Chapter 12:
Gentzen Sequent Calculus for Intuitionistic Logic

Part 1: LI System

The proof system LI was published by Gentzen in 1935 as a particular case of his proof system **LK** for the classical logic.

We discussed a version of the original Gentzen's system **LK** in the previous chapter.

We present now the proof system **LI** and then we show how it can be extended to the original Gentzen system **LK**.

Language of LI

We consider the set of all Gentzen sequents built out of the formulas of our language $\mathcal{L}$ and the additional symbol $\longrightarrow$, as defined in the previous section:

$$SEQ = \{ \Gamma \longrightarrow \Delta : \Gamma, \Delta \in \mathcal{F}^* \}.$$

In the intuitionistic logic we deal only with sequents of the form

$$\Gamma \longrightarrow \Delta,$$

where Δ consists of at most one formula.

The intuitionistic sequents are elements of a following subset IS of the set SEQ of all sequents.

$ISEQ = \{\Gamma \longrightarrow \Delta :$ Δ consists of at most one formula $\}$.

Axioms of LI consists of any sequent from the set $ISEQ$ which contains a formula that appears on both sides of the sequent arrow $\longrightarrow$, i.e any sequent of the form

$$\Gamma_1, A, \Gamma_2 \longrightarrow A.$$

Inference rules of LI

The set inference rules is divided into two groups: the structural rules and the logical rules.

Structural Rules of **LI**

Weakening

$$(\rightarrow weakening) \quad \frac{\Gamma \ \longrightarrow}{\Gamma \ \longrightarrow \ A}$$

A **is called** the weakening formula.

Contraction

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \;\longrightarrow\; \Delta}{A, \Gamma \;\longrightarrow\; \Delta}$$

A **is called** the contraction formula.

Δ **contains** at most one formula.

Exchange

$$(exchange \rightarrow) \quad \frac{\Gamma_1, A, B, \Gamma_2 \;\longrightarrow\; \Delta}{\Gamma_1, B, A, \Gamma_2 \;\longrightarrow\; \Delta},$$

Δ **contains** at most one formula.

Logical Rules of LI

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \longrightarrow \Delta}{(A \cap B), \Gamma \longrightarrow \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \longrightarrow A \; ; \; \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cap B)},$$

Disjunction rules

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow A}{\Gamma \longrightarrow (A \cup B)},$$

$$(\rightarrow \cup)_1 \quad \frac{\Gamma \longrightarrow B}{\Gamma \longrightarrow (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \longrightarrow \Delta \; ; \; B, \Gamma \longrightarrow \Delta}{(A \cup B), \Gamma \longrightarrow \Delta},$$

Δ **contains** at most one formula.

Implication rules

$$(\rightarrow\Rightarrow) \quad \frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow (A \Rightarrow B)},$$

$$(\Rightarrow\rightarrow) \quad \frac{\Gamma \longrightarrow A \;;\; B, \Gamma \longrightarrow \Delta}{(A \Rightarrow B), \Gamma \longrightarrow \Delta},$$

Δ **contains** at most one formula.

Negation rules

$$(\neg \rightarrow) \quad \frac{\Gamma \longrightarrow A}{\neg A, \Gamma \longrightarrow},$$

$$(\rightarrow \neg) \quad \frac{A, \Gamma \longrightarrow}{\Gamma \longrightarrow \neg A}.$$

We define

$\mathbf{LI} = (\mathcal{L}, ISEQ,\ AL,$ Structural rules, Logical rules $\})$.

LK - Original Gentzen system for the classical propositional logic.

Language of LK: $\mathcal{L} = \mathcal{L}_{\{\neg, \cap, \cup, \Rightarrow\}}$, and $\mathcal{E} = SEQ$, for

$$SEQ = \{\Gamma \longrightarrow \Delta : \quad \Gamma, \Delta \in \mathcal{F}^*\}.$$

Axioms of LK: any sequent of the form

$$\Gamma_1, A, \Gamma_2 \longrightarrow \Gamma_3, A, \Gamma_4.$$

Rules of inference of LK

1. We adopt all rules of **LI** with no intuitionistic restriction that the sequence Δ in the succedent of the sequence is at most one formula.

