Chapter 8
Hilbert Systems, Deduction Theorem

Hilbert Systems The Hilbert proof systems are based on a language with implication and contain a Modus Ponens rule as a rule of inference.

Modus Ponens is the oldest of all known rules of inference as it was already known to the Stoics (3rd century B.C.).

It is also considered as the most "natural" to our intuitive thinking and the proof systems containing it as the inference rule play a special role in logic.

Hilbert System H_1 :

$$H_1 = (\ \mathcal{L}_{\{\Rightarrow\}},\quad \mathcal{F},\quad \{A1, A2\},\quad MP\ )$$

A1 $(A \Rightarrow (B \Rightarrow A))$,

A2 $((A \Rightarrow (B \Rightarrow C))$
$\Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

MP

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B}$$

Finding formal proofs in this system requires some ingenuity.

The formal proof of $(A \Rightarrow A)$ in H_1 is a sequence

$$B_1,\ B_2,\ B_3,\ B_4, B_5$$

as defined below.

$B_1 = ((A \Rightarrow ((A \Rightarrow A) \Rightarrow A)) \Rightarrow ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
axiom A2 for $A = A$, $B = (A \Rightarrow A)$, and $C = A$

$B_2 = (A \Rightarrow ((A \Rightarrow A) \Rightarrow A))$,
axiom A1 for $A = A$, $B = (A \Rightarrow A)$

$B_3 = ((A \Rightarrow (A \Rightarrow A)) \Rightarrow (A \Rightarrow A)))$,
MP application to B_1 and B_2

$B_4 = (A \Rightarrow (A \Rightarrow A))$,
axiom A1 for $A = A$, $B = A$

$B_5 = (A \Rightarrow A)$
MP application to B_3 and B_4

A general procedure for searching for proofs in a proof system S can be stated is as follows.

Given an expression B of the system S. If it has a proof, it must be conclusion of the inference rule. Let's say it is a rule r.

We find its premisses, with B being the conclusion, i.e. we evaluate $r^{-1}(B)$.

If all premisses are axioms, the proof is found.

Otherwise we repeat the procedure for any non-axiom premiss.

Search for proof by the means of MP The MP rule says: given two formulas A and $(A \Rightarrow B)$ we can conclude a formula B.

Assume now that we have a formula B and want to find its proof.

If B is an axiom, we have the proof: the formula itself.

If it is not an axiom, it had to be obtained by the application of the Modus Ponens rule, to certain two formulas A and $(A \Rightarrow B)$.

But there is infinitely many of formulas A and $(A \Rightarrow B)$. I.e. for any B, the inverse image of B under the rule MP, $MP^{-1}(B)$ is countably infinite.

The proof system H_1 is not syntactically decidable.

Semantic Link 1 System H_1 is sound under classical semantics and is not sound under **Ł** semantics.

Soundness Theorem of H_1 For any $A \in \mathcal{F}$ of H_1,

$$If \quad \vdash_{H_1} A, \quad then \quad \models A.$$

Semantic Link 2 The system H_1 **is not complete** under classical semantics.

Not all classical tautologies have a proof in H_1.

$\models (\neg\neg A \Rightarrow A)$ and $\not\vdash_{H_1} (\neg\neg A \Rightarrow A)$.

Exercise: show that

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

We construct a formal proof

$$B_1, B_2, ...B_7,$$

as follows.

$B_1 = (B \Rightarrow C)$, hypothesis $B_2 = (A \Rightarrow B)$, hypothesis

$B_3 = ((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,
axiom A2

$B_4 = ((B \Rightarrow C) \Rightarrow (A \Rightarrow (B \Rightarrow C)))$,
axiom A1 for $A = (B \Rightarrow C), B = A$

$B_5 = (A \Rightarrow (B \Rightarrow C))$,
B_1 and B_4 and MP

$B_6 = ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$,
B_3 and B_5 and MP

$B_7 = (A \Rightarrow C)$
B_2, B_6 and MP

In mathematical arguments, one often a statement B on the assumption (hypothesis) of some other statement A and then concludes that we have proved the implication "if A, then B".

This reasoning is justified by the following theorem, called a Deduction Theorem.

