

## CHAPTER 8

### Hilbert Proof Systems, Formal Proofs, Deduction Theorem

The Hilbert proof systems are systems based on a language with implication and contain a Modus Ponens rule as a rule of inference. They are usually called Hilbert style formalizations. We will call them here Hilbert style proof systems, or Hilbert systems, for short.

Modus Ponens is probably the oldest of all known rules of inference as it was already known to the Stoics (3rd century B.C.). It is also considered as the most "natural" to our intuitive thinking and the proof systems containing it as the inference rule play a special role in logic. The Hilbert proof systems put major emphasis on logical axioms, keeping the rules of inference to minimum, often in propositional case, admitting only Modus Ponens, as the sole inference rule.

#### 1 Hilbert System $H_1$

Hilbert proof system  $H_1$  is a simple proof system based on a language with implication as the only connective, with two axioms (axiom schemas) which characterize the implication, and with Modus Ponens as a sole rule of inference.

We define  $H_1$  as follows.

$$H_1 = ( \mathcal{L}_{\{\Rightarrow\}}, \mathcal{F} \{A1, A2\} MP ) \quad (1)$$

where  $A1, A2$  are axioms of the system, MP is its rule of inference, called Modus Ponens, defined as follows:

$$\mathbf{A1} \quad (A \Rightarrow (B \Rightarrow A)),$$

$$\mathbf{A2} \quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$$

**MP**

$$(MP) \frac{A ; (A \Rightarrow B)}{B},$$

and  $A, B, C$  are any formulas of the propositional language  $\mathcal{L}_{\{\Rightarrow\}}$ .

Finding formal proofs in this system requires some ingenuity. Let's construct, as an example, the formal proof of such a simple formula as  $A \Rightarrow A$ .

### Example 1

The formal proof of  $(A \Rightarrow A)$  in  $H_1$  is a sequence

$$B_1, B_2, B_3, B_4, B_5 \tag{2}$$

as defined below.

$$B_1 = ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A))),$$

axiom A2 for  $A = A$ ,  $B = (A \Rightarrow A)$ , and  $C = A$

$$B_2 = (A \Rightarrow ((A \Rightarrow A) \Rightarrow A)),$$

axiom A1 for  $A = A$ ,  $B = (A \Rightarrow A)$

$$B_3 = ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)),$$

MP application to  $B_1$  and  $B_2$

$$B_4 = (A \Rightarrow (A \Rightarrow A)),$$

axiom A1 for  $A = A$ ,  $B = A$

$$B_5 = (A \Rightarrow A)$$

MP application to  $B_3$  and  $B_4$

We have hence proved the following.

**Lemma 1.1** *For any  $A \in \mathcal{F}$ ,*

$$\vdash_{H_1}(A \Rightarrow A)$$

*and the sequence 2 constitutes its formal proof.*

It is easy to see that the above proof wasn't constructed automatically. The main step in its construction was the choice of a proper form (substitution) of logical axioms to start with, and to continue the proof with. This choice is far from obvious for un-experienced prover and impossible for a machine, as the number of possible substitutions is infinite.

**Observe** that the systems  $S_1 - S_4$  from the previous chapter were syntactically decidable for one simple reason. Their inference rules were such that it was

possible to "reverse" their use; to use them in the reverse manner in order to search for proofs, and we were able to do so in a blind, fully automatic way. We were able to conduct an argument of the type: *if this formula has a proof the only way to construct it is from such and such formulas by the means of one of the inference rules, and that formula can be found automatically.*

We will see now, that one can't apply the above argument to the proof search in Hilbert proof systems, which contain Modus Ponens as an inference rule.

**A general procedure** for searching for proofs in a proof system  $S$  can be stated as follows. Given an expression  $B$  of the system  $S$ . If it has a proof, it must be conclusion of the inference rule. Let's say it is a rule  $r$ . We find its premisses, with  $B$  being the conclusion, i.e. we evaluate  $r^{-1}(B)$ . If all premisses are axioms, the proof is found. Otherwise we repeat the procedure for any non-axiom premiss.

