CHAPTER 4

CLASSICAL PROPOSITIONAL SEMANTICS

1 Language

There are several propositional languages that are routinely called classical propositional logic languages. It is due to the functional dependency of classical connectives discussed briefly in chapter 2. They all share the same logical meaning, called semantics. They also define the same set of universally true formulas called tautologies.

We adopt here as classical propositional language the language $\mathcal{L}$ with the full set of connectives $CON = \{\neg, \cup, \cap, \Rightarrow, \Leftrightarrow, \Leftrightarrow\}$ i.e. the language

$$\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow, \Leftrightarrow, \Leftrightarrow\}}.$$

As the choice of the set of connectives is now fixed, we will use the symbol

$$\mathcal{L}$$

to denote the language $\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow,\}}$.

In previous chapters we have already established how we *read* and what are the natural language *names* of the connectives of $\mathcal{L}$.

For example, we **read** the symbol $\neg$ as *"not", "not true"* and its **name** is **negation**.

We **read** the formula $(A \Rightarrow B)$ as *" if A, then B", "A implies B",* or *"from the fact that A we deduce B"*, and the **name** of the connective $\Rightarrow$ is **implication**.

We refer to the established reading of propositional connectives and formulas involving them as **a natural language meaning**.

Propositional logic defines and studies their **a logical meaning** called **a semantics** of the language $\mathcal{L}$. We define here a **classical semantics**, some other semantics are defined in next chapter (Chapter 5).

2 Classical Semantics, Satisfaction

We based the classical logic, i.e. **classical semantics** on the following two assumptions.

TWO VALUES: there are only two logical values. We denote them T (for true) and F (for false). Other common notations are $1, \top$ for true and $0, \bot$ for false.

EXTENSIONALITY: the logical value of a formula depends only on a main connective and logical values of its sub-formulas.

We define a **classical semantics** for $\mathcal{L}$ in terms of two factors: classical truth tables (reflexes the extensionality of connectives) and a truth assignment. In Chapter 2 we provided a motivation for the notion of classical logical connectives and introduced their informal definitions. We summarize here the truth tables for propositional connectives defined in chapter 2 in the following one table.

A	B	$\neg A$	$(A \cap B)$	$(A \cup B)$	$(A \Rightarrow B)$	$(A \Leftrightarrow B)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

(1)

The first truth values row of the above table 2 reads:

For any formulas A, B, if the logical value of $A = T$ and $B = T$, then logical values of $\neg A = T$, $(A \cap B) = T$, $(A \cup B) = T$ and $(A \Rightarrow B) = T$.

We read and write the other rows in a similar manner.

The table 2 indicates that the logical value of of propositional formulas depends on the logical values of its factors; i.e. fulfils the condition of extensionality. Moreover, it shows that the logical value of of propositional connectives depends only on the logical values of its factors; i.e. it is **independent of the formulas** A, B. It gives us the following important property of our propositional connectives.

EXTENSIONAL CONNECTIVES: The logical value of a given connective depend only of the logical values of its factors.

We now write the table in an even simpler form of the following equations.

$$
\begin{gathered}
\neg T = F, \quad \neg F = T; \\
(T \cap T) = T, \quad (T \cap F) = F, \quad (F \cap T) = F, \quad (F \cap F) = F; \\
(T \cup T) = T, \quad (T \cup F) = T, \quad (F \cup T) = T, \quad (F \cup F) = F; \\
(T \Rightarrow T) = T, \quad (T \Rightarrow F) = F, \quad (F \Rightarrow T) = T, \quad (F \Rightarrow F) = T; \\
(T \Leftrightarrow T) = T, \quad (T \Leftrightarrow F) = F, \quad (F \Leftrightarrow T) = F, \quad (T \Leftrightarrow T) = T.
\end{gathered} \tag{2}
$$

Observe that the equations 2 describe a set of unary and binary operations (functions) defined on a set $\{T, F\}$, or on a set $\{T, F\} \times \{T, F\}$ as follows.

Negation $\neg$ is a function function:

$$\neg : \quad \{T, F\} \longrightarrow \{T, F\},$$

such that $\neg T = F$, $\neg F = T$.

Conjunction $\cap$ is a function:

$$\cap : \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that $(T \cap T) = T, \quad (T \cap F) = F, \quad (F \cap T) = F, \quad (F \cap F) = F$

Disjunction $\cup$ is a function:

$$\cup : \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that $(T \cup T) = T, \quad (T \cup F) = T, \quad (F \cup T) = T, \quad (F \cup F) = F.$

Implication $\Rightarrow$ is a function

$$\Rightarrow : \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that $(T \Rightarrow T) = T, \quad (T \Rightarrow F) = F, \quad (F \Rightarrow T) = T, \quad (F \Rightarrow F) = T,$

Observe that if we have have a language $\mathcal{L}_{\{\neg, \cup, \cap, \Rightarrow, \Leftrightarrow\}}$ containing also the equivalence connective $\Leftrightarrow$ we define

Equivalence $\Leftrightarrow$ as a function:

$$\Leftrightarrow : \quad \{T, F\} \times \{T, F\} \longrightarrow \{T, F\},$$

such that $(T \Leftrightarrow T) = T, \quad (T \Leftrightarrow F) = F, \quad (F \Leftrightarrow T) = F, \quad (F \Leftrightarrow F) = T.$

We write them in a form of tables, usually called the classical truth tables.

Classical Truth Tables (3)

Negation :

$\neg$	T	F
	F	T

Disjunction :

$\cup$	T	F
T	T	T
F	T	F

Conjunction :

$\cap$	T	F
T	T	F
F	F	F

Implication :

$\Rightarrow$	T	F
T	T	F
F	T	T

Equivalence :

$\Leftrightarrow$	T	F
T	T	F
F	F	T

Definition 2.1 **A truth assignment** *is any function*

$$v : VAR \longrightarrow \{T, F\}.$$

The function v defined in above definition 2.1 is called the truth assignment because it can be thought as an assignment to each variable (which represents a logical sentence) its logical value of T(ruth) of F(alse).

Remark 1 The domain of v is the countably infinite set VAR of all propositional variables.

Remark 2 The truth function v of the above definition 2.1 assigns logical values to the atomic formulas only.

We use the truth tables to extend the truth assignment v to the set of all formulas as follows.

Definition 2.2 (Truth Tables Semantics) *For each truth assignment*

$$v : VAR \longrightarrow \{T, F\}$$

we define its **extension**

$$v^* : \mathcal{F} \longrightarrow \{T, F\}$$

to the set F of all formulas of $\mathcal{L}$ as follows.