Notation: $\Gamma, A \vdash B$ for $\Gamma \cup \{A\} \vdash B$,

In general: $\Gamma, A_1, A_2, ..., A_n \vdash B$

for $\Gamma \cup \{A_1, A_2, ..., A_n\} \vdash B$.

Deduction Theorem for H_1

$$\Gamma, \ A \vdash_{H_1} B \quad iff \ \Gamma \vdash_{H_1} (A \Rightarrow B).$$

In particular ,

$$A \vdash_{H_1} B \quad iff \vdash_{H_1} (A \Rightarrow B).$$

Lemma :

$$\textbf{(a)} \quad (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C),$$

$$\textbf{(b)} \quad (A \Rightarrow (B \Rightarrow C)), B \vdash_{H_1} (A \Rightarrow C).$$

First we construct a formal proof for part (a):

$$B_1, B_2, B_3, B_4, B_5$$

as follows.

$B_1 = (A \Rightarrow B)$, hypothesis	$B_2 = (B \Rightarrow C)$, hypothesis	$B_3 = A$ hypothesis
$B_4 = B$ B_1, B_3 and MP	$B_5 = C$ B_2, B_4 and MP	

Thus we proved :

$$(A \Rightarrow B), (B \Rightarrow C), A \vdash_{H_1} C.$$

By Deduction Theorem, we get

$$(A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C).$$

Proof Of The Deduction Theorem

DEDUCTION THEOREM (Herbrand, 1930)
For any formulas A, B,

$$if \quad A \vdash B, \quad then \quad \vdash (A \Rightarrow B).$$

General case for H_1:

$$\Gamma,\ A \vdash B \ \ iff \ \Gamma \vdash (A \Rightarrow B).$$

Proof:

Part 1. We first prove:

$$If \quad \Gamma,\ A \vdash B \ \ then \ \ \Gamma \vdash (A \Rightarrow B).$$

Assume that

$$\Gamma,\ A\ \vdash B,$$

i.e. that we have a formal proof

$$B_1, B_2, ..., B_n$$

of B from the set of formulas $\Gamma \cup \{A\}$, we have to show that

$$\Gamma \vdash (A \Rightarrow B).$$

In order to prove that

$$\Gamma \vdash (A \Rightarrow B)$$

follows from $\Gamma,\ A \vdash B$, we prove that a stronger statement, namely that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for any B_i $(1 \leq i \leq n)$ in the formal proof $B_1, B_2, ..., B_n$ of B also follows from $\Gamma,\ A \vdash B$.

Hence in particular case, when $i = n$, we will obtain that

$$\Gamma \vdash (A \Rightarrow B)$$

follows from $\Gamma,\ A \vdash B$, and that will end the proof of Part 1.

The proof of Part 1 is conducted by induction on i for $1 \leq i \leq n$.

Step $i = 1$ (base step).

Observe that when $i = 1$, it means that the formal proof

$$B_1, B_2, ..., B_n$$

contains only one element B_1.

By the definition of the formal proof from $\Gamma \cup \{A\}$, we have that

(1) B_1 is a logical axiom, or $B_1 \in \Gamma$, or

(2) $B_1 = A$.

This means that $B_1 \in \{A1, A2\} \cup \Gamma \cup \{A\}$.

Now we have two cases to consider.

Case 1: $B_1 \in \{A1, A2\} \cup \Gamma$.

Observe that

$$(B_1 \Rightarrow (A \Rightarrow B_1))$$

is the axiom $A1$ and by assumption

$$B_1 \in \{A1, A2\} \cup \Gamma.$$

We get the required proof of $(A \Rightarrow B_1)$ from Γ by the following application of the Modus Ponens rule

$$(MP)\ \frac{B_1\ ;\ (B_1 \Rightarrow (A \Rightarrow B_1))}{(A \Rightarrow B_1)}.$$

Case 2: $B_1 = A$

When $B_1 = A$ then to prove $\Gamma \vdash (A \Rightarrow B)$ means to prove

$$\Gamma \vdash (A \Rightarrow A),$$

what holds by the monotonicity of the consequence and the fact that we have shown that

$$\vdash (A \Rightarrow A).$$

The above cases conclude the proof of

$$\Gamma \vdash (A \Rightarrow B_i)$$

for $i = 1$.