**Search for proof in Hilbert Systems** must involve the Modus Ponens. The rule says: given two formulas  $A$  and  $(A \Rightarrow B)$  we can conclude a formula  $B$ .

Assume now that we have a formula  $B$  and want to find its proof. If it is an axiom, we have the proof: the formula itself. If it is not an axiom, it had to be obtained by the application of the Modus Ponens rule, to certain two formulas  $A$  and  $(A \Rightarrow B)$ . But there is infinitely many of formulas  $A$  and  $(A \Rightarrow B)$ . I.e. for any  $B$ , the inverse image of  $B$  under the rule  $MP$ ,  $MP^{-1}(B)$  is countably infinite.

Obviously, we have the following.

**Fact 1.1** *Any Hilbert proof system is not syntactically decidable, in particular, the system  $H_1$  is not syntactically decidable.*

**Semantic Link 1** System  $H_1$  is obviously sound under classical semantics and is sound under **L**, **H** semantics and not sound under **K** semantics.

We leave the proof of the following theorem (by induction with respect of the length of the formal proof) as an easy exercise to the reader.

**Theorem 1.1 (Soundness of  $H_1$ )** *For any  $A \in \mathcal{F}$  of  $H_1$ ,*

$$\text{If } \vdash_{H_1} A, \text{ then } \models A.$$

**Semantic Link 2** The system  $H_1$  is **not complete** under classical semantics.

It means that not all classical tautologies have a proof in  $H_1$ . We have proved that one needs negation and one of other connectives  $\cup, \cap, \Rightarrow$  to express all classical connectives, and hence all classical tautologies. Our language contains only implication and one can't express negation in terms of implication and hence we can't provide a proof of any tautology i.e. its logically equivalent form in our language.

We have constructed a formal proof 2 of  $(A \Rightarrow A)$  in  $H_1$  on a base of logical axioms, as an example of complexity of finding proofs in Hilbert systems.

In order to make the construction of formal proofs easier by the use of previously proved formulas we use the notions of a formal proof from some hypotheses  $\Gamma$  (and logical axioms), as defined in chapter 7. Here is a simple example.

**Example 2**

Construct a proof of  $(A \Rightarrow C)$  from hypotheses  $\Gamma = \{(A \Rightarrow B), (B \Rightarrow C)\}$ . I.e. show that

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

**The formal proof** is a sequence

$$B_1, B_2, \dots, B_7 \tag{3}$$

such that

$$B_1 = (B \Rightarrow C),$$

hypothesis

$$B_2 = (A \Rightarrow B),$$

hypothesis

$$B_3 = ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C))),$$

axiom A1 for  $A = (B \Rightarrow C), B = A$

$$B_4 = (A \Rightarrow (B \Rightarrow C))$$

$B_1, B_3$  and MP

$$B_5 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$$

axiom A2

$$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)),$$

$B_5$  and  $B_4$  and MP

$$B_7 = (A \Rightarrow C)$$

$B_2$  and  $B_6$  and MP

**Example 2**

Show, by constructing a formal proof that

$$A \vdash_{H_1} (A \Rightarrow A)$$

**The formal proof** is a sequence

$$B_1, B_2, B_3 \tag{4}$$

such that

$$B_1 = A,$$

hypothesis

$$B_2 = (A \Rightarrow (A \Rightarrow A)),$$

Axiom **A1** for  $B = A$ ,

$$B_3 = (A \Rightarrow A)$$

$B_1, B_2$  and MP.

We can even further simplify the task of constructing formal proofs by the use of the Deduction Theorem, which is presented and proved in the next section.

