Chapter 9
Completeness Theorem Proofs

We consider a sound proof system (under classical semantics)

$$S = (\ \mathcal{L}_{\{\Rightarrow,\neg\}},\quad \mathcal{AL},\quad MP\ ),$$

such that the formulas listed below are provable in S.

1. $(A \Rightarrow (B \Rightarrow A))$,

2. $((A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)))$,

3. $((\neg B \Rightarrow \neg A) \Rightarrow ((\neg B \Rightarrow A) \Rightarrow B))$,

4. $(A \Rightarrow A)$,

5. $(B \Rightarrow \neg\neg B)$,

6. $(\neg A \Rightarrow (A \Rightarrow B))$,

7. $(A \Rightarrow (\neg B \Rightarrow \neg(A \Rightarrow B)))$,

8. $((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$,

9. $((\neg A \Rightarrow A) \Rightarrow A)$,

We present here two proofs of the following theorem.

Completeness Theorem For any formula A of S,

$$\models A \quad \text{if and only if} \quad \vdash_S A.$$

OBSERVATION 1 All the above formulas have proofs in the system H_2 and the system H_2 is sound, hence the Completeness Theorem for the system S implies the completeness of the system H_2.

OBSERVATION 2 We have assumed that the system S is sound, i.e. that the following theorem holds for S.

Soundness Theorem

For any formula A of S,

$$\text{if } \vdash_S A, \quad \text{then} \models A.$$

It means that in order to prove the Completeness Theorem we need to prove only the following implication.

For any formula A of S,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash_S A.$$

Both proofs of the Completeness Theorem relay strongly of the Deduction Theorem, as discussed and proved in the previous chapter.

Deduction theorem was proved for the system H_1 that is different that S, but all formulas that were used in its proof are provable in S, so it is valid for S as well, as it was for the system H_2, i.e. the following theorem holds.

Deduction Theorem for S

For any formulas A, B of S and Γ be any subset of formulas of S,

$$\Gamma,\ A \vdash_S B \ \text{ if and only if } \quad \Gamma \vdash_S (A \Rightarrow B).$$

It is possible to prove the Completeness Theorem independently from the Deduction Theorem and we will present two of such a proof in later chapters.

The first proof presented here is similar in its structure to the proof of the deduction theorem and is due to **Kalmar, 1935.**

It shows how one can use the assumption that a formula A is a tautology in order to construct its formal proof. It is hence called **a proof - construction method**.

The second proof is a proof of the equivalent opposite implication to the Completeness part, i.e. we show how one can deduce that a formula A is not a tautology from the fact that it does not have a proof. It is hence called **a counter-model construction method**.

Completeness Theorem

A Proof - Construction Method

We first present one definition and prove one lemma.

We write $\vdash A$ instead of $\vdash_S A$, as the system S is fixed.

Definition Let A be a formula and $b_1, b_2, ..., b_n$ be all propositional variables that occur in A.

Let v be variable assignment $v : VAR \longrightarrow \{T, F\}$.

DEFINITION 1

We define, for $A, b_1, b_2, ..., b_n$ and v a corresponding formulas A', $B_1, B_2, ..., B_n$ as follows:

$$A' = \begin{cases} A & \text{if } v^*(A) = T \\ \neg A & \text{if } v^*(A) = F \end{cases}$$

$$B_i = \begin{cases} b_i & \text{if } v(b_i) = T \\ \neg b_i & \text{if } v(b_i) = F \end{cases}$$

for $i = 1, 2, ..., n$.

Example 1: let A be a formula $(a \Rightarrow \neg b)$.

Let v be such that

$$v(a) = T, \quad v(b) = F.$$

In this case: $b_1 = a$, $b_2 = b$, and $v^*(A) = v^*(a \Rightarrow \neg b) = v(a) \Rightarrow \neg v(b) = T \Rightarrow \neg F = T$.

The corresponding A', B_1, B_2 are:

$A' = A$ (as $v^*(A) = T$),

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

Example 2

Let A be a formula

$$((\neg a \Rightarrow \neg b) \Rightarrow c)$$

and let v be such that

$$v(a) = T, \quad v(b) = F, v(c) = F.$$

Evaluate A', $B_1, ...B_n$ as defined by the definition 1.

In this case $n = 3$ and

$$b_1 = a,\ b_2 = b, b_3 = c,$$

and we evaluate

$$v^*(A) = v^*((\neg a \Rightarrow \neg b) \Rightarrow c) =$$

$$((\neg v(a) \Rightarrow \neg v(b)) \Rightarrow v(c)) =$$

$$((\neg T \Rightarrow \neg F) \Rightarrow F) = (T \Rightarrow F) = F.$$

The corresponding A', B_1, B_2, B_2 are:

$$A' = \neg A = \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

as $v^*(A) = F$,

$B_1 = a$ (as $v(a) = T$),

$B_2 = \neg b$ (as $v(b) = F$).

$B_3 = \neg c$ (as $v(c) = F$).

The lemma stated below describes a method of transforming a semantic notion of a tautology into a syntactic notion of provability. It defines, for any formula A and a variable assignment v a corresponding deducibility relation.