INDUCTIVE STEP

Assume that

$$\Gamma \vdash (A \Rightarrow B_k)$$

for all $k < i$ (strong induction),

we will show that using this fact we can conclude that also

$$\Gamma \vdash (A \Rightarrow B_i).$$

Consider a formula B_i in the formal proof $B_1, B_2, ..., B_n$

By the definition of the formal proof we have to show the following:

Case 1 $B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$ or

Case 2: B_i follows by MP from certain B_j, B_m such that $j < m < i$.

We have to consider these cases.

Case 1:

$B_i \in \{A1, A2\} \cup \Gamma \cup \{A\}$.

The proof of $(A \Rightarrow B_i)$ from Γ in this case is obtained from the proof of the **Step** $i = 1$ by replacement B_1 by B_i and will be omitted here as a straightforward repetition.

Case 2:

B_i is a conclusion of MP.

If B_i is a conclusion of MP, then we must have two formulas B_j, B_m in the formal proof

$B_1, B_2, ..., B_n$ such that $j < m < i$ and

$$(MP)\ \frac{B_j\ ;\ B_m}{B_i}.$$

By the inductive assumption,

the formulas B_j, B_m are such that

$$\Gamma \vdash (A \Rightarrow B_j)$$

and

$$\Gamma \vdash (A \Rightarrow B_m).$$

Moreover, by the definition of Modus Ponens rule, the formula B_m has to have a form

$$(B_j \Rightarrow B_i),$$

i.e.

$$B_m = (B_j \Rightarrow B_i),$$

and the inductive assumption can be rewritten as follows.

$$\Gamma \vdash (A \Rightarrow (B_j \Rightarrow B_i))$$

, for $j < i$.

Observe now that the formula

$$((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))$$

is a substitution of the axiom A2 and hence **has a proof** in our system.

By the monotonicity of the consequence, it also has a proof from the set Γ, i.e.

We know that

$$\Gamma \vdash ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))).$$

Applying the rule MP i.e. performing the following

$$\frac{(A \Rightarrow (B_j \Rightarrow B_i)) \ ; \ ((A \Rightarrow (B_j \Rightarrow B_i)) \Rightarrow ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)))}{((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}$$

we get that also

$$\Gamma \vdash ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i)).$$

Applying again the rule MP i.e. performing the following

$$\frac{(A \Rightarrow B_j)\ ;\ ((A \Rightarrow B_j) \Rightarrow (A \Rightarrow B_i))}{(A \Rightarrow B_i)})$$

we get that

$$\Gamma \vdash (A \Rightarrow B_i)$$

what ends the proof of the inductive step.

By the mathematical induction principle, we hence have proved that

$$\Gamma \vdash (A \Rightarrow B_i)$$

for all i such that $1 \leq i \leq n$.

In particular it is true for $i = n$, what means for $B_n = B$.

This ends the proof of the fact that

if $\Gamma, A \vdash B$, then $\Gamma \vdash (A \Rightarrow B)$.

The proof of the inverse implication:

$$If \Gamma \vdash_{H_1} (A \Rightarrow B), \quad then \quad \Gamma,\ A \vdash_{H_1} B$$

is straightforward and goes as follows.

Assume that

$\Gamma \vdash (A \Rightarrow B)$.

By the monotonicity of the consequence we have also that

$\Gamma, A \ \vdash (A \Rightarrow B)$.

Obviously

$\Gamma, A \ \vdash A$.

Applying Modus Ponens to the above, we get the proof of B from $\{\Gamma, A\}$ i.e.

we have proved that

$$\Gamma, A \vdash B.$$

THIS ENDS the proof of the deduction theorem for any set $\Gamma \subseteq \mathcal{F}$ and any formulas $A, B \in \mathcal{F}$.

The particular case of the theorem is obtained from the above by assuming that the set Γ is empty.

System H_2 and Formal Proofs

Hilbert System H_2

The system H_1 is sound and strong enough to prove the Deduction Theorem, but it is not complete.

We extend now its set of logical axioms to **a complete set of axioms**, i.e. we define a system H_2 that is **complete** with respect to classical semantics.

