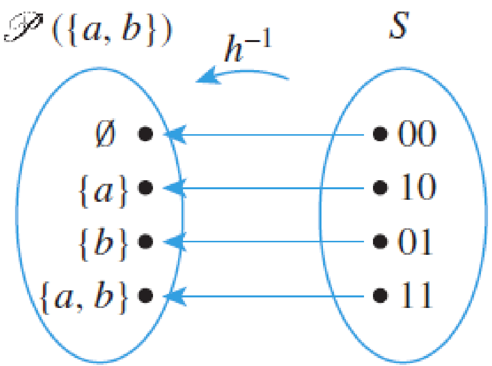


Inverse Functions

- Example:



the inverse function for h is h^{-1}:



$h^{-1}(00) = \emptyset \quad h^{-1}(10) = \{a\}$

$h^{-1}(01) = \{b\} \quad h^{-1}(11) = \{a, b\}$

Inverse Functions

- Example: $f : \mathbf{R} \to \mathbf{R}$, $f(x) = 4x - 1$ for all real numbers x.

The inverse function for f is $f^{-1} : \mathbf{R} \to \mathbf{R}$,

For any [particular but arbitrarily chosen] y in $\mathbf{R}$

$f^{-1}(y)$ = that unique real number x such that $f(x)=y$.

$f(x) = y \Leftrightarrow 4x - 1 = y \Leftrightarrow x = (y + 1)/4$

Hence $f^{-1}(y) = (y + 1)/4$.

Inverse Functions

- If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

Proof:

F^{-1} is one-to-one: Suppose y_1 and y_2 are elements of Y, s.t. $F^{-1}(y_1) = F^{-1}(y_2)$

Let $x = F^{-1}(y_1) = F^{-1}(y_2)$, $x \in X$.

By definition of F^{-1}, $F(x) = y_1$ and $F(x) = y_2$, so $y_1 = y_2$

F^{-1} is onto: Suppose $x \in X$.

Let $y = F(x)$, $y \in Y$

By definition of F^{-1}, $F^{-1}(y) = x$.

One-to-One and Onto for Finite Sets

- Let X and Y be finite sets with the **same number of elements** and suppose f is a function from X to Y .

f is one-to-one ⇔ f is onto

Proof: Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_m\}$

(1) f is one-to-one → f is onto

$f(x_1), f(x_2), \ldots, f(x_m)$ are all distinct, and $S = \{y \in Y \mid \forall x \in X, f(x) \neq y\}$

$\{f(x_1)\}, \{f(x_2)\}, \ldots, \{f(x_m)\}$ and S are mutually disjoint

$m = N(Y) = N(\{f(x_1)\}) + N(\{f(x_2)\}) + \ldots + N(f(x_m)\}) + N(S) = m + N(S)$

⇔ $N(S) = 0$, there is no element of Y that is not the image of some element of X

f is onto

(2) f is onto → f is one-to-one

$N(f^{-1}(y_i)) \geq 1$ for all $i = 1, \ldots, m$ →

$m = N(X) = N(f^{-1}(y_1)) + \ldots + N(f^{-1}(y_m))$, m terms → $N(f^{-1}(y_i)) = 1$,