2. We add two structural rules to the system **LI**.

We add one more contraction rule:

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A,}{\Gamma \longrightarrow \Delta, A},$$

We add one more exchange rule:

$$(\rightarrow exchange) \quad \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}.$$

Observe that they both become obsolete in **LI** .

The rules of inference of **LK** are hence as follows.

Structural Rules of LK

Weakening

$$(weakening \rightarrow) \quad \frac{\Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta},$$

$$(\rightarrow weakening) \quad \frac{\Gamma \longrightarrow}{\Gamma \longrightarrow \Delta, A} .$$

A **is called** the weakening formula.

Contraction

$$(contr \rightarrow) \quad \frac{A, A, \Gamma \longrightarrow \Delta}{A, \Gamma \longrightarrow \Delta},$$

$$(\rightarrow contr) \quad \frac{\Gamma \longrightarrow \Delta, A, A,}{\Gamma \longrightarrow \Delta, A},$$

A **is called** the contraction formula.

Exchange

$$(exchange \rightarrow) \quad \frac{\Gamma_1, A, B, \Gamma_2 \longrightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \longrightarrow \Delta},$$

$$(\rightarrow exchange) \quad \frac{\Delta \longrightarrow \Gamma_1, A, B, \Gamma_2}{\Delta \longrightarrow \Gamma_1, B, A, \Gamma_2}.$$

Logical Rules of LK

Conjunction rules

$$(\cap \rightarrow) \quad \frac{A, B, \Gamma \;\longrightarrow\; \Delta}{(A \cap B), \Gamma \;\longrightarrow\; \Delta},$$

$$(\rightarrow \cap) \quad \frac{\Gamma \;\longrightarrow \Delta,\; A \;;\; \Gamma \;\longrightarrow \Delta,\; B\Delta}{\Gamma \;\longrightarrow\; \Delta, (A \cap B)}.$$

Disjunction rules

$$(\rightarrow \cup) \quad \frac{\Gamma \;\longrightarrow\; \Delta, A, B}{\Gamma \;\longrightarrow\; \Delta, (A \cup B)},$$

$$(\cup \rightarrow) \quad \frac{A, \Gamma \;\longrightarrow\; \Delta \;;\; B, \Gamma \;\longrightarrow\; \Delta}{(A \cup B), \Gamma \;\longrightarrow\; \Delta}.$$

Implication rules

$$(\longrightarrow\Rightarrow)\quad \frac{A,\Gamma\ \longrightarrow\ \Delta, B}{\Gamma\ \longrightarrow\ \Delta, (A\Rightarrow B)},$$

$$(\Rightarrow\longrightarrow)\quad \frac{\Gamma\ \longrightarrow\ \Delta, A\ ;\ B,\Gamma\ \longrightarrow\ \Delta}{(A\Rightarrow B),\Gamma\ \longrightarrow\ \Delta}.$$

Negation rules

$$(\neg\ \longrightarrow)\quad \frac{\Gamma\ \longrightarrow\ \Delta,\ A}{\neg A,\Gamma\ \longrightarrow\ \Delta},$$

$$(\longrightarrow\ \neg)\quad \frac{A,\Gamma\ \longrightarrow\ \Delta}{\Gamma\ \longrightarrow\ \Delta,\neg A}.$$

We define formally $\mathbf{LK} = (\mathcal{L}, , SEQ,\ AL,$ Structural rules, Logical rules).

PART 2: Examples of proof search decomposition trees in LI

Search for proofs in **LI** is a much more complicated process then the one in classical logic systems **RS** or **GL**.

Proof search procedure consists of building the decomposition trees.

Remark 1: in **RS** the decomposition tree $\mathbf{T}_A$ of any formula A is always unique.

Remark 2: in **GL** the "blind search" defines, for any formula A a finite number of decomposition trees, but it can be proved that the search can be reduced to examining only one of them, due to the absence of structural rules.