The proof of completeness will be presented in the next chapter.

Hilbert System $H_2 = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\ \ A1, A2, A3,\ \ MP\ )$

A1 (Law of simplification)
$(A \Rightarrow (B \Rightarrow A))$,

A2 (Frege's Law)
$((A \Rightarrow (B \Rightarrow C))$
$\Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

A3 $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B)))$

MP (Rule of inference)

$$(MP)\ \frac{A\ ;\ (A \Rightarrow B)}{B},$$

and A, B, C are any formulas of the propositional language $\mathcal{L}_{\{\Rightarrow,\neg\}}$.

We write :

$$\vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 (from the set of logical axioms $A1, A2, A3$), and

$$\Gamma \vdash_{H_2} A$$

to denote that a formula A has a formal proof in H_2 from a set of formulas Γ (and the set of logical axioms $A1, A2, A3$.

Observe that system H_2 was obtained by adding axiom A_3 to the system H_1. Hence the Deduction Theorem holds for system H_2 as well. I.e the following theorem holds.

Deduction Theorem for H_2: For any subset Γ of the set of formulas $\mathcal{F}$ of H_2 and for any formulas $A, B \in \mathcal{F}$,

$$\Gamma,\ A \vdash_{H_2} B \ \ if\ and\ only\ if \ \ \Gamma \vdash_{H_2} (A \Rightarrow B).$$

In particular,

$$A \vdash_{H_2} B \ \ if\ and\ only\ if \ \ \vdash_{H_2} (A \Rightarrow B).$$

Soundness Theorem for H_2:

For every formula $A \in \mathcal{F}$,

$$if \;\; \vdash_{H_2} A, \;\; then \;\; \models A.$$

The soundness theorem proves that the system ”produces” only tautologies. We show, in the next chapter, that our proof system H_2 ”produces” not only tautologies, but that all tautologies are provable in it. This is called a **completeness theorem** for classical logic.

Completeness Theorem for H_2

For every $A \in \mathcal{F}$,

$$\vdash_{H_2} A, \quad \text{if and only if} \quad \models A.$$

The proof of completeness theorem (for a given semantics) is always a main point in any logic creation.

There are many ways (techniques) to prove it, depending on the proof system, and on the semantics we define for it.

We present in the next chapter two proofs of the completeness theorem for our system H_2.

The proofs use very different techniques, hence the reason of presenting both of them.

In fact the proofs are valid for any proof system for classical propositional logic in which one can prove all formulas proved in the next section.

FORMAL PROOFS IN H_2
Examples and Exercises

We present here some examples of formal proofs in H_2. There are two reasons for presenting them.

First reason is that all formulas we prove here to be provable play a crucial role in the proof of Completeness Theorem for H_2, or are needed to find formal proofs of those needed.

The second reason is that they provide a "training" ground for a reader to learn how to develop formal proofs.

For this reason we write some proofs in a full detail and we leave some for the reader to complete in a way explained in the following example.

We write $\vdash$ instead of $\vdash_{H_2}$ for the sake of simplicity.

Reminder In the construction of the formal proofs we very often use Deduction Theorem and the following Lemma (proved in previous section)

Lemma 1 :

$$\textbf{(a)} \quad (A \Rightarrow B), (B \Rightarrow C) \vdash_{H_1} (A \Rightarrow C),$$

$$\textbf{(b)} \quad (A \Rightarrow (B \Rightarrow C)) \vdash_{H_1} ((B \Rightarrow (A \Rightarrow C)).$$

EXAMPLE 1

Here are consecutive steps

$B_1, ..., B_5, B_6$

of the proof (in H_2) of $(\neg\neg B \Rightarrow B)$.

$$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$$

$$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$$

$$B_3 = (\neg B \Rightarrow \neg B)$$

$$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$$

$$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$$

$$B_6 = (\neg\neg B \Rightarrow B).$$

EXERCISE 1

Complete the proof presented in the example 1 by providing comments how each step of the proof was obtained.

ATTENTION The solution presented here shows you how you will have to write details of YOUR solutions on the TESTS.

Solutions of other problems presented later are less detailed. Use them as exercises to write a detailed, complete solution.