For any $a \in VAR$,

$$v^*(a) = v(a).$$

For any $A, B \in \mathcal{F}$,

$$\begin{aligned} v^*(\neg A) &= \neg v^*(A); \\ v^*(A \cap B) &= v^*(A) \cap v^*(B); \\ v^*(A \cup B) &= v^*(A) \cup v^*(B); \\ v^*(A \Rightarrow B) &= v^*(A) \Rightarrow v^*(B), \\ v^*(A \Leftrightarrow B) &= v^*(A) \Leftrightarrow v^*(B), \end{aligned} \tag{4}$$

where

the symbols on the **left-hand side** *of the equations 4 represent connectives in their* **natural language meaning** *and*

the symbols on the **right-hand side** *represent connectives in their* **logical meaning** *given by the classical truth tables 5 or equations 2 representing them.*

Example 1

Let A be a formula $((a \cup \neg b) \Rightarrow a)$ and $v : VAR \longrightarrow \{T, F\}$ be such that $v(a) = T, v(b) = F$ and $v(x) = T$ for all other variables. We evaluate $v^*(A)$ as follows.

$v^*((a \cup \neg b) \Rightarrow a) = (v^*(a \cup \neg b) \Rightarrow v^*(a)) = ((v^*(a) \cup v^*(\neg b)) \Rightarrow v^*(a)) = ((v(a) \cup \neg v^*(b)) \Rightarrow v(a)) = ((v(a) \cup \neg v(b)) \Rightarrow v(a)) = ((T \cup \neg F) \Rightarrow T) = ((T \cup T) \Rightarrow T) = (T \Rightarrow T) = T.$

We write the computation of logical value of the formula A in a short-hand notation using the last set of equations that involve only T, F symbols as follows. $v^*((a \cup \neg b) \Rightarrow a) = ((T \cup \neg F) \Rightarrow T) = ((T \cup T) \Rightarrow T) = (T \Rightarrow T) = T.$

Definition 2.3 (Satisfaction relation) *For any truth assignment* v*,*

v **satisfies** *a formula* $A \in \mathcal{F}$ *if and only if* $v^*(A) = T$.

We denote it by

$$v \models A.$$

If $v^*(A) \neq T$*, we say that* v **does not satisfy** *the formula* A *and denote it by*

$$v \not\models A.$$

Observe, that by the definition of our classical semantics $v^*(A) \neq T$ if and only if $v^*(A) = F$ and we say in this case that v **falsifies** A.

Example 2

Let A be a formula

$$((a \Rightarrow b) \cup \neg a))$$

and v be a truth assignment

$$v : VAR \longrightarrow \{T, F\}$$

such that $v(a) = T, v(b) = F$ and $v(x) = F$ for all $x \in VAR - \{a, b\}$.

By the definition, we calculate the logical value of the formula A as follows:

$v^*(A) = v^*((a \Rightarrow b) \cup \neg a)) =$
$(v^*(a \Rightarrow b) \cup v^*(\neg a)) = ((v(a) \Rightarrow v(b)) \cup \neg v(a)) = ((T \Rightarrow F) \cup \neg T) =$
$(F \cup F) = F$.
It proves that

$$v \not\models ((a \Rightarrow b) \cup \neg a)).$$

As we remarked before, in practical cases we use a short-hand notation for while evaluating the logical value of a given formula. Here is a short procedure for v and A defined above.

Short-hand evaluation

(1) We write $a = T$, $b = F$ for $v(a) = T, v(b) = F$.

(2) We replace a by T and b by F in the formula $((a \Rightarrow b) \cup \neg a))$ i.e. we write $((T \Rightarrow F) \cup \neg T)$.

(3) We use equations 2 or tables 5 to evaluate the logical value $v^*(A)$ as follows. $((T \Rightarrow F) \cup \neg T) = (F \cup F) = F$.

(4) We write

$$v \not\models ((a \Rightarrow b) \cup \neg a))$$

and say v *falsifies* A¿

Example 3

Let A be a formula

$$((a \cap \neg b) \cup \neg c)$$

and v be a truth assignment

$$v : VAR \longrightarrow \{T, F\}$$

such that $v(a) = T, v(b) = F, v(c) = T$ and $v(x) = T$ for all $x \in VAR - \{a, b, c\}$.

By the definition, we calculate (in the short-hand notation) the logical value of the formula A as follows:
$((T \cap \neg F) \cup \neg T) = ((T \cap T) \cup F) = (T \cup F) = T$
This proves that the truth assignment v *satisfies the formula* A and we write

$$v \models ((a \cap \neg b) \cup \neg c).$$

Example 4 Consider now the formula $A = ((a \cap \neg b) \cup \neg c)$ from the previous example. Let v_1, v_2, v_3 be the following truth assignments.

$v_1 : VAR \longrightarrow \{T, F\}$, such that $v_1(a) = T, v_1(b) = F, v_1(c) = T, v_1(x) = F$, for all $x \in VAR - \{a, b, c\}$, and

For $v_2 : VAR \longrightarrow \{T, F\}$, such that $v_2(a) = T, v_2(b) = F, v_2(c) = T, v_2(d) = T, v_2(x) = F$, for all $x \in VAR - \{a, b, c, d\}$, and

$v_3 : VAR \longrightarrow \{T, F\}$ such that $v_3(a) = T, v_3(b) = F, v_3(c) = T, v_3(d) = T, v_3(e) = T, v_3(x) = F$, for all $x \in VAR - \{a, b, c, d, e\}$.

We evaluate $v_i^*(A)$ and get that

$$v_i \models ((a \cap \neg b) \cup \neg c), \quad for \quad i = 1, 2, 3.$$

But instead of performing evaluation let's observe that v_1, v_2, v_3 have the same values on a, b, c as v, i.e.

$$v_i(a) = v(a), v_i(b) = v(b), v_i(c) = v(c),$$

and as we have already proved that $v \models ((a \cap \neg b) \cup \neg c)$, so we conclude that so do all v_i.

Moreover, for any $w : VAR \longrightarrow \{T, F\}$, such that $w(a) = v(a), w(b) = v(b), w(c) = v(c)$, we also have that

$$w \models ((a \cap \neg b) \cup \neg c).$$

We are going to prove that there are as many of such w's an real numbers. But they don't bring anything more then the initial v to our knowledge about A; they are all *the same* as v with respect to the formula A.