MAIN LEMMA For any formula A and a variable assignment v, if $A^{'}$, B_1 , B_2, ..., B_n are corresponding formulas defined by our definition, then

$$B_1, B_2, ..., B_n \ \vdash \ A'.$$

Example 3 Let A, v be as defined by the Example 1, then the Lemma asserts that

$$a, \neg b \ \vdash \ (a \Rightarrow \neg b).$$

Example 4 Let A, v be as defined in Example 2, then the lemma asserts that

$$a, \neg b, \neg c \ \vdash \ \neg((\neg a \Rightarrow \neg b) \Rightarrow c)$$

Proof of the MAIN LEMMA The proof is by induction on the degree of A i.e. a number n of logical connectives in A.

Case: $n = 0$

In the case that $n = 0$ A is atomic and so consists of a single propositional variable, say a.

Clearly, if $v^*(A) = T$ then we $A' = A = a$, $B_1 = a$.

We obtain that

$$a \vdash a$$

by the Deduction Theorem and the fact that $\vdash (A \Rightarrow A)$, i.e. also $\vdash (a \Rightarrow a)$.

In case when $v^*(A) = F$ we have that

$$A' = \neg A = \neg a,$$

$$B_1 = \neg a, .$$

We obtain that

$$\neg a \ \vdash \ \neg a$$

also by the Deduction Theorem and assumption $\vdash (A \Rightarrow A)$ in S.

This proves that Lemma holds for $n = 0$

Now assume that the lemma holds for any A with $j < n$ connectives.

Prove: lemma holds for A with n connectives.

There are several subcases to deal with.

Case: A is $\neg A_1$

If A is of the form $\neg A_1$ then A_1 has less then n connectives.

By the inductive assumption we have the formulas

$$A_1', \quad B_1, B_2, ..., B_n$$

corresponding to the A_1 and the propositional variables $b_1, b_2, ..., b_n$ in A_1, such that

$$B_1, B_2, ..., B_n \ \vdash \ A_1'$$

Observe, that the formulas A and $\neg A_1$ have the same propositional variables.

So the corresponding formulas B_1 , B_2, ..., B_n are the same for both of them.

We are going to show that the inductive assumption allows us to prove that the lemma holds for A, ie. that

$$B_1, B_2, ..., B_n \vdash A^{'}.$$

There two cases to consider.

Case: $v^*(A_1) = T$

If $v^*(A_1) = T$ then by definition

$$A_1^{'} = A_1$$

and by the inductive assumption

$$B_1, B_2, ..., B_n \vdash A_1$$

.

In this case: $v^*(A) = v^*(\neg A_1) = \neg v^*(A_1) = \neg T = F$

So we have that $A' = \neg A = \neg\neg A_1$.

Since we have assumed about S that

$$\vdash \ (A_1 \Rightarrow \neg\neg A_1)$$

we obtain by the monotonicity that also

$$B_1, B_2, ..., B_n \ \vdash \ (A_1 \Rightarrow \neg\neg A_1).$$

By inductive assumption and Modus Ponens we have that also

$$B_1, B_2, ..., B_n \ \vdash \ \neg\neg A_1,$$

and as $A^{'} = \neg A = \neg\neg A_1$ we get

$$B_1, B_2, ..., B_n \ \vdash \ \neg A,$$

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}.$$

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then $A_1^{'} = \neg A_1$ and $v^*(A) = T$ so $A^{'} = A$.

Therefore by the inductive assumption we have that

$$B_1, B_2, ..., B_n \ \vdash \ \neg A_1$$

that is (as $A = \neg A_1$ and $A^{'} = A$)

$$B_1, B_2, ..., B_n \ \vdash \ A^{'}.$$

Case: A **is** $(A_1 \Rightarrow A_2)$

If A is $(A_1 \Rightarrow A_2)$ then A_1 and A_2 have less than n connectives.

By the inductive assumption and monotonicity we have
$B_1, B_2, ..., B_n \vdash A_1{}'$
and
$B_1, B_2, ..., B_n \vdash A_2{}'$,

where $B_1, B_2, ..., B_n$ are formulas corresponding to the propositional variables in A.

Now we have the following subcases to consider.

Case: $v^*(A_1) = v^*(A_2) = T$

If $v^*(A_1) = T$ then ${A_1}^{'}$ is A_1 and if $v^*(A_2) = T$ then ${A_2}^{'}$ is A_2.
We also have $v^*(A_1 \Rightarrow A_2) = T$ and so $A^{'}$ is $(A_1 \Rightarrow A_2)$.

By the above and the inductive assumption, $B_1, B_2, ..., B_n \vdash A_2$ and since we have assumed about S that $\vdash (A_2 \Rightarrow (A_1 \Rightarrow A_2))$,

we have by monotonicity and Modus Ponens, that $B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2)$, that is

$$B_1, B_2, ..., B_n \vdash A^{'}.$$

Case: $v^*(A_1) = T, v^*(A_2) = F$

If $v^*(A_1) = T$ then ${A_1}' = A_1$ and

if $v^*(A_2) = F$ then ${A_2}' = \neg A_2$.