Solution

The comments that complete the proof are as follows.

$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and lemma 1 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$, i.e.

$((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B)) \vdash ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

$B_3 = (\neg B \Rightarrow \neg B)$
We proved for H_1 and hence for H_2 that $\vdash (A \Rightarrow A)$ and we substitute $A = \neg B$

$B_4 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$
B_2, B_3 and MP

$B_5 = (\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$
Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_6 = (\neg\neg B \Rightarrow B)$

B_4, B_5 and Lemma 1 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$; i.e.

$(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B)),\ ((\neg B \Rightarrow \neg\neg B) \Rightarrow B) \ \vdash\ (\neg\neg B \Rightarrow B)$.

GENERAL REMARK

In step B_2, B_3, B_5, B_6 we call previously proved facts and use their results as a part of our proof. We can insert previously constructed formal proofs into our formal proof.

For example we adopt previously constructed proof of $(A \Rightarrow A)$ in H_1 to the proof of $(\neg B \Rightarrow \neg B)$ in H_2 by replacing A by $\neg B$ and we insert the proof of $(\neg B \Rightarrow \neg B)$ after B_2.

The "old" step B_3 becomes now B_7, the "old" step B_4 becomes now B_8, etc.....

$B_1 = ((\neg B \Rightarrow \neg\neg B) \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow B))$
Axiom $A3$ for $A = \neg B, B = B$

$B_2 = ((\neg B \Rightarrow \neg B) \Rightarrow ((\neg B \Rightarrow \neg\neg B) \Rightarrow B))$

B_1 and lemma 1 **b** for $A = (\neg B \Rightarrow \neg\neg B), B = (\neg B \Rightarrow \neg B), C = B$,

$B_3 = ((\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B)) \Rightarrow ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,
axiom A2 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$, and $C = \neg B$

$B_4 = (\neg B \Rightarrow ((\neg B \Rightarrow \neg B) \Rightarrow \neg B))$,
axiom A1 for $A = \neg B$, $B = (\neg B \Rightarrow \neg B)$

$B_5 = ((\neg B \Rightarrow (\neg B \Rightarrow \neg B)) \Rightarrow (\neg B \Rightarrow \neg B)))$,
MP application to B_4 and B_3

$B_6 = (\neg B \Rightarrow (\neg B \Rightarrow \neg B))$,
axiom A1 for $A = \neg B$, $B = \neg B$

$B_7 =$ ("old" B_3)$(\neg B \Rightarrow \neg B)$
MP application to B_5 and B_4

$B_8 =$ ("old" B_4) $((\neg B \Rightarrow \neg\neg B) \Rightarrow B)$
B_2, B_3 and MP

$B_9 =$ ("old B_5) $(\neg\neg B \Rightarrow (\neg B \Rightarrow \neg\neg B))$
Axiom $A1$ for $A = \neg\neg B, B = \neg B$

$B_{10} =$ ("old B_6) $(\neg\neg B \Rightarrow B)$
B_8, B_9 and Lemma 1 **a** for $A = \neg\neg B, B = (\neg B \Rightarrow \neg\neg B), C = B$

We repeat our procedure by replacing the step B_2 by its formal proof as defined in the proof of the lemma 1 **b**, and continue the process for all other steps which involved application of lemma 1 until we get a full formal proof from the axioms of H_2 only.

Usually we don't need to do it, but it is important to remember that it always can be done, if we wished to take time and space to do so.

EXAMPLE 2

Here are consecutive steps

$B_1, ..., B_5$ in a proof of $(B \Rightarrow \neg\neg B)$.

$$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$$

$$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$$

$$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$$

$$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$$

$$B_5 = (B \Rightarrow \neg\neg B)$$

EXERCISE 2

Complete the proof presented in Example 2 by providing detailed comments how each step of the proof was obtained.

Solution

The comments that complete the proof are as follows.