## 2 Deduction Theorem

In mathematical arguments, one often assumes a statement  $B$  on the assumption (hypothesis) of some other statement  $A$  and then concludes that we have proved the implication "if  $A$ , then  $B$ ". This reasoning is justified by the following theorem, called a Deduction Theorem. It was first formulated and proved for a proof system for the classical propositional logic by Herbrand in 1930.

**Theorem 2.1 (Herbrand,1930)**    *For any formulas  $A, B$ ,*

$$\text{if } A \vdash B, \text{ then } \vdash (A \Rightarrow B).$$

We are going to prove now that for our system  $H_1$  is strong enough to prove the Deduction Theorem for it. In fact we prove a more general version of Herbrand

theorem. To formulate it we introduce the following notation. We write  $\Gamma, A \vdash B$  for  $\Gamma \cup \{A\} \vdash B$ , and in general we write  $\Gamma, A_1, A_2, \dots, A_n \vdash B$  for  $\Gamma \cup \{A_1, A_2, \dots, A_n\} \vdash B$ .

**Theorem 2.2 (Deduction Theorem for  $H_1$ )** For any subset  $\Gamma$  of the set of formulas  $\mathcal{F}$  of  $H_1$  and for any formulas  $A, B \in \mathcal{F}$ ,

$$\Gamma, A \vdash_{H_1} B \text{ if and only if } \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_1} B \text{ if and only if } \vdash_{H_1} (A \Rightarrow B).$$

**Proof.** We use in the proof the symbol  $\vdash$  instead of  $\vdash_{H_1}$ .

Assume that  $\Gamma, A \vdash B$ , i.e. that we have a formal proof

$$B_1, B_2, \dots, B_n \tag{5}$$

of  $B$  from the set of formulas  $\Gamma \cup \{A\}$ . In order to prove that  $\Gamma \vdash (A \Rightarrow B)$  we will prove a little bit stronger statement, namely that  $\Gamma \vdash (A \Rightarrow B_i)$  for any  $B_i$  ( $1 \leq i \leq n$ ) in the formal proof 5 of  $B$ . And hence, in particular case, when  $i = n$ , we will obtain that also  $\Gamma \vdash (A \Rightarrow B)$ .

The proof is conducted by induction on  $i$  ( $1 \leq i \leq n$ ).

**Step  $i = 1$ .** When  $i = 1$ , it means that the formal proof 5 contains only one element  $B_1$ . By the definition of the formal proof from  $\Gamma \cup \{A\}$ , we have that  $B_1$  must be an logical axiom, or in  $\Gamma$ , or  $B_1 = A$ , i.e.  $B_1 \in \{A_1, A_2\} \cup \Gamma \cup \{A\}$ . Here we have two cases.

**Case 1:**  $B_1 \in \{A_1, A_2\} \cup \Gamma$ . Observe that  $(B_1 \Rightarrow (A \Rightarrow B_1))$  is the axiom  $A1$  and by assumption  $B_1 \in \{A_1, A_2\} \cup \Gamma$ , hence we get the required proof of  $(A \Rightarrow B_1)$  from  $\Gamma$  by the following application of the Modus Ponens rule

$$(MP) \frac{B_1 ; (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

**Case 2:**  $B_1 = A$ . When  $B_1 = A$ , then to prove  $\Gamma \vdash (A \Rightarrow B)$  means to prove  $\Gamma \vdash (A \Rightarrow A)$ , what holds by the monotonicity of the consequence and the fact that we have shown that  $\vdash(A \Rightarrow A)$ .

The above cases conclude the proof of  $\Gamma \vdash (A \Rightarrow B_i)$  for  $i = 1$ .

**Inductive step.** Assume that  $\Gamma \vdash (A \Rightarrow B_k)$  for all  $k < i$ , we will show that using this fact we can conclude that also  $\Gamma \vdash (A \Rightarrow B_i)$ .

Consider a formula  $B_i$  in the sequence 5. By the definition,  $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$  or  $B_i$  follows by MP from certain  $B_j, B_m$  such that  $j < m < i$ . We have to consider again two cases.