Also we have in this case $v^*(A_1 \Rightarrow A_2) = F$ and so $A' = \neg(A_1 \Rightarrow A_2)$.

By the above and the inductive assumption, therefore, $B_1, B_2, ..., B_n \vdash \neg A_2$. Since we have assumed $\vdash (A_1 \Rightarrow (\neg A_2 \Rightarrow \neg(A_1 \Rightarrow A_2)))$, we have by monotonicity and Modus Ponens twice, that $B_1, B_2, ..., B_n \vdash \neg(A_1 \Rightarrow A_2)$, that is

$$B_1, B_2, ..., B_n \vdash A'.$$

Case: $v^*(A_1) = F$

If $v^*(A_1) = F$ then ${A_1}' = \neg A_1$ and, whatever value v gives A_2, we have that $v^*(A_1 \Rightarrow A_2) = T$ and so $A' = (A_1 \Rightarrow A_2)$.

Therefore,

$$B_1, B_2, ..., B_n \vdash \neg A_1$$

and since $\vdash (\neg A_1 \Rightarrow (A_1 \Rightarrow A_2))$, by monotonicity and Modus Ponens we get that

$$B_1, B_2, ..., B_n \vdash (A_1 \Rightarrow A_2),$$

that is

$$B_1, B_2, ..., B_n \vdash A'.$$

With that we have covered all cases and, by induction on n, the proof of the lemma is complete.

Proof of the Completeness Theorem

Assume that $\models A$.

Let $b_1, b_2, ..., b_n$ be all propositional variables that occur in A, i.e. $A = A(b_1, b_2, ..., b_n)$.

By the lemma we know that, for any variable assignment v, the corresponding formulas A', B_1 , B_2, ..., B_n can be found such that

$$B_1, B_2, ..., B_n \vdash A'$$

.

Note here that A' of the definition is A for any v since $\models A$.

Hence, if v is such that $v(b_n) = T$, then B_n is b_n and

$$B_1, B_2, ..., b_n \ \vdash \ A.$$

If w is such that $w(b_n) = F$, then B_n is $\neg b_n$ and by the lemma

$$B_1, B_2, ..., \neg b_n \ \vdash \ A.$$

So, by the Deduction Theorem, we have

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (b_n \Rightarrow A)$$

and

$$B_1, B_2, ..., B_{n-1} \ \vdash \ (\neg b_n \Rightarrow A).$$

By monotonicity and assumed formula 9

$$\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$$

we have that
$B_1, B_2, ..., B_{n-1} \vdash ((b_n \Rightarrow A) \Rightarrow ((\neg b_n \Rightarrow A) \Rightarrow A))$.

Applying Modus Ponens twice we get that $B_1, B_2, ..., B_{n-1} \vdash A$.

Similarly, $v^*(B_{n-1})$ may be T or F, and, again applying Deduction Theorem, monotonicity, and $\vdash_S ((A \Rightarrow B) \Rightarrow ((\neg A \Rightarrow B) \Rightarrow B))$, and Modus Ponens twice we can eliminate B_{n-1} just as we eliminated B_n.

After n steps, we finally obtain proof of A in S, i.e. we have that

$$\vdash A.$$

Observe that our proof of the fact that $\vdash A$ is a constructive one. Moreover, we have used in it only Main Lemma and Deduction Theorem which both have a constructive proofs.

We can hence reconstruct proofs in each case when we apply these theorems back to the original axioms $A1 - A3$ of H_2. The same applies to the proofs in H_2 of all formulas 1 -9 of the system S.

It means that for any A, such that $\models A$, each v restricted to A provides us the method of a construction of the formal proof of A in H_2, or in any system S in which formulas 1 -9 are provable.

EXAMPLE As an example of how the Completeness Theorem proof works, we consider the case in which A is a tautology

$$(a \Rightarrow (\neg a \Rightarrow b))$$

and show how the construction described in the Proof 1 works; i.e how we construct the proof of A.

Step 1. We apply Main Lemma to all different variable assignments for A. We have 4 cases to consider. As $\models A$ in all cases we have that $A^{'} = A$.

Case 1: $v(a) = T, v(b) = T$.
In this case $B_1 = a, B_2 = b$ and, as in all cases $A^{'} = A$.

By the Main Lemma,

$$a, b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 2: $v(a) = T, v(b) = F$.

In this case $B_1 = a, B_2 = \neg b$, $A^{'} = A$ and by the Main Lemma,

$$a, \neg b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 3: $v(a) = F, v(b) = T$.

In this case $B_1 = \neg a, B_2 = b$, $A^{'} = A$ and by the Main Lemma,

$$\neg a, b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

Case 4: $v(a) = F, v(b) = F$.

In this case $B_1 = \neg a, B_2 = \neg b$, $A^{'} = A$ and by the Main Lemma,

$$\neg a, \neg b \ \vdash \ (a \Rightarrow (\neg a \Rightarrow b)).$$

We apply Deduction Theorem on formulas $b, \neg b$ to all the cases 1-4. This is the case of B_n elimination in the Proof 1.