When we ask a question: *"How many truth assignments satisfy/fasify a formula A?"* we mean to find all assignment that are *different on the formula A*, not just different on a set VAR of all variables, as all of our w's were. To address and to answer this question formally we first introduce some notations and definitions.

Definition 2.4 *For any $A \in \mathcal{F}$, let VAR_A be a set of all propositional variables appearing in A. Any function*

$$v : VAR_A \longrightarrow \{T, F\},$$

is called **a truth assignment restricted to** *A.*

Example 5

Let $A = ((a \Rightarrow \neg b) \cup \neg c)$. The set of variables of A is $VAR_A = \{a, b, c\}$.
By definition 2.4 the truth assignment **restricted to** A is any function:

$$v : \{a, b, c\} \longrightarrow \{T, F\}.$$

We use the following theorem to count all possible truth assignment restricted to A.

Theorem 2.1 (Counting Functions (1)) *For any* **finite sets** *A and B, if A has* **n** *elements and B has* **m** *elements, then there are* $\mathbf{m^n}$ *possible functions that map A into B.*

So there are $2^3 = 8$ possible truth assignment restricted to $A = ((a \Rightarrow \neg b) \cup \neg c)$. We usually list them, and their value on the formula A in a form of an extended truth table below.

v	a	b	c	$v^*(A)$ *computation*	$v^*(A)$
v_1	T	T	T	$(T \Rightarrow T) \cup \neg T = T \cup F = T$	T
v_2	T	T	F	$(T \Rightarrow T) \cup \neg F = T \cup T = T$	T
v_3	T	F	F	$(T \Rightarrow F) \cup \neg F = F \cup T = T$	T
v_4	F	F	T	$(F \Rightarrow F) \cup \neg T = T \cup F = T$	T
v_5	F	T	T	$(F \Rightarrow T) \cup \neg T = T \cup F = T$	T
v_6	F	T	F	$(F \Rightarrow T) \cup \neg F = T \cup T = T$	T
v_7	T	F	T	$(T \Rightarrow F) \cup \neg T = F \cup F = F$	F
v_8	F	F	F	$(F \Rightarrow F) \cup \neg F = T \cup T = T$	T

(5)

By the same argument and theorem 2.1 we get proof of the following.

Theorem 2.2 *For any $A \in \mathcal{F}$ there are*

$$2^{card(VAR_A)}$$

possible truth assignments restricted to A.

Directly from the definition 2.4 we get that the following theorem hold.

Theorem 2.3 *For any formula $A \in \mathcal{F}$ and any v restricted to A, i.e.*

$$v : VAR_A \longrightarrow \{T, F\}, \tag{6}$$

the following holds.

$$v \models A \quad (v \not\models A)$$

if and only if for any truth assignment w such that

$$w : VAR \longrightarrow \{T, F\}$$
$$and \quad w(a) = v(a) \quad for \quad all \quad a \in VAR_A, \tag{7}$$

we have that

$$w \models A \quad (w \not\models A).$$

We put the relationship between v and w defined in the above theorem in a form of the following definition.

Definition 2.5 *The function w defined by 7 is called an* **extension of** *v to the set VAR.*
The function v defined by 6 is called an **restriction of** *w to the set VAR_A.*

A natural question arises: for given v, and a formula A (or any finite set of formulas) how many are there extension of v? Observe, that v has a finite domain and all w's have a countably infinite domain. We write it as $card(Dom(w) = \aleph_0$. In order to count all possible functions w we recall a second "counting the functions" theorem for infinite sets, similar to the first theorem 2.1.

Theorem 2.4 (Counting Functions (2)) *Let B^A be the set of all functions that map A into B, i.e.*

$$B^A = \{f : \quad f : A \longrightarrow B\}.$$

For any sets A and B,

$$card(B^A) = card(B)^{card(A)} = \mathcal{M}^{\mathcal{N}}.$$

In particular, when A is infinitely countable ie. $card(A) = \aleph_0$ and $card(B) = 2$, we get that there are

$$2^{\aleph_0} = \mathcal{C}$$

functions that map A into B, where $\mathcal{C} = card(R)$ for R being a set of real umbers.

Of course, theorem 2.1 is a particular case of the above theorem 2.4.

For any formula A, the set VAR_A is finite, and hence the set $VAR - VAR_A$ is countably infinite, i.e. $card(VAR - VAR_A) = 2^{\aleph_0} = \mathcal{C}$.

Given any truth assignment v restricted to A, all of its extensions w are defined by the formula 7 and differ on elements from the countable infinite set $VAR - VAR_A$. So there as many of them as functions from By the set $VAR - VAR_A$ into the set $\{T, F\}$. By theorem 2.4 there $2^{\aleph_0} = \mathcal{C}$ of them. This proves the following theorem, similar to the theorem 2.2.

Theorem 2.5 *For any $A \in \mathcal{F}$, for any any truth assignment v restricted to A, there are*

$$2^{\aleph_0} = \mathcal{C}$$

possible truth assignments that are extensions of v to the set VAR of all propositional variables.

We generalize theorems 2.2, 2.5 and theorem 2.3 to any finite set $\mathcal{F}_{FIN}$ of formulas as follows.

Theorem 2.6 *Let $\mathcal{F}_{FIN} \subseteq \mathcal{F}$ be a finite set of formulas and VAR_{FIN} be the set of all variables appearing in all formulas from $\mathcal{F}_{FIN}$, i.e.*

$$VAR_{FIN} = \bigcup_{A \in \mathcal{F}_{FIN}} VAR_A.$$

Let $v : VAR_{FIN} \longrightarrow \{T, F\}$ be any variable assignment restricted to $\mathcal{F}_{FIN}$. Let and S_v be a set of all extensions of v, i.e.

$$S_v = \{w : VAR \longrightarrow \{T, F\} : \forall a \in VAR_{FIN}(v(a) = w(a))\}. \tag{8}$$

The following conditions holds.