**Case 1:**  $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$ . The proof of  $(A \Rightarrow B_i)$  from  $\Gamma$  in this case is obtained from the proof of the **Step**  $i = 1$  by replacement  $B_1$  by  $B_i$  and will be omitted here as a straightforward repetition.

**Case 2:**  $B_i$  is a conclusion of MP. If  $B_i$  is a conclusion of MP, then we must have two formulas  $B_j, B_m$  in the sequence 5 such that  $j < m < i$  and  $(MP) \frac{B_j; B_m}{B_i}$ . By the inductive assumption, the formulas  $B_j, B_m$  are such that

$$\Gamma \vdash (A \Rightarrow B_j) \quad (6)$$

and

$$\Gamma \vdash (A \Rightarrow B_m). \quad (7)$$

Moreover, by the definition of the Modus Ponens rule, the formula  $B_m$  has to have a form  $(B_j \Rightarrow B_i)$ , i.e.  $B_m = (B_j \Rightarrow B_i)$ , and the inductive assumption 7 can be re-written as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i)), \quad \text{for } j < i. \quad (8)$$

Observe now that the formula  $((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$  is a substitution of the axiom schema A2 and hence has a proof in our system. By the monotonicity of the consequence, it also has a proof from the set  $\Gamma$ , i.e.

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))). \quad (9)$$

Applying the rule MP to formulas 9 and 8, i.e. performing the following

$$(MP) \frac{(A \Rightarrow (B_j \Rightarrow B_i)) ; ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{(A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)). \quad (10)$$

Applying again the rule MP to formulas 6 and 10, i.e. performing the following

$$(MP) \frac{(A \Rightarrow B_j) ; ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)}$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what ends the proof of the inductive step. By the mathematical induction principle, we hence have proved that  $\Gamma \vdash (A \Rightarrow B_j)$  for all  $i$  such that  $1 \leq i \leq n$ . In particular it is true for  $i = n$ , what means for  $B_n = B$ . This ends the proof of the fact that if  $\Gamma, A \vdash B$ , then  $\Gamma \vdash (A \Rightarrow B)$ .

The proof of the inverse implication is straightforward. Assume that  $\Gamma \vdash (A \Rightarrow B)$ , hence by the monotonicity of the consequence we have also that  $\Gamma, A \vdash (A \Rightarrow B)$ . Obviously,  $\Gamma, A \vdash A$ . Applying Modus Ponens to the above, we get the proof of  $B$  from  $\{\Gamma, A\}$  i.e. we have proved that  $\Gamma, A \vdash B$ . That ends the proof of the deduction theorem for any set  $\Gamma \subseteq \mathcal{F}$  and any formulas  $A, B \in \mathcal{F}$ . The particular case is obtained from the above by assuming that the set  $\Gamma$  is empty.

The proof of the following Lemma provides a good example of multiple applications of Deduction Theorem.

**Lemma 2.1** *For any  $A, B, C \in \mathcal{F}$ ,*

- (a)  $(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$ ,
- (b)  $(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$ .

**Proof** of (a).

Deduction theorem says:

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C) \text{ if and only if } (A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C.$$

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5$$

of  $(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C$  as follows.

$$B_1 = (A \Rightarrow B),$$

hypothesis

$B_2 = (B \Rightarrow C)$ ,  
hypothesis

$B_3 = A$   
hypothesis

$B_4 = B$   
 $B_1, B_3$  and MP

$B_5 = C$   
 $B_2, B_4$  and MP

Thus

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C)$$

by Deduction Theorem.

**Proof of (b).**

By Deduction Theorem,

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C)) \text{ if and only if } (A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C).$$

We construct a formal proof

$$B_1, B_2, B_3, B_4, B_5, B_6, B_7$$

of  $(A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C)$ . as follows.