D1 (Cases 1 and 2)

$$a \ \vdash \ (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$a \ \vdash \ (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

D2 (Cases 3 and 4)

$$\neg a \ \vdash \ (b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))),$$

$$\neg a \ \vdash \ (\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))).$$

By the monotonicity and proper substitution of the formula 8 we have that

$a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$,

$\neg a \vdash ((b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg b \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$.

Applying Modus Ponens twice to **D1**, **D2** and these above, respectively, gives us

$$a \vdash (a \Rightarrow (\neg a \Rightarrow b)) \text{ and}$$

$$\neg a \vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Applying the Deduction Theorem to the above we obtain

D3 $\vdash (a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$ and

D4 $\vdash (\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b)))$.

Applying Modus Ponens twice to **D3** and **D4** and the following form of formula 8,

$$\vdash ((a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow ((\neg a \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))) \Rightarrow (a \Rightarrow (\neg a \Rightarrow b))))$$

we get finally that

$$\vdash (a \Rightarrow (\neg a \Rightarrow b)).$$

Completeness Theorem: Proof 2
A Counter- Model Existence Method

We prove now the Completeness Theorem by proving the opposite implication:

$$If \quad \nvdash A, \; then \; \not\models A$$

We will show now how one can define of **a counter-model** for A from the fact that A is **not provable.**

This means that we deduce that a formula A is not a tautology from the fact that it does not have a proof.

We hence call it a **a counter-model existence method**.

The construction of a counter-model for any non-provable A is much more general (and less constructive) then in the case of our first proof.

It can be generalized to the case of predicate logic, and many of non-classical logics; propositional and predicate.

It is hence a much more general method then the first one and this is the reason we present it here.

We remind that $\not\models\ A$ means that there is a variable assignment $v : VAR \longrightarrow \{T, F\}$, such that $v^*(A) \neq T$, i.e. in classical semantics that $v^*(A) = F$. a Such v is called a counter-model for A, hence the proof provides a counter-model construction method.

Since we assume that A does not have a proof in S ($\not\vdash\ A$) the method uses this information in order to show that A is not a tautology, i.e. to define v such that $v^*(A) = F$.

We also have to prove that all steps in that method are correct. This is done in the following steps.

Step 1: Definition of Δ^*

We use the information $\nvdash A$ to define a special set Δ^*, such that $\neg A \in \Delta^*$.

Step 2: Counter - model definition

We define the variable assignment $v : VAR \longrightarrow \{T, F\}$ as follows:

$$v(a) = \begin{cases} T & \text{if} \quad \Delta^* \vdash a \\ F & \text{if} \quad \Delta^* \vdash \neg a. \end{cases}$$

Step 3: Prove that v is a counter-model

We first prove a more general property, namely we prove that the set Δ^* and v defined in the steps 1 and 2, respectively, are such that for every formula $B \in \mathcal{F}$,

$$v^*(B) = \begin{cases} T & \text{if } \Delta^* \vdash B \\ F & \text{if } \Delta^* \vdash \neg B. \end{cases}$$

Then we use the **Step 1** to prove that $v^*(A) = F$.

The definition and the properties of the set Δ^*, and hence the **Step 1**, are the most essential for the proof.

The other steps have only technical character.

The main notions involved in this step are: **consistent set, complete set** and a **consistent complete extension of a set.**

We are going now to introduce them and to prove some essential facts about them.

Consistent and Inconsistent Sets

There exist two definitions of consistency; semantical and syntactical.

Semantical definition uses the notion of a model and says:

a set is consistent if it has a model.

Syntactical definition uses the notion of provability and says:

a set is consistent if one can't prove a contradiction from it.

In our proof of the Completeness Theorem we use assumption that a given formula A does not have a proof to deduce that A is not a tautology.

We hence use the following syntactical definition of consistency.

Consistent set

We say that a set $\Delta \subseteq \mathcal{F}$ of formulas is **consistent** if and only if **there is no** a formula $A \in \mathcal{F}$ such that

$$\Delta \vdash A \quad \text{and} \quad \Delta \vdash \neg A.$$

Inconsistent set

A set $\Delta \subseteq \mathcal{F}$ is **inconsistent** if and only if **there is** a formula $A \in \mathcal{F}$ such that $\Delta \vdash A$ and $\Delta \vdash \neg A$.

The notion of consistency, as defined above, is characterized by the following lemma.

LEMMA: Consistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **consistent**,

(ii) there is a formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$.

Proof: The implications:

(i) $\triangle$ is **consistent**, implies

(ii) there is a formula $A \in \mathcal{F}$ such that $\boldsymbol{\Delta} \nvdash A$ and vice-versa are proved by showing the corresponding opposite implications.

I.e. to establish the equivalence of **(i)** and **(ii)**, we first show that

Case 1: not **(ii)** implies not **(i)**, and then that

Case 2: not **(i)** implies not **(ii)**.

Case 1

Assume that not **(ii)**.

It means that **for all formulas** $A \in \mathcal{F}$ we have that $\Delta \vdash A$.

In particular it is true for a certain $A = B$ and $A = \neg B$ and hence it proves that Δ is inconsistent,

i.e. not **(i)** holds.

Case 2

Assume that not **(i)**, i.e that **Δ** is inconsistent.