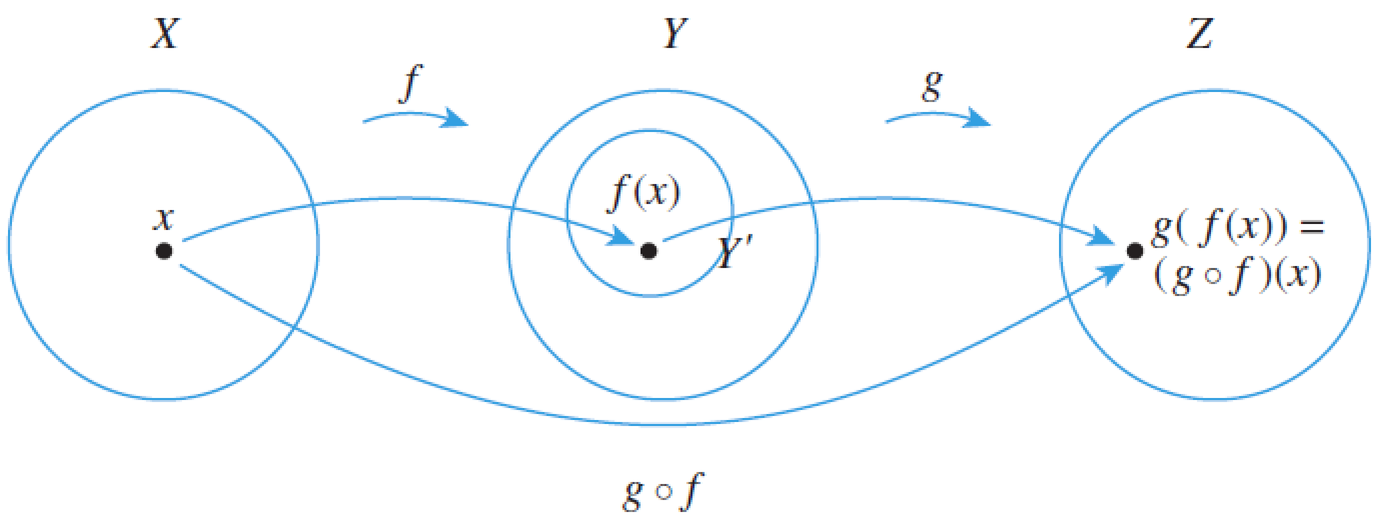
f is one-to-one

Composition of Functions

- Let $f : X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g: $Y' \subseteq Y$

The composition of f and g is a function $g \circ f : X \to Z$:

$$(g \circ f)(x) = g(f(x)) \qquad \text{for all } x \in X$$



Composition of Functions

- Example composition of functions:

Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$ and $g: \mathbf{Z} \rightarrow \mathbf{Z}$

$$f(n) = n + 1, \text{ for all } n \in \mathbf{Z}$$

$$g(n) = n^2, \text{ for all } n \in \mathbf{Z}$$

$$(g \circ f)(n) = g(f(n)) = g(n+1) = (n + 1)^2, \text{ for all } n \in \mathbf{Z}$$

$$(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1, \text{ for all } n \in \mathbf{Z}$$

$$(g \circ f)(1) = (1 + 1)^2 = 4$$

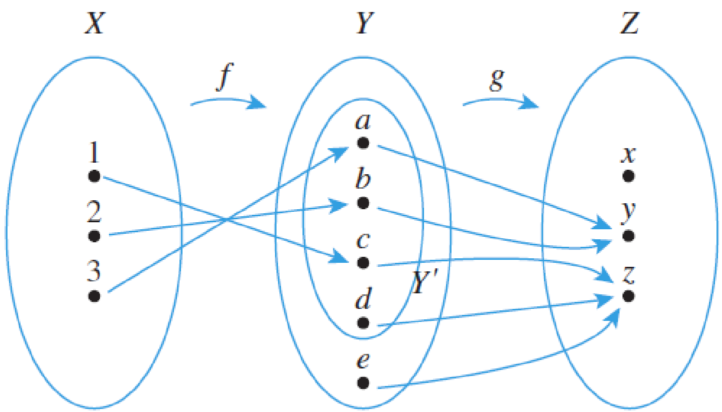
$$(f \circ g)(1) = 1^2 + 1 = 2$$

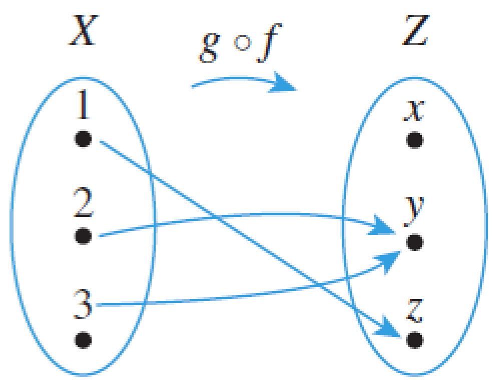
$$f \circ g \neq g \circ f$$

Composition of Functions

- Example composition of functions:

Let f : {1,2,3} → {a,b,c,d} and g: {a,b,c,d,e} → {x,y,z}

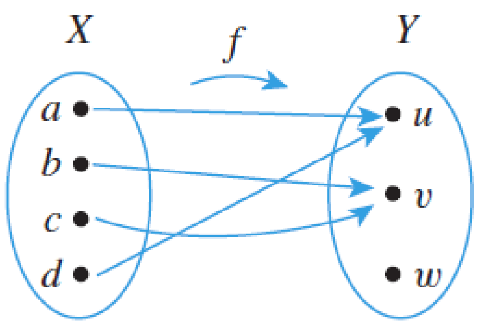




Composition of Functions

- Example composition of functions:

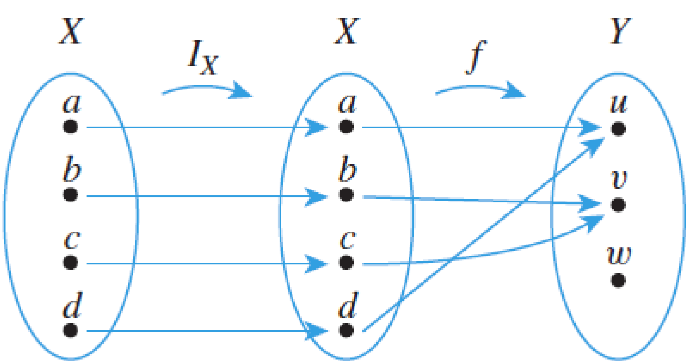
Let $X = \{a, b, c, d\}$ and $Y = \{u, v, w\}$, $f : X \rightarrow Y$



I_X: $X \rightarrow X$ is an identity function

$I_X(x) = x$, for all $x \in X$

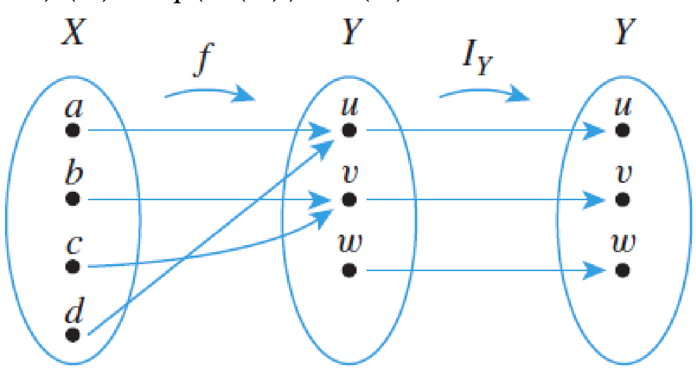
$(f \circ I_X)(x) = f(I_X(x)) = f(x)$, for all $x \in X$



I_Y: $Y \rightarrow Y$ is an identity function

$I_Y(y) = y$, for all $y \in Y$

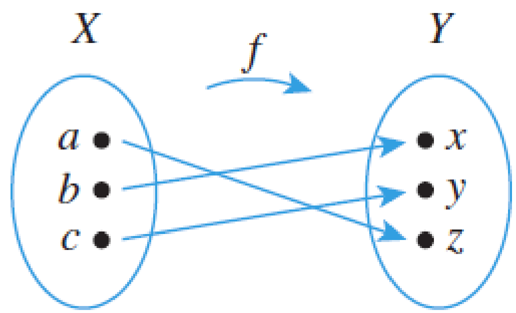
$(I_Y \circ f)(x) = I_Y(f(x)) = f(x)$, for all $x \in X$



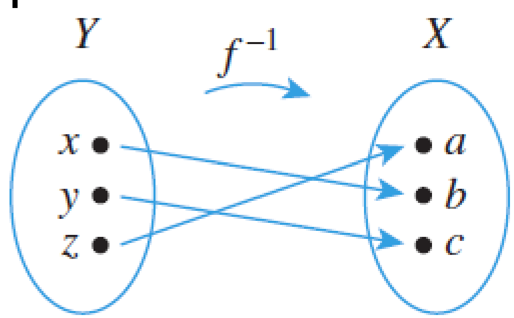
Composition of Functions

- **Composing a Function with Its Inverse:**

  Let $f : \{a, b, c\} \to \{x, y, z\}$ be a one-to-one and onto function



f is one-to-one correspondence ➔ $f^{-1} : \{x, y, z\} \to \{a, b, c\}$



$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(z) = a$

$(f^{-1} \circ f)(b) = f^{-1}(f(b)) = f^{-1}(x) = b$ ➔ $\mathbf{f^{-1} \circ f = I_X}$

$(f^{-1} \circ f)(c) = f^{-1}(f(c)) = f^{-1}(y) = c$ **also** $\mathbf{f \circ f^{-1} = I_Y}$

Composition of Functions

- **Composing a Function with Its Inverse:**

If $f: X \to Y$ is a one-to-one and onto function with inverse function $f^{-1}: Y \to X$, then **(a)** $f^{-1} \circ f = I_X$ and **(b)** $f \circ f^{-1} = I_Y$

Proof (a):

Let x be any element in X: $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x' \in X$ (*)

Definition of inverse function:

$f^{-1}(b) = a \Leftrightarrow f(a) = b$ for all $a \in X$ and $b \in Y$

$\rightarrow f^{-1}(f(x)) = x' \Leftrightarrow f(x') = f(x)$

Since f is one-to-one, this implies that $x' = x$.