$B_1 = (A \Rightarrow (B \Rightarrow C))$   
hypothesis

$B_2 = B$   
hypothesis

$B_3 = ((B \Rightarrow (A \Rightarrow B)))$   
A1 for  $A = B, B = A$

$B_4 = (A \Rightarrow B)$   
 $B_2, B_3$  and MP

$B_5 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$   
*axiomA2*

$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$   
 $B_1, B_5$  and MP

$$B_7 = (A \Rightarrow C)$$

Thus

$$(A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} (B \Rightarrow (A \Rightarrow C))$$

by Deduction Theorem.

### 3 Hilbert System $H_2$

The system  $H_1$  presented in the previous section is sound and strong enough to prove the Deduction Theorem for it, but it is not complete.

We extend now its set of logical axioms to a *complete set of axioms*, i.e. we define a system  $H_2$  that is *complete* with respect to classical semantics. The proof of completeness will be presented in the next chapter.

$H_2$  is the following proof system:

$$H_2 = ( \mathcal{L}_{\{\Rightarrow, \neg\}}, A1, A2, A3, MP ) \quad (11)$$

where  $A1, A2, A3$  are axioms of the system defined below, MP is its rule of inference, called Modus Ponens is called a Hilbert proof system for the classical propositional logic. The axioms  $A1 - A3$  are defined as follows.

$$\mathbf{A1} \quad (A \Rightarrow (B \Rightarrow A)),$$

$$\mathbf{A2} \quad ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))),$$

$$\mathbf{A3} \quad ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

**MP** (Rule of inference)

$$(MP) \frac{A ; (A \Rightarrow B)}{B},$$

and  $A, B, C$  are any formulas of the propositional language  $\mathcal{L}_{\{\Rightarrow, \neg\}}$ .

We write, as before

$$\vdash_{H_2} A$$

to denote that a formula  $A$  has a formal proof in  $H_2$  (from the set of logical axioms  $A_1, A_2, A_3$ ). We write

$$\Gamma \vdash_{H_2} A$$

to denote that a formula  $A$  has a formal proof in  $H_2$  from a set of formulas  $\Gamma$  (and the set of logical axioms  $A_1, A_2, A_3$ ).

**Observe** that system  $H_2$  was obtained by adding axiom  $A_3$  to the system  $H_1$ . Hence the Deduction Theorem holds for system  $H_2$ .

**Theorem 3.1 (Deduction Theorem for  $H_2$ )** *For any subset  $\Gamma$  of the set of formulas  $\mathcal{F}$  of  $H_2$  and for any formulas  $A, B \in \mathcal{F}$ ,*

$$\Gamma, A \vdash_{H_2} B \text{ if and only if } \Gamma \vdash_{H_2} (A \Rightarrow B).$$

*In particular,*

$$A \vdash_{H_2} B \text{ if and only if } \vdash_{H_2} (A \Rightarrow B).$$

Obviously, the axioms  $A_1, A_2, A_3$  are tautologies, and the Modus Ponens rule leads from tautologies to tautologies, hence our proof system  $H_2$  is *sound* i.e. the following holds.

**Theorem 3.2 (Soundness Theorem for  $H_2$ )** *For every formula  $A \in \mathcal{F}$ ,*

$$\text{if } \vdash_{H_2} A, \text{ then } \models A.$$

The soundness theorem proves that the system "produces" only tautologies. We show, in the next chapter, that our proof system  $H_2$  "produces" not only tautologies, but that all tautologies are provable in it. This is called a *completeness theorem for classical logic*.

**Theorem 3.3 (Completeness Theorem for  $H_2$ )** *For every  $A \in \mathcal{F}$ ,*

$$\vdash_{H_2} A, \text{ if and only if } \models A.$$

The proof of completeness theorem (for a given semantics) is always a main point in any logic creation. There are many ways (techniques) to prove it, depending on the proof system, and on the semantics we define for it.