(0) *There are $2^{|VAR_{FIN}|}$ possible variable assignment restricted to the set $\mathcal{F}_{FIN}$.*

(1) *For any v, there are $card(S_v) = \mathcal{C}$ possible extensions of v to the set VAR of all possible propositional variables.*

(2) *For any v, for any $A \in \mathcal{F}_{FIN}$,*
$v \models A$, *if and only if* $w \models A$, *for any $w \in S_v$.*

(3) *For any v, for any $A \in \mathcal{F}_{FIN}$,*
$v \not\models A$, *if and only if* $w \not\models A$, *for any $w \in S_v$.*

3 Model, Counter-Model, Tautology

A notion of a model is an important, if not the most important notion, of modern logic. It is always defined in terms of the notion of satisfaction. In classical propositional logic, it means in the classical semantics for a classical propositional language these two notions are the same. The use of expressions *" v satisfies A"* and *"v is a model for A"* is interchangeable. This is also a case for some non-classical semantics, like 3-valued semantics discussed in the next chapter, but it is not the same in the intuitionistic semantics, modal semantics. They are not interchangeable for predicate languages semantics. We introduce the notion of a model for classical propositional formulas as a separate definition to stress its importance and its non interchangeable dependence from the notion of satisfaction in predicate logic.

Definition 3.1 (Model) *Given a formula $A \in \mathcal{F}$, a truth assignment $v : VAR \longrightarrow \{T, F\}$,*

$$v \;\; is\ a \;\; model \;\; for \;\; A \quad iff \quad v \models A$$

When $dom(v) = VAR_A$ we call v a **model restricted to** *A.*

Definition 3.2 (Counter- Model) *Given a formula $A \in \mathcal{F}$, a truth assignment $v : VAR \longrightarrow \{T, F\}$ such that falsifies A, i.e.*

$$v \not\models A$$

is called a counter- model for a formula A.

When $dom(v) = VAR_A$ we call v a **counter-model restricted to** *A.*

We often need to find a model, or a set of all models, or a counter-model, or the set of all possible counter-models for a given formula A. In this case we habitually, as we did in the case of truth assignments satisfying a given formula, find a model (counter-model) restricted to A, or list all models (counter-models) restricted to A and we stop at this.

We can proceed like that because we know from theorems 2.3, 2.6 for any restricted model v of A and any of its extensions w (definition 2.5 we have that

$$v \models A \;\; if \;\; and \;\; only \;\; if \;\; w \models A. \tag{9}$$

The same holds for any counter-model restricted to A, and any of its extensions w, i.e. the following holds.

$$v \not\models A \;\; if \;\; and \;\; only \;\; if \;\; w \not\models A. \tag{10}$$

The above properties 9 and 10 justify the following.

Remark 1 *We use, as it is habitually used, the words of* **model, counter-model** *for* **model, counter-model restricted to**.

Example 1

Consider a formula

$$A = ((a \Rightarrow \neg b) \cup \neg c).$$

We read all models and counter-models for A (it means, by Remark 1 all models and counter-models restricted to A) from the table 5.

We say that the set of **all models** for A is

$$\{w_1, w_2, w_3, w_4, w_5, w_6, w_8\},$$

and w_7 is a **counter-model** for A.

Definition 3.3 (Tautology) *For any formula* $A \in \mathcal{F}$,

A is a **tautology** *if and only if all truth assignments*

$$v : VAR \longrightarrow \{T, F\}$$

are models for A i.e.

$$v \models A, \quad for \quad all \quad v.$$

A formula A is **not a tautology** *if and only if A has a counter- model, i.e.*

$$v \not\models A, \quad for \quad some \quad v.$$

Tautology symbol

If a formula A is a tautology so by the definition 3.3 A satisfied by all v. This means that it is independent of v and it justifies a notation

$$\models A, \qquad \not\models A \tag{11}$$

for " A is **a tautology**" and " A is **not a tautology**", respectively.

The theorem 2.3 and properties 9 and 10 prove the following.

Theorem 3.1 (Tautology) *For any formula* $A \in \mathcal{F}$,

$$\models A \quad if \ and \ only \ if$$

$$v \models A \quad for \ all \ v, \ such \ that \quad v : VAR_A \longrightarrow \{T, F\}.$$

The theorem 3.6 is a formal justification of the truth table method of tautology verification. It says that in order to verify whether a given formula A is a tautology it is sufficient to construct a truth table for A, following the pattern established by the table 5, that lists all possible truth assignments restricted to A. If all rows are evaluated to T, the formula A is a tautology, otherwise it is not a tautology.

Example 2

Consider a formula

$$A = (a \Rightarrow (a \cup b)).$$

To verify whether $\models A$ we construct a table of all v, such that $v : VAR_A \longrightarrow \{T, F\}$. If all rows are evaluated to T, the formula A is a tautology, otherwise it is not a tautology.

v	a	b	$v^*(A)$ *computation*	$v^*(A)$
v_1	T	T	$(T \Rightarrow (T \cup T)) = (T \Rightarrow T) = T$	T
v_2	T	F	$(T \Rightarrow (T \cup F)) = (T \Rightarrow T) = T$	T
v_3	F	T	$(F \Rightarrow (F \cup T)) = (F \Rightarrow T) = T$	T
v_4	F	F	$(F \Rightarrow (F \cup F)) = (F \Rightarrow F) = T$	T

All rows are evaluated to T, hence by theorems 3.6 and 2.6 it proves that

$$\models (a \Rightarrow (a \cup b)).$$

The theorem 3.6 provides, of course a method of decision whether formula is not a tautology, as well. We list it as a separate fact as follows.

Theorem 3.2 (Not Tautology) *For any formula $A \in \mathcal{F}$,*

$$\not\models A \quad if\ and\ only\ if$$

$$v \not\models A \quad for\ some\ v,\ such\ that\ \ v: VAR_A \longrightarrow \{T, F\}.$$

Example 3

The formula $A = ((a \Rightarrow \neg b) \cup \neg c)$ is **not a tautology**, i.e.

$$\not\models ((a \Rightarrow \neg b) \cup \neg c)$$

because the truth assignment w_7 of table 5 is a counter-model for A, i.e.

$$w_7 \not\models A.$$

The theorems 3.6, 3.2, and 2.6 prove also that the notion of classical propositional tautology is decidable, i.e. that the following holds.

Theorem 3.3 (Decidability) *For any formula $A \in \mathcal{F}$, one has examine at most*

$$2^{VAR_A}$$

truth assignments $v: VAR_A \longrightarrow \{T, F\}$ in order to decide whether

$$\models A, \quad or \quad \not\models A.$$

I.e. the notion of classical propositional tautology is decidable.

3.1 Tautologies Verification Methods

There are three basic tautologies verification methods: Truth Table Method, Proof by Contradiction Method and Substitution Method presented below, or mixture of the above. We start we the most common one, the truth tables method.

Truth Table Method

The verification method, called **a truth-table method** consists of examination, for any formula A, all possible variable assignments restricted to A, i.e we have to perform at most $2^{card(VAR_A)}$ steps. If we find an assignment which evaluates A to F, we stop the process and give answer: $\not\models A$. Otherwise we continue. If all assignments ($2^{card(VAR_A)}$ of them) evaluate A to T, we give answer: $\models A$.