(*) $\rightarrow (f^{-1} \circ f)(x) = x$

Composition of Functions

- **Composition of One-to-One Functions:**

If $f : X \rightarrow Y$ and $g: Y \rightarrow Z$ are both one-to-one functions, then $g \circ f$ is also one-to-one.

Proof (by the method of direct proof):

Suppose $f : X \rightarrow Y$ and $g: Y \rightarrow Z$ are both one-to-one functions.

Suppose $x_1, x_2 \in X$ such that: $(g \circ f)(x_1) = (g \circ f)(x_2)$

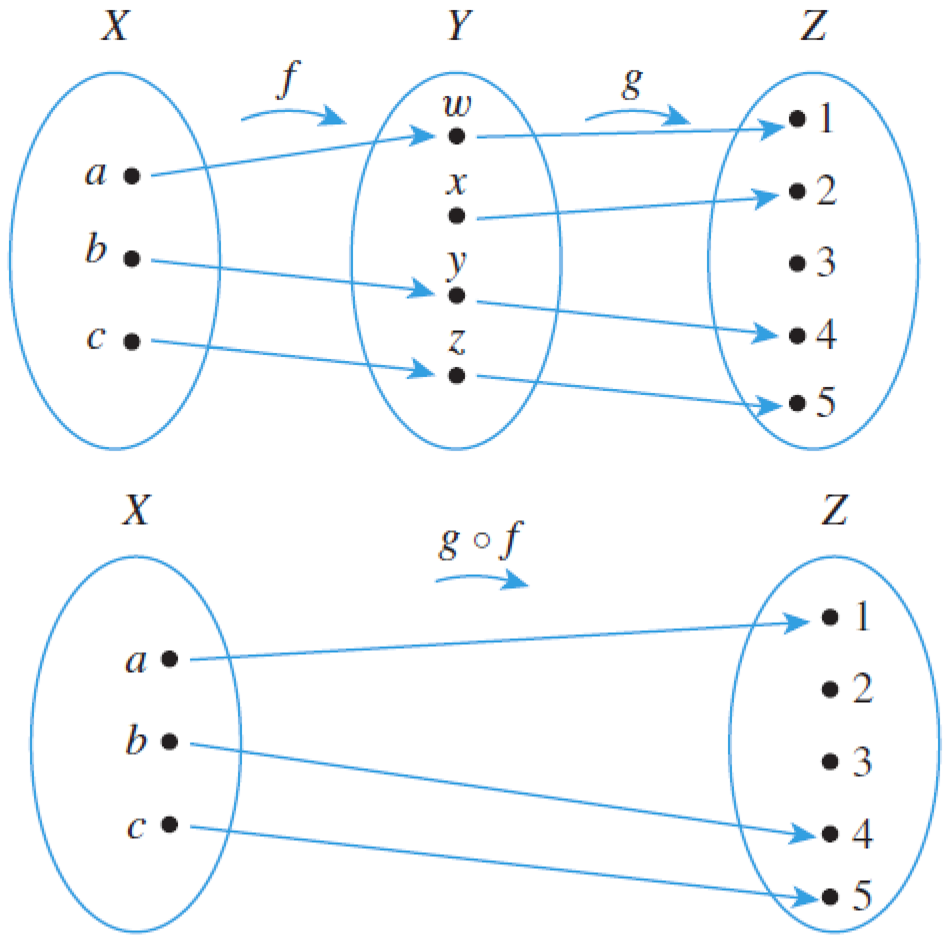
By definition of composition of functions, $g(f(x_1)) = g(f(x_2))$.

Since g is one-to-one, $f(x_1) = f(x_2)$.

Since f is one-to-one, $x_1 = x_2$.

Composition of Functions

- **Composition of One-to-One Functions Example:**



Composition of Functions

- **Composition of Onto Functions:**

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both onto functions, then $g \circ f$ is onto.

Proof:

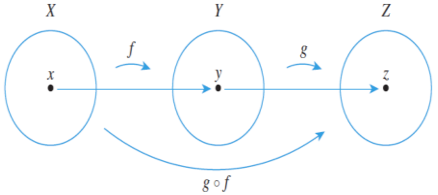
Suppose $f : X \rightarrow Y$ and $g: Y \rightarrow Z$ are both onto functions.

Let z be a [particular but arbitrarily chosen] element of Z.

Since g is onto, there is an element y in Y such that $g(y) = z$.

Since f is onto, there is an element x in X such that $f(x) = y$.

$z = g(y) = g(f(x)) = (g \circ f)(x)$ ➔ $g \circ f$ is onto



Composition of Functions

- **Composition of Onto Functions Example:**