Then there is a formula A such that **Δ** $\vdash A$ and **Δ** $\vdash \neg A$.

Let B be any formula. Since $(\neg A \Rightarrow (A \Rightarrow B))$ is provable in S (formula 6),

hence by monotonicity and applying Modus Ponens twice and by detaching from it $\neg A$ first, and A next, we obtain a formal proof of B from the set **Δ**.

This proves that **Δ** $\vdash B$ for any formula B. Thus not **(ii)**.

The inconsistent sets are hence characterized by the following fact.

LEMMA: Inconsistency Condition

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **inconsistent**,

(ii) for all formulas $A \in \mathcal{F}$, $\Delta \vdash A$.

We remind here the property of the finiteness of the consequence operation.

LEMMA: Finite Consequence

For every set Δ of formulas and for every formula $A \in \mathcal{F}$,

$\Delta \vdash A$ if and only if there is a finite subset $\Delta_0 \subseteq \Delta$ such that $\Delta_0 \vdash A$.

Proof:

If $\Delta_0 \vdash A$ for a certain $\Delta_0 \subseteq \Delta$, then by the monotonicity of the consequence, also $\Delta \vdash A$.

Assume now that $\Delta \vdash A$ and let

$$A_1, A_2, ..., A_n$$

be a formal proof of A from Δ.

Let $\Delta_0 = \{A_1, A_2, ..., A_n\} \cap \Delta$.

Obviously, Δ_0 is finite and $A_1, A_2, ..., A_n$ is a formal proof of A from Δ_0.

The following theorem is a simply corollary of the above Finite Consequence Lemma.

Finite Inconsistency THEOREM

1. If a set Δ is **inconsistent**, then there is a finite subset $\Delta_0 \subseteq \Delta$ which is inconsistent.

It follows therefore from that

2. if every finite subset of a set Δ is consistent, then the set Δ is also consistent.

Proof:

If Δ is inconsistent, then for some formula A,

$$\Delta \vdash A \text{ and } \Delta \vdash \neg A.$$

By the Finite Consequence Lemma , there are finite subsets Δ_1 and Δ_2 of Δ such that

$$\Delta_1 \vdash A \text{ and } \Delta_2 \vdash \neg A.$$

By monotonicity, the union $\Delta_1 \cup \Delta_2$ is a finite subset of Δ, such that

$$\Delta_1 \cup \Delta_2 \vdash A \text{ and } \Delta_1 \cup \Delta_2 \vdash \neg A.$$

Hence $\Delta_1 \cup \Delta_2$ is a **finite inconsistent subset** of Δ.

The second implication is the opposite to the one just proved and hence also holds.

The following lemma links the notion of non-provability and consistency.

It will be used as an important step in our proof of the Completeness Theorem.

LEMMA

For any formula $A \in \mathcal{F}$,

if $\nvdash A$, then the set $\{\neg A\}$ is consistent.

Proof: If $\{\neg A\}$ is inconsistent, then by the Inconsistency Condition Lemma we have $\{\neg A\} \vdash A$.

$\{\neg A\} \vdash A$ and the Deduction Theorem imply
$\vdash (\neg A \Rightarrow A)$.

Applying the Modus Ponens rule to $(\neg A \Rightarrow A)$ and assumed provable formula 9
$((\neg A \Rightarrow A) \Rightarrow A)$,

we get that $\vdash A$, contrary to the assumption of the lemma.

Complete and Incomplete Sets

Another important notion, is that of a **complete set** of formulas. Complete sets, as defined here are sometimes called **maximal**, but we use the first name for them.

They are defined as follows.

Complete set

A set Δ of formulas is called complete if **for every** formula $A \in \mathcal{F}$,

$$\Delta \vdash A \quad \text{or} \quad \Delta \vdash \neg A.$$

The complete sets are characterized by the following fact.

Complete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **complete**,

(ii) for every formula $A \in \mathcal{F}$, if

$$\Delta \not\vdash A,$$

then the set

$$\Delta \cup \{A\}$$

is **inconsistent**.

Proof: We consider two cases.

Case 1 We show that **(i)** implies **(ii)** and

Case 2 we show that **(ii)** implies **(i)**.

Proof of Case 1:

Assume that **(i)** and that for every formula $A \in \mathcal{F}$, $\mathbf{\Delta} \nvdash A$.

We have to show that in this case $\mathbf{\Delta} \cup \{A\}$ is inconsistent.

But if $\mathbf{\Delta} \nvdash A$, then from the definition of complete set and assumption that $\mathbf{\Delta}$ is complete set, we get that

$$\mathbf{\Delta} \vdash \neg A.$$

By the monotonicity of the consequence we have that

$$\Delta \cup \{A\} \ \vdash \ \neg A.$$

By formula 4 $\vdash (A \Rightarrow A)$ and monotonicity we get $\Delta \vdash (A \Rightarrow A)$ and by Deduction Theorem

$$\Delta \cup \{A\} \ \vdash \ A.$$

This proves that $\Delta \cup \{A\}$ is inconsistent. Hence **(ii)** holds.