We present in the next chapter two proofs of the completeness theorem for our system  $H_2$ . The proofs use very different techniques, hence the reason of presenting both of them. In fact the proofs are valid for any proof system for classical propositional logic in which one can prove all formulas proved in the next section and stated in lemma 4.1.

## 4 Formal Proofs in $H_2$

We present here some examples of formal proofs in  $H_2$ . There are two reasons for presenting them. First reason is that all formulas we prove here to be provable play a crucial role in the proof of Completeness Theorem for  $H_2$ , or are needed to find formal proofs of those needed. The second reason is that they provide a "training" ground for a reader to learn how to develop formal proofs. For this second reason we write some proofs in a full detail and we leave some others for the reader to complete in a way explained in the following example.

We write  $\vdash$  instead of  $\vdash_{H_2}$  for the sake of simplicity.

### Example 1

Here are consecutive steps

$$B_1, \dots, B_5, B_6 \tag{12}$$

of the proof (in  $H_2$ ) of  $(\neg\neg B \Rightarrow B)$ .

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

$$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$$B_3 = (\neg B \Rightarrow \neg B)$$

$$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

$$B_6 = (\neg\neg B \Rightarrow B)$$

### Exercise 1

Complete the proof 12 by providing comments how each step of the proof was obtained.

### Solution

**The comments** that complete the proof are as follows.

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

Axiom A3 for  $A = \neg B, B = B$

$$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$B_1$  and lemma 2.1 **b** for  $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$ , i.e.

$$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$$B_3 = (\neg B \Rightarrow \neg B)$$

Lemma 1.1 for  $A = \neg B$

$$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$B_2, B_3$  and MP

$$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

Axiom A1 for  $A = \neg\neg B, B = \neg B$

$$B_6 = (\neg\neg B \Rightarrow B)$$

$B_4, B_5$  and Lemma 2.1 **a** for  $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$ ; i.e.

$$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)), ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \vdash (\neg\neg B \Rightarrow B).$$

### General Remark

In step  $B_2, B_3, B_5, B_6$  we call previously proved facts and use their results as a part of our proof. We can insert previously constructed formal proofs into our formal proof. For example we adopt previously constructed proof 2 of  $(A \Rightarrow A)$  in  $H_1$  to the proof of  $(\neg B \Rightarrow \neg B)$  in  $H_2$  by replacing  $A$  by  $\neg B$  and we insert the proof of  $(\neg B \Rightarrow \neg B)$  after  $B_2$ . The "old" step  $B_3$  becomes now  $B_7$ , the "old" step  $B_4$  becomes now  $B_8$ , etc.....

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

Axiom A3 for  $A = \neg B, B = B$

$$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$B_1$  and lemma 2.1 **b** for  $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$ ,

$$B_3 = ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B))),$$

axiom A2 for  $A = \neg B, B = (\neg B \Rightarrow \neg B)$ , and  $C = \neg B$

$$B_4 = (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)),$$

axiom A1 for  $A = \neg B, B = (\neg B \Rightarrow \neg B)$

$$B_5 = ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)),$$

MP application to  $B_4$  and  $B_3$

$$B_6 = (\neg B \Rightarrow (\neg B \Rightarrow \neg B)),$$

axiom A1 for  $A = \neg B, B = \neg B$

$$B_7 = (\text{"old" } B_3)(\neg B \Rightarrow \neg B)$$

MP application to  $B_5$  and  $B_4$

$$B_8 = (\text{"old" } B_4)((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$B_2, B_3$  and MP

$$B_9 = (\text{"old" } B_5) (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

Axiom A1 for  $A = \neg\neg B, B = \neg B$

$$B_{10} = (\text{"old" } B_6) (\neg\neg B \Rightarrow B)$$

$B_8, B_9$  and Lemma 2.1 **a** for  $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$

We repeat our procedure by replacing the step  $B_2$  by its formal proof as defined in the proof of the lemma 2.1 **b**, and continue the process for all other steps which involved application of lemma 2.1 until we get a full formal proof from the axioms of  $H_2$  only.