We usually list all assignments v in a form of a truth table similar to the table 5, hence the name of the method.

The complexity of the truth table methods grows exponentially, too fast even for modern computers to handle formulas with a great number of variables, not to mention humans. In practice, if we need, we use often much shorter methods of verification presented below.

Proof by Contradiction Method

In this method, in order to decide whether $\models A$, $\not\models A$ we work backwards. We try to find a truth assignment v which makes a formula A **false**.

If we find one, it means that A is not a tautology. If we prove that **it is impossible** by getting a contradiction, it means that the formula A is a tautology. Hence the name of the method.

Example 1

Consider

$$A = (a \Rightarrow (a \cup b)).$$

Step 1 Assume that $\not\models A$. It means that there is v, such that $v(A) = F$.

Step 2 Analyze Step 1 (in short-hand notation):

$$(a \Rightarrow (a \cup b)) = F \;\; iff \;\; a = T \;\; and \;\; (a \cup b) = F.$$

Step 3 Analyze Step 2: $a = T$ and $(a \cup b) = F$, means that $(T \cup b) = F$. This is a contradiction with the definition of $\cup$, hence the formula A is **a tautology.**

We write it symbolically:

$$\models \quad (a \Rightarrow (a \cup b)).$$

Substitution Method

Observe that exactly the same reasoning as in the above example proves that for any formulas $A, B \in \mathcal{F}$,

$$\models (A \Rightarrow (A \cup B)),$$

i.e. let's assume $(A \Rightarrow (A \cup B)) = F$. This holds only when $A = T$ and $(A \cup B) = F$, i.e. $(T \cup B) = F$. Contradiction. It proves $\models (A \Rightarrow (A \cup B))$.

Instead of repeating the same argument as with formula $(a \Rightarrow (a \cup b))$ over again, we make a simple observation that we obtain $(A \Rightarrow (A \cup B))$ from $(a \Rightarrow (a \cup b))$ by a substitution (replacement) of A for a and B for b in the formula $C = (a \Rightarrow (a \cup b))$, what we write symbolically as

$$C(a/A, b/B).$$

We are going to prove in theorem 3.4 stated below that such substitutions lead always from a tautology to a tautology, hence we know that $\models (A \Rightarrow (A \cup B))$.

In particular, making substitution in $C = (a \Rightarrow (a \cup b))$ we get new tautologies as follows.

1. By substitution

$$C(a/((a \Rightarrow b) \cap \neg c), b/\neg d)$$

we get that

$$\models ((((a \Rightarrow b) \cap \neg c) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup \neg d)).$$

2. By substitution

$$C(a/((a \Rightarrow b) \cap \neg C), b/((a \Rightarrow \neg e))$$

we get that also

$$\models (((a \Rightarrow b) \cap \neg C) \cup d) \cap \neg e) \Rightarrow (((a \Rightarrow b) \cap \neg C) \cup d) \cap \neg e) \cup ((a \Rightarrow \neg e))).$$

Example 2

Now let's look at this substitution process backward. Assume that we are given the formulas

$$((((a \Rightarrow b) \cap \neg c) \Rightarrow ((((a \Rightarrow b) \cap \neg c) \cup \neg d)),$$

$$(((a \Rightarrow b) \cap \neg C) \cup d) \cap \neg e) \Rightarrow (((a \Rightarrow b) \cap \neg C) \cup d) \cap \neg e) \cup ((a \Rightarrow \neg e)))$$

and a task of finding whether they are, or not, tautologies. We observe that both of them are of the form

$$(A \Rightarrow (A \cup B))$$

and hence substitutions described above of our initial tautology $(a \Rightarrow (a \cup b)$ and this proves that they are tautologies.

This observation and theorem 3.4 saves us from examining $2^4 = 16$ and $2^5 = 36$ cases of combination of logical values of the propositional variables, not mentioning further calculations.

The theorem 3.4 stated and proved below describes validity of a method of constructing new tautologies from given tautologies. First we formally introduce needed notations.

Let $A \in \mathcal{F}$ be a formula and $VAR_A = \{a_1, a_2, ...a_n\}$ be the set of all propositional variables appearing in A. We will denote it by

$$A(a_1, a_2, ...a_n).$$

Let $A_1, ...A_n$ be any formulas. We denote by

$$A(a_1/A_1, ..., a_n/A_n)$$

the result of simultaneous replacement (substitution) in A variables $a_1, a_2, ...a_n$ by formulas $A_1, ...A_n$, respectively.

Theorem 3.4 *For any formulas $A, A_1, ...A_n \in \mathcal{F}$,*

If

$$\models A(a_1, a_2, ...a_n) \quad \textit{and} \quad B = A(a_1/A_1, ..., a_n/A_n),$$

then

$$\models B.$$

Proof. Let $B = A(a_1/A_1, ..., a_n/A_n)$. Let $b_1, b_2, ...b_m$ be all those propositional variables which occur in $A_1, ...A_n$. Given a truth assignment $v : VAR \longrightarrow \{T, F\}$, any values $v(b_1), v(b_2), ...v(b_m)$ defines the logical value of $A_1, ...A_n$, i.e. $v^*(A_1), ...v^*(A_n)$ and, in turn, $v^*(B)$.

Let $w : VAR \longrightarrow \{T, F\}$ be a truth assignment such that $w(a_1) = v^*(A_1), w(a_2) = v^*(A_2), ...w(a_n) = v^*(A_n)$. Obviously, $v^*(B) = w^*(A)$. Since A is a propositional tautology, $w^*(A) = T$, for all possible w, hence $v^*(B) = w^*(A) = T$ for all truth assignments w and B is also a tautology.

Mixed Methods

We may, in some cases, as in example below, apply the substitution method, and then truth tables, or proof by contradiction method.

Example 3

Show that $v \models (\neg((a \cap \neg b) \Rightarrow ((c \Rightarrow (\neg f \cup d)) \cup e)) \Rightarrow ((a \cap \neg b) \cap (\neg(c \Rightarrow (\neg f \cup d)) \cap \neg e)))$, for all $v : VAR \longrightarrow \{T, F\}$, i.e. that

$\models (\neg((a \cap \neg b) \Rightarrow ((c \Rightarrow (\neg f \cup d)) \cup e)) \Rightarrow ((a \cap \neg b) \cap (\neg(c \Rightarrow (\neg f \cup d)) \cap \neg e)))$.