Case 2

Assume that **(ii)**, i.e. for every formula $A \in \mathcal{F}$, if $\Delta \nvdash A$, then the set $\Delta \cup \{A\}$ is inconsistent.

Let A be any formula. We want to show **(i)**, i.e. to show that the condition:

$$\Delta \vdash A \text{ or} \Delta \vdash \neg A$$

is satisfied.

If

$$\Delta \vdash \neg A,$$

then the condition is obviously satisfied.

If, on other hand,

$$\Delta \nvdash \neg A,$$

then we are going to show now that it must be, under the assumption of **(ii)**, that $\Delta \vdash A$, i.e. that **(i)** holds.

Assume that

$$\Delta \nvdash \neg A,$$

then by **(ii)**, the set $\Delta \cup \{\neg A\}$ is inconsistent.

It means, by the Consistency Condition Lemma, that

$$\Delta \cup \{\neg A\} \vdash A.$$

By the Deduction Theorem, this implies that

$$\Delta \vdash (\neg A \Rightarrow A).$$

Observe that

$$((\neg A \Rightarrow A) \Rightarrow A)$$

is a provable formula 4 in S.

By monotonicity,

$$\Delta \vdash ((\neg A \Rightarrow A) \Rightarrow A).$$

Detaching $(\neg A \Rightarrow A)$, we obtain that

$$\Delta \vdash A.$$

This ends the proof that **(i)** holds.

Incomplete set

A set Δ of formulas is called incomplete if it is not complete, i.e. if **there exists** a formula $A \in \mathcal{F}$ such that

$$\Delta \not\vdash A \text{ and } \Delta \not\vdash \neg A$$

We get as a direct consequence of the Complete Set Condition Lemma the following characterization of incomplete sets.

Incomplete Set Condition Lemma

For every set $\Delta \subseteq \mathcal{F}$ of formulas, the following conditions are equivalent:

(i) Δ is **incomplete**,

(ii) there is formula $A \in \mathcal{F}$ such that $\Delta \not\vdash A$, and the set $\Delta \cup \{A\}$ is **consistent**.

Main Lemma: Complete Consistent Extension

Now we are going to prove a lemma that is essential to the construction of the special set Δ^* mentioned in the **Step 1** of the proof of the Completeness Theorem, and hence to the proof of the theorem itself.

Let's first introduce one more notion.

Extension Δ^* of the set Δ.

A set Δ^* of formulas is called an **extension** of a set Δ of formulas if the following condition holds:

$$\{A \in \mathcal{F} : \quad \Delta \ \vdash \ A\} \subseteq \{A \in \mathcal{F} : \quad \Delta^* \ \vdash \ A\}.$$

In this case we say also that Δ **extends** to the set of formulas Δ^*.

The Main Lemma states as follows.

Complete Consistent Extension Lemma

Every consistent set Δ of formulas can be extended to a complete consistent set Δ^* of formulas.

Proof: Assume that the lemma does not hold, i.e. that there is a consistent set Δ, such that all its consistent extensions are not complete.

In particular, as Δ is an consistent extension of itself, we have that Δ is not complete.

The proof consists of a construction of a particular set Δ^* and proving that it forms a complete consistent extension of Δ, contrary to the assumption that all its consistent extensions are not complete.

CONSTRUCTION of Δ^*.

As we know, the set $\mathcal{F}$ of all formulas is enumerable. They can hence be put in an infinite sequence

F $A_1, A_2, ..., A_n, ...$
such that every formula of $\mathcal{F}$ occurs in that sequence exactly once.

We define, by **mathematical induction**, an infinite sequence $\{\Delta_n\}_{n \in N}$ of **consistent** subsets of formulas together with a sequence $\{B_n\}_{n \in N}$ of formulas as follows.

Initial Step

In this step we define the sets Δ_1, Δ_2 and the formula B_1 and prove that Δ_1 and Δ_2 are consistent, incomplete extensions of Δ.

We take as the first set, the set Δ, i.e. we define

$$\Delta_1 = \Delta.$$

By assumption the set Δ, and hence also Δ_1 is **not complete**.

From the Incomplete Set Condition we get that there is a formula $B \in \mathcal{F}$ such that

$$\Delta_1 \nvdash B \text{ and } \Delta_1 \cup \{B\} \text{ is } \textbf{consistent}.$$

Let

$$B_1$$

be the first formula with this property in the sequence **F** of all formulas;

We define

$$\Delta_2 = \Delta_1 \cup \{B_1\}.$$

Observe that the set Δ_2 is consistent and

$$\Delta_1 = \Delta \subseteq \Delta_2,$$

so by the monotonicity, Δ_2 is a **consistent extension** of Δ.

Hence, as we assumed that all consistent extensions of Δ are not complete, we get that Δ_2 cannot be complete, i.e.

Δ_2 is **incomplete**.

Inductive Step

Suppose that we have defined a sequence

$$\Delta_1, \Delta_2, ..., \Delta_n$$

of **incomplete, consistent extensions** of Δ, and a sequence

$$B_1, B_2, ...B_{n-1}$$

of formulas, for $n \geq 2$.