Usually we don't need to do it, but it is important to remember that it always can be done, if we wished to take time and space to do so.

### Example 2

Here are consecutive steps

$$B_1, \dots, B_5 \tag{13}$$

in a proof of

$$(B \Rightarrow \neg\neg B).$$

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

$$B_5 = (B \Rightarrow \neg\neg B)$$

### Exercise 2

Complete the proof sequence 13 by providing comments how each step of the proof was obtained.

### Solution

The comments that complete the proof 13 are as follows.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

Axiom A3 for  $A = B, B = \neg\neg B$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

Example 1 for  $B = \neg B$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$B_1, B_2$  and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B); ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

Axiom A1 for  $A = B, B = \neg\neg\neg B$

$$B_5 = (B \Rightarrow \neg\neg B)$$

$B_3, B_4$  and lemma 2.1a for  $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$ , i.e.

$$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$$

### Example 3

Here are consecutive steps

$$B_1, \dots, B_{12} \tag{14}$$

in a proof of

$$(\neg A \Rightarrow (A \Rightarrow B)).$$

$$B_1 = \neg A$$

$$B_2 = A$$

$$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$$

$$B_4 = (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_5 = (\neg B \Rightarrow A)$$

$$B_6 = (\neg B \Rightarrow \neg A)$$

$$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

$$B_8 = ((\neg B \Rightarrow A) \Rightarrow B)$$

$$B_9 = B$$

$$B_{10} = \neg A, A \vdash B$$

$$B_{11} = \neg A \vdash (A \Rightarrow B)$$

$$B_{12} = (\neg A \Rightarrow (A \Rightarrow B))$$

### Exercise 3

(1) Complete the proof sequence 14 by providing comments how each step of the proof was obtained.

(2) Prove that  $\neg A, A \vdash B$ .

### Example 4

Here are consecutive steps

$$B_1, \dots, B_7 \tag{15}$$

in a proof of

$$((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B)).$$

$$B_1 = (\neg B \Rightarrow \neg A)$$

$$B_2 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$$

$$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$$

$$B_4 = ((\neg B \Rightarrow A) \Rightarrow B)$$

$$B_5 = (A \Rightarrow B)$$

$$B_6 = (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$$

$$B_6 = ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$$

### Exercise 4

Complete the proof sequence 15 by providing comments how each step of the proof was obtained.

### Example 5

Here are consecutive steps

$$B_1, \dots, B_9 \tag{16}$$

in a proof of

$$((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)).$$

$$B_1 = (A \Rightarrow B)$$

$$B_2 = (\neg\neg A \Rightarrow A)$$

$$B_3 = (\neg\neg A \Rightarrow B)$$

$$B_4 = (B \Rightarrow \neg\neg B)$$

$$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$$

$$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_7 = (\neg B \Rightarrow \neg A)$$

$$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$$

$$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

### Exercise 5

Complete the proof sequence 16 by providing comments how each step of the proof was obtained.

### Solution

$$B_1 = (A \Rightarrow B)$$

*Hypothesis*

$$B_2 = (\neg\neg A \Rightarrow A)$$

Example 1 for  $B = A$

$$B_3 = (\neg\neg A \Rightarrow B)$$

Lemma 2.1 **a** for  $A = \neg\neg A, B = A, C = B$

$$B_4 = (B \Rightarrow \neg\neg B)$$

Example 2

$$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$$

Lemma 2.1 **a** for  $A = \neg\neg A, B = B, C = \neg\neg B$

$$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

Example 4 for  $B = \neg A, A = \neg B$

$$B_7 = (\neg B \Rightarrow \neg A) \\ B_5, B_6 \text{ and MP}$$

$$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A) \\ B_1 - B_7$$

$$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A)) \\ \text{Deduction Theorem}$$