$B_1 = ((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))$
Axiom $A3$ for $A = B, B = \neg\neg B$

$B_2 = (\neg\neg\neg B \Rightarrow \neg B)$
Example 1 for $B = \neg B$

$B_3 = ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)$

B_1, B_2 and MP, i.e.

$$\frac{(\neg\neg\neg B \Rightarrow \neg B);((\neg\neg\neg B \Rightarrow \neg B) \Rightarrow ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B))}{((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B)}$$

$B_4 = (B \Rightarrow (\neg\neg\neg B \Rightarrow B))$

Axiom $A1$ for $A = B, B = \neg\neg\neg B$

$B_5 = (B \Rightarrow \neg\neg B)$

B_3, B_4 and lemma 1**a** for $A = B, B = (\neg\neg\neg B \Rightarrow B), C = \neg\neg B$, i.e.

$(B \Rightarrow (\neg\neg\neg B \Rightarrow B)), ((\neg\neg\neg B \Rightarrow B) \Rightarrow \neg\neg B) \vdash_{H_2} (B \Rightarrow \neg\neg B)$

EXAMPLE 3

Here are consecutive steps $B_1, ..., B_{12}$ in a proof of $(\neg A \Rightarrow (A \Rightarrow B))$.

$B_1 = \neg A$

$B_2 = A$

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 = (\neg A \Rightarrow (\neg B \Rightarrow \neg A))$

$B_5 = (\neg B \Rightarrow A)$

$B_6 = (\neg B \Rightarrow \neg A)$

$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_8 = ((\neg B \Rightarrow A) \Rightarrow B)$

$B_9 = B$

$B_{10} = \neg A, A \vdash B$

$B_{11} = \neg A \vdash (A \Rightarrow B)$

$B_{12} = (\neg A \Rightarrow (A \Rightarrow B))$

EXERCISE 3

1. Complete the proof from the example 3 by providing comments how each step of the proof was obtained.

2. Prove that $\neg A, A \vdash B$.

EXAMPLE 4

Here are consecutive steps $B_1, ..., B_7$ in a proof of $((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$.

$B_1 = (\neg B \Rightarrow \neg A)$

$B_2 = ((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$

$B_3 = (A \Rightarrow (\neg B \Rightarrow A))$

$B_4 = ((\neg B \Rightarrow A) \Rightarrow B)$

$B_5 = (A \Rightarrow B)$

$B_6 = (\neg B \Rightarrow \neg A) \vdash (A \Rightarrow B)$

$B_7 = ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

Exercise 4

Complete the proof from example 4 by providing comments how each step of the proof was obtained.

EXAMPLE 5

Here are consecutive steps $B_1, ..., B_9$ in a proof of $((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$.

$B_1 = (A \Rightarrow B)$

$B_2 = (\neg\neg A \Rightarrow A)$

$B_3 = (\neg\neg A \Rightarrow B)$

$$B_4 = (B \Rightarrow \neg\neg B)$$

$$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$$

$$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$$

$$B_7 = (\neg B \Rightarrow \neg A)$$

$$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$$

$$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$$

EXERCISE 5

Complete the proof of example 5 by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg\neg A \Rightarrow A)$
Example 1 for $B = A$

$B_3 = (\neg\neg A \Rightarrow B)$
Lemma 1 **a** for $A = \neg\neg A, B = A, C = B$

$B_4 = (B \Rightarrow \neg\neg B)$
Example 2

$B_5 = (\neg\neg A \Rightarrow \neg\neg B)$

Lemma 1 **a** for $A = \neg\neg A, B = B, C = \neg\neg B$

$B_6 = ((\neg\neg A \Rightarrow \neg\neg B) \Rightarrow (\neg B \Rightarrow \neg A))$

Example 4 for $B = \neg A, A = \neg B$

$B_7 = (\neg B \Rightarrow \neg A)$

B_5, B_6 and MP

$B_8 = (A \Rightarrow B) \vdash (\neg B \Rightarrow \neg A)$

$B_1 - B_7$

$B_9 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

Deduction Theorem

EXERCISE 6

Prove that $\vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B))))$.

Solution Here are consecutive steps of building the formal proof.