Observe that $VAR_A = \{a, b, c, d, e, f\}$, so there are $2^6 = 64$ truth assignments to consider. Much too many to apply the truth table method.

The "proof by contradiction" method may be shorter, but before we apply it let's look closer at the sub-formulas of A and patterns they form inside the formula A, i.e. we apply the substitution method first.
We denote (substitute):

$$B = (a \cap \neg b),\ C = (c \Rightarrow (\neg f \cup d)),\ D = e.$$

We re-write A as

$$(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))).$$

Now we apply "proof by contradiction" method.

Step 1: Assume $(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))) = F$. It is possible **only** when $(B \Rightarrow (C \cup D)) = F$ and $(B \cap (\neg C \cap \neg D)) = F$.
Step 2: $(B \Rightarrow (C \cup D)) = F$ **only** when

$$B = T, C = F, D = F.$$

Step 3: From **Step 1** we have that

$$(B \cap (\neg C \cap \neg D)) = F.$$

We now evaluate its logical value for $B = T, C = F, D = F$ obtained in **Step 2**, i.e. compute:

$$(T \cap (\neg F \cap \neg F)) = F,$$
$$(T \cap (T \cap T)) = F,$$
$$T = F.$$

Contradiction. This proves that

$$\models (\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))),$$

and hence

$$\models (\neg((a \cup b) \Rightarrow ((c \Rightarrow d) \cup e)) \Rightarrow ((a \cup b) \cap (\neg(c \Rightarrow d) \cap \neg e))).$$

All truth assignments are models for A, i.e. A does not have a counter-model.

3.2 Sets of Tautologies and Contradictions

Definition 3.4 *We define the sets* $\mathbf{T} \subseteq \mathcal{F}$ *of* **all tautologies**, *and the set* $\mathbf{C} \subseteq \mathcal{F}$ *of* **all contradictions** *as follows:*

$\mathbf{T}= \{A \in \mathcal{F} : \ \models A\}$,

$\mathbf{C}= \{A \in \mathcal{F} : \ \forall v \ (v \not\models A)\}$.

Example 1

The following formulas are contradictions i.e. are elements of the set $\mathbf{C}$.

$$(a \cap \neg a), \quad (a \cap \neg(a \cup b)), (a \Rightarrow \neg a).$$

Following the proof of theorem 3.4 we get similar theorem for contradictions, and hence a method of constructing new contradictions from already known ones.

Theorem 3.5 *For any formulas* $A, A_1, ...A_n \in \mathcal{F}$,

If $A(a_1, a_2, ...a_n) \in \mathbf{C}$ *and* $B = A(a_1/A_1, ..., a_n/A_n)$, *then* $B \in \mathbf{C}$.

Observe, that are formulas which neither in $\mathbf{T}$ nor in $\mathbf{C}$, for example $(a \cup b)$. The valuation $v(a) = F, v(b) = F$ falsifies our formula, what proves that it is not a tautology, a valuation $v(a) = T, v(b) = T$ satisfies the formula, what proves that it is not a contradiction.

We put now the facts we have discussed here and we know and about the sets $\mathbf{T}$ and $\mathbf{C}$ in two theorems.

Theorem 3.6 (Tautology) *For any formula* $A \in \mathcal{F}$ *the following conditions are equivalent.*

(1) $A \in \mathbf{T}$

(2) $\neg A \in \mathbf{C}$

(3) *For any* v, $v^*(A) = T$

(4) *For any* v, $v \models A$

(7) *Every* v, *is a model for* A

Theorem 3.7 (Contradiction) *For any formula* $A \in \mathcal{F}$ *the following conditions are equivalent.*

(1) $A \in \mathbf{C}$

(2) $\neg A \in \mathbf{T}$

(3) *For any* v, $v^*(A) = F$

(4) *For any* v, $v \not\models A$

(5) *A does not have a model*

Here is one more general fact about classical tautologies.

Theorem 3.8 *For any formula* $A, B \in \mathcal{F}$*, if* $A \in \mathbf{T}$ *and* $(A \Rightarrow B) \in \mathbf{T}$*, then* $B \in \mathbf{T}$.

Proof. We know that for all v, $v(A) = T$ and $v^*((A \Rightarrow B)) = T$, hence $(T \Rightarrow v^*(B) = T$ for all v. This is is true only when $v^*(B) = T$ for all v, i.e. only when B is a tautology.

Consider $\mathcal{L} = \mathcal{L}_{CON}$ and let $\mathcal{S}$ be a set $\mathcal{S} \subseteq \mathcal{F}$ of formulas of $\mathcal{L}$. We adopt the following definition.

Definition 3.5 *A truth truth assignment*

$$v : VAR \longrightarrow \{T, F\}$$

is a **model for the set** $\mathcal{S}$ *of formulas if and only if*

$$v \models A \quad for \quad all \quad A \in \mathcal{S}.$$

We write

$$v \models \mathcal{S}$$

to denote the v *is a model for the set* $\mathcal{S}$ *of formulas.*

Definition 3.6 *The restriction of the model* v *to the domain*

$$VAR_{\mathcal{S}} = \bigcup_{A \in \mathcal{S}} VAR_A \subseteq VAR$$

is called a **restricted model** *for* $\mathcal{S}$.

Definition 3.7 *A truth assignment*

$$v : VAR \longrightarrow \{T, F\}$$

is a **counter-model for the set** $\mathcal{S}$ *of formulas if and only if*

$$v \models A \quad for \quad some \quad formula \quad A \in \mathcal{S}.$$

We write

$$v \not\models \mathcal{S}$$

to denote the v *is a counter- model for the set* $\mathcal{S}$ *of formulas.*

Definition 3.8 *The restriction of the counter-model* v *to the domain* $VAR_{\mathcal{S}}$ *is called a* **restricted counter-model** *for* $\mathcal{S}$.

Example
Let $\mathcal{L} = \mathcal{L}_{\{\neg,\cap\}}$ and $\mathcal{S} = \{a, (a \cap \neg b), c, \neg b\}$. $VAR_{\mathcal{S}} = \{a, b, c\}$. $v : VAR_{\mathcal{S}} \rightarrow \{T, F\}$ such that $v(a) = T, v(c) = T, v(b) = F\}$ is a restricted model for $\mathcal{S}$ and $v : VAR_{\mathcal{S}} \rightarrow \{T, F\}$ such that $v(a) = F$ is a restricted counter-model for $\mathcal{S}$.