Since Δ_n is **incomplete**, it follows from the Incomplete Set Condition that

there is a formula $B \in \mathcal{F}$ such that $\Delta_n \not\vdash B$, then and the set $\Delta_n \cup \{B\}$ is **consistent**.

Let B_n be the first formula with this property in the sequence **F** of all formulas.

We define:

$$\Delta_{n+1} = \Delta_n \cup \{B_n\}.$$

By the definition,

$$\Delta \subseteq \Delta_n \subseteq \Delta_{n+1}$$

and the set Δ_{n+1} is a consistent extension od Δ.

Hence by our assumption that all all consistent extensions of Δ are incomplete we get that Δ_{n+1} is an **incomplete consistent extension** of Δ.

By the principle of mathematical induction we have defined an infinite sequence

D $\Delta = \Delta_1 \subseteq \Delta_2 \subseteq ..., \subseteq \Delta_n \subseteq \Delta_{n+1} \subseteq ....$

such that for all $n \in N$, Δ_n is **consistent**, and each Δ_n an **incomplete consistent extension** of Δ.

Moreover, we have also defined a sequence

B $B_1, B_2, ..., B_n, ...$

of formulas, such that for all $n \in N$,

$\Delta_n \not\vdash B_n$, and the set $\Delta_n \cup \{B_n\}$ is **consistent**.

Observe that $B_n \in \Delta_{n+1}$ for all $n \geq 1$.

Now we are ready to define Δ^*.

Definition of Δ^*

$$\Delta^* = \bigcup_{n \in N} \Delta_n.$$

To complete the proof our theorem we have now to prove that

Δ^* is a **complete consistent extension** of Δ.

Obviously, by the definition,

Δ^* is an extension of Δ.

Fact 1 Δ^* is **consistent**.

proof: assume that Δ^* is **inconsistent**. By the Finite Inconsistency theorem there is a finite subset Δ_0 of Δ^* that is **inconsistent**, i.e.

$$\Delta_0 = \{C_1, ..., C_n\} \subseteq \bigcup_{n \in N} \Delta_n$$

and Δ_0 is **inconsistent**.

By the definition, $C_i \in \Delta_{k_i}$ for certain Δ_{k_i} in the sequence **D** and $1 \leq i \geq n$.

Hence $\Delta_0 \subseteq \Delta_m$ for $m = max\{k_1, k_2, ..k_n\}$.

But all sets of the sequence **D** are **consistent**.

This contradicts the fact that Δ_m is **inconsistent**, as it contains an inconsistent subset Δ_0.

Hence Δ^* must be **consistent**.

Fact 2 Δ^* is **complete**.

proof: assume that Δ^* is **not complete**. By the Incomplete Set Condition, there is a formula $B \in \mathcal{F}$ such that

$\Delta^* \nvdash B$,and the set $\Delta^* \cup \{B\}$ is **consistent**.

By definition **D** of the sequence Δ_n, for every $n \in N$, $\Delta_n \nvdash B$ and the set $\Delta_n \cup \{B\}$ is **consistent**.

Since the formula B is one of the formulas of the sequence **B** and it would have to be one of the formulas of the sequence i.e. $B = B_j$ for certain j.

By definition, $B_j \in \Delta_{j+1}$, it proves that $B \in \Delta^* = \bigcup_{n \in N} \Delta_n$.

But this means that

$$\Delta^* \vdash B,$$

contrary to the assumption.

This proves that Δ^* is a **complete consistent extension** of Δ and completes the proof out our lemma.

Now we are ready to prove the **completeness theorem** for the system S.

Proof of the Completeness Theorem

As by assumption our system S is sound, we have to prove only the Completeness part of the Completeness Theorem, i.e for any formula A,

$$\text{If} \quad \models A, \quad \text{then} \quad \vdash A$$

We prove it by proving the opposite implication

$$\text{If} \quad \not\vdash A, \quad \text{then} \quad \not\models A.$$

Reminder: $\not\models A$ means that there is a variable assignment $v : VAR \longrightarrow \{T, F\}$, such that $v^*(A) \neq T$.

In classical case it means that $v^*(A) = F$, i.e. that there is a variable assignment that falsifies A . Such v is also called a **counter-model** for A.

Assume that A doesn't have a proof in S, we want to define a **counter-model** for A.

But if $\not\vdash A$, then by the Inconsistency Lemma the set $\{\neg A\}$ is consistent.

By the Main Lemma there is a complete, consistent extension of the set $\{\neg A\}$, i.e. there is a set set Δ^* such that $\{\neg A\} \subseteq \Delta^*$, i.e.

E $\quad \neg A \in \Delta^*$.

Since Δ^* is a consistent, complete set, it satisfies the following form consistency condition, which says that for any A,

$$\Delta^* \not\vdash A \text{ or } \Delta^* \not\vdash \neg A.$$

It also satisfies the completeness condition, which says that for any A,

$$\Delta^* \vdash A \text{ or } \Delta^* \vdash \neg A.$$

This means that for any A, **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash A$, or

(2) $\Delta^* \vdash \neg A$.

In particular, for every propositional variable $a \in VAR$ **exactly one** of the following conditions is satisfied:

(1) $\Delta^* \vdash a$, or

(2) $\Delta^* \vdash \neg a$.