$B_1 = A, (A \Rightarrow B) \vdash B$
by MP

$B_2 = A \vdash ((A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

$B_3 = \vdash (A \Rightarrow ((A \Rightarrow B) \Rightarrow B))$
Deduction Theorem

$B_4 = \vdash (((A \Rightarrow B) \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$
Example 5 for $A = (A \Rightarrow B), B = B$

$B_5 = \vdash (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

3. and 4. and lemma 2**a** for $A = A, B = ((A \Rightarrow B) \Rightarrow B), C = (\neg B \Rightarrow (\neg(A \Rightarrow B))$

EXAMPLE 7

Here are consecutive steps $B_1, ..., B_{12}$ in a proof of $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$.

$B_1 = (A \Rightarrow B)$

$B_2 = (\neg A \Rightarrow B)$

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

$B_4 = (\neg B \Rightarrow \neg A)$

$$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$$

$$B_6 = (\neg B \Rightarrow \neg\neg A)$$

$$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$$

$$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$$

$$B_9 = B$$

$$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$$

$$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$$

$$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

EXERCISE 7

Complete the proof in example 7 by providing comments how each step of the proof was obtained.

Solution

$B_1 = (A \Rightarrow B)$
Hypothesis

$B_2 = (\neg A \Rightarrow B)$
Hypothesis

$B_3 = ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$
Example 5

$B_4 = (\neg B \Rightarrow \neg A)$
B_1, B_3 and MP

$B_5 = ((\neg A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg\neg A))$
Example 5 for $A = \neg A, B = B$

$B_6 = (\neg B \Rightarrow \neg\neg A)$
$B2, B_5$ and MP

$B_7 = ((\neg B \Rightarrow \neg\neg A) \Rightarrow ((\neg B \Rightarrow \neg A) \Rightarrow B)))$
Axiom A3 for $B = B, A = \neg A$

$B_8 = ((\neg B \Rightarrow \neg A) \Rightarrow B)$
B_6, B_7 and MP

$B_9 = B$
B_4, B_8 and MP

$B_{10} = (A \Rightarrow B), (\neg A \Rightarrow B) \vdash B$
$B1 - B9$

$B_{11} = (A \Rightarrow B) \vdash ((\neg A \Rightarrow B) \Rightarrow B)$
Deduction Theorem

$B_{12} = ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$ Deduction Theorem

EXAMPLE 8

Here are consecutive steps $B_1, ..., B_3$ in a proof of $((\neg A \Rightarrow A) \Rightarrow A)$.

$$B_1 = ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$$

$$B_2 = (\neg A \Rightarrow \neg A)$$

$$B_3 = ((\neg A \Rightarrow A) \Rightarrow A))$$

EXERCISE 8

Complete the proof of example 8 by providing comments how each step of the proof was obtained.

Solution

$B_1 = ((\neg A \Rightarrow \neg A) \Rightarrow ((\neg A \Rightarrow A) \Rightarrow A)))$
Axiom A3 for $B = A$

$B_2 = (\neg A \Rightarrow \neg A)$
Proved $(A \Rightarrow A)$ for $A = \neg A$

$B_3 = ((\neg A \Rightarrow A) \Rightarrow A))$
B_1, B_2 and MP

Examples 1 - 8, and the example 1 of previous section provide a proof of the following lemma.

LEMMA 2 For any formulas A, B, C of the system H_2,

1. $\vdash_{H_2} (A \Rightarrow A)$

2. $\vdash_{H_2} (\neg\neg B \Rightarrow B)$

3. $\vdash_{H_2} (B \Rightarrow \neg\neg B)$

4. $\vdash_{H_2} (\neg A \Rightarrow (A \Rightarrow B))$

5. $\vdash_{H_2} ((\neg B \Rightarrow \neg A) \Rightarrow (A \Rightarrow B))$

6. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow (\neg B \Rightarrow \neg A))$

7. $\vdash_{H_2} (A \Rightarrow (\neg B \Rightarrow (\neg(A \Rightarrow B)))$

8. $\vdash_{H_2} ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$

9. $\vdash_{H_2} ((\neg A \Rightarrow A) \Rightarrow A$

The set of provable formulas from the above lemma 2 includes a set of provable formulas (formulas 1, 3, 4, and 7-9) needed, with H_2 axioms to execute two proofs of the Completeness Theorem for H_2.

We present these proofs in the next chapter. They represent two diametrally different methods of proving Completeness Theorem.