3.3 Exercises and Homework Problems

Exercise 1

(1) Write the following natural language statement

From the fact that it is possible that $2 + 2 \neq 4$ *we deduce that it is not possible that* $2 + 2 \neq 4$ *or, if it is possible that* $2 + 2 \neq 4$, *then it is not necessary that you go to school.*

as a formula
1. $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg,\mathbf{C},\mathbf{I},\cap,\cup,\Rightarrow\}}$,
2. $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg,\cap,\cup,\Rightarrow\}}$.

(2) Can you find a model, or a counter-model for $A_1 \in \mathcal{F}_1$ of $\mathcal{L}_1$?

(3) Find a model v restricted to the formula $A_2 \in \mathcal{F}_2$ of $\mathcal{L}_2$.

(4) Find 3 models w of A_2 of $\mathcal{L}_2$, such that $v^*(A_2) = w^*(A_2)$, for v from **(3)**. How many of such models exist?

(5) Find a counter-model restricted to formula $A_2 \in \mathcal{F}_2$ (if exists).

(6) Find 3 counter-models for A_2.

(7) Find all models, counter-models (restricted) for A_2 of $\mathcal{L}_2$.

(8) Is $A_2 \in$**C**?, is $A_2 \in$**T**?

Solution

(1) We translate our statement into a formula

1. $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a denotes statement $2+2=4$, b denotes a statement *you go to school.*
Propositional Modal Connectives: **C** denotes statement *it is possible that*, **I** denotes statement *it is necessary that.*
Translation:

$$A_1 = (\mathbf{C}\neg a \Rightarrow (\neg \mathbf{C}\neg a \cup (\mathbf{C}\neg a \Rightarrow \neg \mathbf{I} b))).$$

Now we translate our statement into a formula

2. $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$ as follows.

Propositional Variables: a denotes statement *it is possible that* $2+2 \neq 4$, b denotes a statement *it is necessary that you go to school.*

Translation:

$$A_2 = (a \Rightarrow (\neg a \cup (a \Rightarrow \neg b))).$$

(2) Can you find a model, or a counter-model for A_1?
Maybe, but at this stage we don't know yet any modal connectives semantics. Moreover, as we will see later, there are over hundred different semantics for the language $\mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$. In some of them the formula A_1 can have a model, or a counter-model, in some not.

(3) v is a restricted model for A_2 if and only if

$$v : VAR_{A_2} = \{a, b\} \longrightarrow \{T, F\}$$

such that $v^*(a \Rightarrow (\neg a \cup (a \Rightarrow \neg b))) = T$
Observe that if we put $v(a) = F$ then $v^* A_2 = T$ for any value of b. We need one model, so we can choose for example v such that $v(b) = T$. We write is using short-hand notation as follows.

We want to evaluate $A_2 = T$, it means $(a \Rightarrow (\neg a \cup (a \Rightarrow \neg b))) = T$. Let's put $a = F$. Then by definition of implication, $A_2 = T$ for any logical value b. We choose $b = T$ and the model is any v such that $v(a) = F, v(b) = T$, in short-hand $a = F, b = F$.

(4) A model for A_2 by definition, is any function

$$w : VAR \longrightarrow \{T, F\},$$

so to make a restricted model v defined in **(3)** a model, we have to extend it to the set of all propositional variables VAR. Here are three of such extensions.

Model 1:

$$w(a) = v(a) = F,\ \ w(b) = v(b) = T\ \ and\, w(x) = T,\ \ for\ \ all\ \ x \in VAR - \{a, b\}.$$

Model 2:

$$w(a) = v(a) = F,\ w(b) = v(b) = T,\ w(c) = F\ and\ w(x) = T,$$

$$for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

Model 3:

$$w(a) = v(a) = F,\ w(b) = v(b) = T,\ w(c) = T\ and\ w(x) = F,$$

$$for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

There is an many of such models, being extensions of v to the set VAR, as real numbers.

(5) To find a restricted counter- model for A_2 we must evaluate it to F. We write it in a short hand notation

$$(a \Rightarrow (\neg a \cup (a \Rightarrow \neg b))) = F.$$

This is possible only when $a = T$ and $(\neg a \cup (a \Rightarrow \neg b)) = F$, i.e. $(F \cup (T \Rightarrow \neg b)) = F$. This happens only when $(T \Rightarrow \neg b)) = F$, i.e. only when $b = F$. The mapping

$$v : \{a, b\} \longrightarrow \{T, F\}$$

such that $v(a) = T$ and $v(b) = F$ is a truth assignment restricted to A_2 and $v^*(A_2) = F$, i.e. v is a restricted counter- model for A_2.

(6) Observe, that v defined in **(5)** is the only restricted counter-model for A_2. All other counter -models must be extensions of it. For example we list three of them below.

Counter- model 1 :

$$w(a) = v(a) = T,\ \ w(b) = v(b) = F\ \ and\, w(x) = T,\ \ for\ \ all\ \ x \in VAR - \{a, b\}.$$

Counter- model 2 :

$$w(a) = v(a) = T,\ w(b) = v(b) = F,\ w(c) = F$$

$$and\ w(x) = T,\ for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

Counter- model 3 :

$$w(a) = v(a) = T,\ w(b) = v(b) = F,\ w(c) = T$$

$$and\ w(x) = F,\ for\ \ all\ \ x \in VAR - \{a, b, c\}.$$

There is as many of counter-models as there are extensions of v to the set VAR, i.e. as many as real numbers.

(7) To find all models or counter-models for A_2 we have only to find all restricted models or counter-models , as all others are always their extensions.

We have just shown in **(6)** that there is only one restricted counter-model (with all its extensions) for A. So there is $2^2 - 1 = 3$ restricted models for A_2.

We have also shown in **(3), (4)** that any v

$$v : VAR \longrightarrow \{T, F\}$$

such that

$$v(a) = F, v(b) = T$$

is a model for A_2. Other two other models:

$$v(a) = F\ \ and\ \ v(b) = F,$$

$$v(a) = T\ \ and\ \ v(b) = F.$$

(8) $A_2 \notin \mathbf{C}$ because A_2 has a model. $A_2 \notin \mathbf{T}$ because A_2 has a counter-model.

Exercise 2

Consider a formula

$$A = (\neg((a \cup b) \Rightarrow ((c \Rightarrow d) \cup e)) \Rightarrow ((a \cup b) \cap (\neg(c \Rightarrow d) \cap \neg e))).$$

Find all models, counter-models for A (if exists). Determine whether

$$\models A.$$

Solution

Observe that $VAR_A = \{a, b, c, d, e\}$, so there are $2^5 = 32$ truth assignments to consider. Much too much to use the truth table method.