This justifies the correctness of the following definition.

Definition of v

We define the variable assignment

$$v : VAR \longrightarrow \{T, F\}$$

as follows:

$$v(a) = \begin{cases} T & \text{if} \quad \Delta^* \vdash a \\ F & \text{if} \quad \Delta^* \vdash \neg a. \end{cases}$$

We show, as a separate lemma below, that such defined variable assignment v has the following property.

Property of v Lemma

Let v be the variable assignment defined above and v^* its extension to the set $\mathcal{F}$ of all formulas.

For every formula $B \in \mathcal{F}$, the following is true

$$v^*(B) = \begin{cases} T & \text{if} \quad \Delta^* \vdash B \\ F & \text{if} \quad \Delta^* \vdash \neg B. \end{cases}$$

Given Property of v Lemma (still to be proved) we now prove that the v is in fact, a **counter model for** any formula A, such that $\not\vdash A$.

Let A be such that $\not\models A$. By **E** $\neg A \in \Delta^*$ and obviously,

$$\Delta^* \vdash \neg A.$$

Hence, by the property of v,

$$v^*(A) = F,$$

what proves that v is a **counter-model** for A and hence **ends** the proof of the completeness theorem.

In order to really complete the proof we still have to show the Property of v Lemma.

Proof of the Lemma (Property of v lemma)

The proof is conducted by the induction on the degree of the formula A.

Initial step If A is a propositional variable, then the Lemma is true holds by definition of v.

Inductive Step If A is not a propositional variable, then A is of the form $\neg C$ or $(C \Rightarrow D)$, for certain formulas C, D.

By the inductive assumption the Lemma holds for the formulas C and D.

Case $A = \neg C$

We have to consider two possibilities:

1. $\Delta^* \vdash A$,

2. $\Delta^* \vdash \neg A$.

Consider case 1. i.e. assume

$$\Delta^* \vdash A.$$

It means that $\Delta^* \vdash \neg C$.

Then from the fact that Δ^* is **consistent** it must be that

$$\Delta^* \nvdash C.$$

By the inductive assumption we have that $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg F = T.$$

Consider case 2. i.e. assume that

$$\Delta^* \vdash \neg A.$$

Then from the fact that Δ^* is **consistent** it must be that $\Delta^* \nvdash A$ and

$$\Delta^* \nvdash \neg C.$$

If so, then $\Delta^* \vdash C$, as the set Δ^* is **complete**.

By the inductive assumption, $v^*(C) = T$, and accordingly

$$v^*(A) = v^*(\neg C) = \neg v^*(C) = \neg T = F.$$

Thus A satisfies the v property Lemma.

Case $A = (C \Rightarrow D)$.

As in the previous case, we assume that the Lemma holds for the formulas C, D and we consider two possibilities:

1. $\Delta^* \vdash A$ and

2. $\Delta^* \vdash \neg A$.

Case 1. Assume $\Delta^* \vdash A$. It means that $\Delta^* \vdash (C \Rightarrow D)$.

If at the same time $\Delta^* \nvdash C$, then $v^*(C) = F$, and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = F \Rightarrow v^*(D) = \mathbf{T}.$$

If at the same time $\mathbf{\Delta}^* \vdash C$, then since $\mathbf{\Delta}^* \vdash (C \Rightarrow D)$, we infer, by Modus Ponens, that

$$\mathbf{\Delta}^* \vdash D.$$

If so, then

$$v^*(C) = v^*(D) = T,$$

and accordingly

$$v^*(A) = v^*(C \Rightarrow D) =$$

$$v^*(C) \Rightarrow v^*(D) = T \Rightarrow T = \mathbf{T}.$$

Thus, if $\mathbf{\Delta}^* \vdash A$, then $v^*(A) = T$.

Case 2. Assume now, as before, that

$$\Delta^* \ \vdash \ \neg A.$$

Then from the fact that Δ^* is **consistent** it must be that $\Delta^* \ \nvdash \ A$, i.e.,

$$\Delta^* \ \nvdash \ (C \Rightarrow D).$$

It follows from this that

$$\Delta^* \ \nvdash \ D,$$

for if $\Delta^* \ \vdash \ D$, then, as $(D \Rightarrow (C \Rightarrow D))$ is provable formula 1 in S, by monotonicity also

$$\Delta^* \vdash (D \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we obtain

$$\Delta^* \ \vdash \ (C \Rightarrow D),$$

which is contrary to the assumption.

Also we must have

$$\Delta^* \vdash C,$$

for otherwise, as Δ^* is **complete** we would have $\Delta^* \vdash \neg C$.

But this is impossible, since the formula $(\neg C \Rightarrow (C \Rightarrow D))$ is assumed to be provable formula 9 in S and by monotonicity

$$\Delta^* \vdash (\neg C \Rightarrow (C \Rightarrow D)).$$

Applying Modus Ponens we would get

$$\Delta^* \vdash (C \Rightarrow D),$$

which is contrary to the assumption.

This ends the proof of the lemma and completes the counter- model existence proof of the Completeness Theorem.