### Exercise 6

Prove that

$$\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))).$$

**Solution** Here are consecutive steps of building the formal proof.

$$B_1 = A, (A \Rightarrow B) \vdash B \\ \text{by MP}$$

$$B_2 = A \vdash ((A \Rightarrow B) \Rightarrow B) \\ \text{Deduction Theorem}$$

$$B_3 = \vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B)) \\ \text{Deduction Theorem}$$

$$B_4 = \vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B))) \\ \text{Example 5 for } A = (A \Rightarrow B), B = B$$

$$B_5 = \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))) \\ 3. \text{ and 4. and lemma 2a for } A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B)))$$

### Example 7

Here are consecutive steps

$$B_1, \dots, B_{12} \tag{17}$$

in a proof of

$$((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)).$$

$$B_1 = (A \Rightarrow B)$$

$$B_2 = (\neg A \Rightarrow B)$$

$$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_4 = (\neg B \Rightarrow \neg A)$$

$$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

$$B_6 = (\neg B \Rightarrow \neg\neg A)$$

$$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B))$$

$$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$$B_9 = B$$

$$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

$$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

### Exercise 7

Complete the proof sequence 17 by providing comments how each step of the proof was obtained.

### Solution

$$B_1 = (A \Rightarrow B)$$

Hypothesis

$$B_2 = (\neg A \Rightarrow B)$$

Hypothesis

$$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

Example 5

$$B_4 = (\neg B \Rightarrow \neg A)$$

$B_1, B_3$  and MP

$$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

Example 5 for  $A = \neg A, B = B$

$$B_6 = (\neg B \Rightarrow \neg\neg A)$$

$B_2, B_5$  and MP

$$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B))$$

Axiom A3 for  $B = B, A = \neg A$

$$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$B_6, B_7$  and MP

$$B_9 = B$$

$B_4, B_8$  and MP

$$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$$B_1 - B_9$$

$$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

Deduction Theorem

$$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B)) \text{ Deduction Theorem}$$

**Example 8**

Here are consecutive steps

$$B_1, \dots, B_3 \tag{18}$$

in a proof of

$$((\neg A \Rightarrow A) \Rightarrow A).$$

$$B_1 = ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A))$$

$$B_2 = (\neg A \Rightarrow \neg A)$$

$$B_3 = ((\neg A \Rightarrow A) \Rightarrow A)$$

**Exercise 8**

Complete the proof sequence 18 by providing comments how each step of the proof was obtained.

**Solution**

$$B_1 = ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A))$$

Axiom A3 for  $B = A$

$$B_2 = (\neg A \Rightarrow \neg A)$$

Lemma 1.1 for  $A = \neg A$

$$B_3 = ((\neg A \Rightarrow A) \Rightarrow A)$$

$B_1, B_2$  and MP

The above examples 1 - 8, and the example 1 of previous section provide a proof of the following lemma.

**Lemma 4.1** *For any formulas  $A, B, C$  of the system  $H_2$ ,*

1.  $\vdash_{H_2} (A \Rightarrow A)$

2.  $\vdash_{H_2} (\neg\neg B \Rightarrow B)$
3.  $\vdash_{H_2} (B \Rightarrow \neg\neg B)$
4.  $\vdash_{H_2} (\neg A \Rightarrow (A \Rightarrow B))$
5.  $\vdash_{H_2} ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$
6.  $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
7.  $\vdash_{H_2} (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$
8.  $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$
9.  $\vdash_{H_2} ((\neg A \Rightarrow A) \Rightarrow A)$

The set of provable formulas from the above lemma 4.1 includes a set of provable formulas (formulas 1, 3, 4, and 7-9) needed, with  $H_2$  axioms to execute two proofs of the Completeness Theorem for  $H_2$  which we present in the next chapter. These two proofs represent two diametrically different methods of proving Completeness Theorem.