The "proof by contradiction" method may be shorter, but before we apply it let's look closer at the sub-formulas of A and patterns they form inside the formula A.

We apply first the "substitution method". I.e. we denote : $B = (a \cup b), C = (c \Rightarrow d)$, and $D = e$. We re-write A as

$$(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))).$$

Now we apply "proof by contradiction" method.

Step 1: Assume $(\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))) = F$. It is possible **only** when $(B \Rightarrow (C \cup D)) = F$ and $(B \cap (\neg C \cap \neg D)) = F$.
Step 2: $(B \Rightarrow (C \cup D)) = F$ **only** when

$$B = T, C = F, D = F.$$

Step 3: From **Step 1** we have that

$$(B \cap (\neg C \cap \neg D)) = F.$$

We now evaluate its logical value for $B = T, C = F, D = F$ obtained in **Step 2**, i.e. compute:

$$(T \cap (\neg F \cap \neg F)) = F,$$

$$(T \cap (T \cap T)) = F,$$

$$T = F.$$

Contradiction. This proves that

$$\models (\neg(B \Rightarrow (C \cup D)) \Rightarrow (B \cap (\neg C \cap \neg D))),$$

and hence

$$\models (\neg((a \cup b) \Rightarrow ((c \Rightarrow d) \cup e)) \Rightarrow ((a \cup b) \cap (\neg(c \Rightarrow d) \cap \neg e))).$$

All truth assignments are models for A, i.e. A does not have a counter-model.

Exercise 3

(1) Write the following natural language statement

If it is not believed that quiz is easy or quiz is not easy, then from the fact that $2 + 2 = 5$ *we deduce that it is believed that quiz is easy.*

as a formula

Formula 1 $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \mathbf{B}, \cap, \cup, \Rightarrow\}}$, where **B** is a believe connective. Statement **B**A says: *It is believed that* A.

Formula 2 $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

2. Degree of the formula A_1 is: , degree of the formula A_2 is:

3. List all proper sub-formulas of A_1.

4. List all non-atomic sub-formulas of A_2.

5. Find all counter-models (restricted) for A_2. Use short-hand notation. Don't construct Truth Tables! Explain.

6. Find a restricted model for A_2. Use short-hand notation. Don't construct Truth Tables! Explain your solution.

7. How many are there possible restricted models for A_2? Don't need to list them, just justify your answer.

8. List 2 models (not restricted) for A_2 by extending the model you have found in **6.** to the VAR of all variables.

9. How many are there possible models for A_2?
How many are there possible counter-models for A_2?

HOMEWORK PROBLEMS

Problem 1

(1) Write the following natural language statement

It is believed that yellow flowers are blue, or from the fact that $2 + 2 = 5$ *we deduce that it is not believed that it is not true that yellow flowers are blue.*

as a formula

1. $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \mathbf{B}, \cap, \cup, \Rightarrow\}}$, where **B** is a believe connective. Statement **B**A says: *It is believed that* A.
2. $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

(2) Can you find a model, or a counter-model for $A_1 \in \mathcal{F}_1$ of $\mathcal{L}_1$?

(3) Find a model v restricted to the formula $A_2 \in \mathcal{F}_2$ of $\mathcal{L}_2$.

(4) Find all counter-model restricted to formula $A_2 \in \mathcal{F}_2$ (if exists).

(5) How are there many models restricted to A?

(6) Find 2 extensions w of v from **(3)**, i.e. two models w of A_2 of $\mathcal{L}_2$, such that $v^*(A_2) = w^*(A_2)$, for v from **(3)**.
How many of such models exist?

(7) Find 3 counter-models w for A_2 different from counter models found in **(4)**.

Problem 2

(1) Write the following natural language statement

From the the fact that it is possible that both $2 + 2 = 5$ *and it is not necessary that* $1 + 3 \neq 5$*, we deduce that* $2 + 2 \neq 5$ *or possibly* $1 + 3 = 5$.

as a formula

1. $A_1 \in \mathcal{F}_1$ of a language $\mathcal{L}_1 = \mathcal{L}_{\{\neg, \mathbf{C}, \mathbf{I}, \cap, \cup, \Rightarrow\}}$,

2. $A_2 \in \mathcal{F}_2$ of a language $\mathcal{L}_2 = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$.

(2) Can you find a model, or a counter-model for $A_1 \in \mathcal{F}_1$ of $\mathcal{L}_1$?

(3) Find a model v restricted to the formula $A_2 \in \mathcal{F}_2$ of $\mathcal{L}_2$.

(4) Find all counter-model restricted to formula $A_2 \in \mathcal{F}_2$ (if exists).

(5) How are there many models restricted to A?

(6) Find 2 extensions w of v from **(3)**, i.e. two models w of A_2 of $\mathcal{L}_2$, such that $v^*(A_2) = w^*(A_2)$, for v from **(3)**.
How many of such models exist?

(7) Find 3 counter-models w for A_2 different from counter models found in **(4)**.

Problem 3

Find all models and a counter-model restricted to $\mathcal{S}$ (if exist) for the following sets $\mathcal{S}$ of formulas. Use shorthand notation.

(1) $\mathcal{S}_1 = \{a, (a \cap \neg b), (\neg a \Rightarrow (a \cup b))\}$

(2) $\mathcal{S}_2 = \{(a \Rightarrow b), (c \cap \neg a), b\}$

(3) $\mathcal{S}_3 = \{a, (a \cap \neg b), \neg a, c\}$

Problem 4

For the formulas listed below determine whether they are tautologies or not.
If a formula is not a tautology list its counter-model (restricted).
Use shorthand notation.

(1) $A_1 = (\neg(a \Rightarrow (b \cap \neg c)) \Rightarrow (a \cap \neg(b \cap \neg c)))$

(2) $A_2 = ((a \cap \neg b) \Rightarrow ((c \cap \neg d) \Rightarrow (a \cap \neg b)))$

(3) $A_2 = (\neg(A \cap \neg B) \cup (A \cap \neg B))$

Problem 5

Given $v : VAR \longrightarrow \{T, F\}$ such that $v^*((\neg a \cup b) \Rightarrow (a \Rightarrow \neg c)) = F$
under classical semantics. Evaluate: $v^*(((b \Rightarrow a) \Rightarrow (a \Rightarrow \neg c)) \cup (a \Rightarrow b))$.