Remark 3: In **LI** the structural rules play a vital role in the proof construction and hence, in the proof search.

The fact that a given decomposition tree ends with an axiom leaf does not always imply that the proof does not exist. It might only imply that our search strategy was not good.

The problem of deciding whether a given formula A does, or does not have a proof in **LI** becomes more complex then in the case of Gentzen system for classical logic.

Example 1

Determine whether

$$\vdash_{\mathbf{LI}} \longrightarrow A$$

for $A = ((\neg A \cap \neg B) \Rightarrow \neg(A \cup B))$.

If we find a decomposition tree such that all its leaves are axiom, we have a proof.

If all possible decomposition trees have a non-axiom leaf, proof of A in **LI** does not exist.

Consider the following decomposition tree

$$\mathbf{T1_A}$$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\neg B, (A \cup B) \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$\neg B(A \cup B) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$(A \cup B) \longrightarrow B$$

$$\bigwedge(\cup \longrightarrow)$$

$$A \longrightarrow B \qquad\qquad B \longrightarrow B$$

$$non-axiom \qquad\qquad axiom$$

The tree $\mathbf{T1_A}$ has a non-axiom leaf, so it does not constitute a proof in **LI**.

Observe that the decomposition tree in **LI** is not always unique.

Hence this fact does not yet prove that proof doesn't exist.

Let's consider now the following tree

$$\mathbf{T2_A}$$

$$\longrightarrow ((\neg A \cap \neg B) \Rightarrow (\neg(A \cup B))$$

$$\mid (\longrightarrow\Rightarrow)$$

$$(\neg A \cap \neg B) \longrightarrow \neg(A \cup B)$$

$$\mid (\longrightarrow \neg)$$

$$(A \cup B), (\neg A \cap \neg B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(\neg A \cap \neg B), (A \cup B) \longrightarrow$$

$$\mid (\cap \longrightarrow)$$

$$\neg A, \neg B, (A \cup B) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$\neg A, (A \cup B), \neg B \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$(A \cup B), \neg A, \neg B \longrightarrow$$

$$\bigwedge (\cup \longrightarrow)$$

$$
\begin{array}{cc}
A, \neg A, \neg B \longrightarrow & B, \neg A, \neg B \longrightarrow \\
\mid (exch \longrightarrow) & \mid (exch \longrightarrow) \\
\neg A, A, \neg B \longrightarrow & B, \neg B, \neg A \longrightarrow \\
\mid (\neg \longrightarrow) & \mid (exch \longrightarrow) \\
A, \neg B \longrightarrow A & \neg B, B, \neg A \longrightarrow \\
axiom & \mid (\neg \longrightarrow) \\
 & B, \neg A \longrightarrow B \\
 & axiom
\end{array}
$$

All leaves of $\mathbf{T2_A}$ are axioms, what proves that $\mathbf{T2_A}$ is a proof of A.

Hence we proved that

$$((\neg A \cap \neg B) \Rightarrow \neg(A \cup B)).$$

Example 2: Proof that

Part 1

$$\vdash_{\mathbf{LI}} \longrightarrow (A \Rightarrow \neg\neg A),$$

Part 2

$$\not\vdash_{\mathbf{LI}} \longrightarrow (\neg\neg A \Rightarrow A).$$

Solution of Part 1: We construct some, or all decomposition trees of

$$\longrightarrow (A \Rightarrow \neg\neg A).$$

The tree that ends with all axioms leaves is a proof of $(A \Rightarrow \neg\neg A)$ in **LI**.

Consider the following decomposition tree of $\longrightarrow A$, for $A = (A \Rightarrow \neg\neg A)$..

$$\mathbf{T}_A$$

$$\longrightarrow (A \Rightarrow \neg\neg A).$$

$$\mid (\longrightarrow \Rightarrow)$$

$$A \longrightarrow \neg\neg A$$

$$\mid (\longrightarrow \neg)$$

$$\neg A, A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$A \longrightarrow A$$

$$axiom$$

All leaves of $\mathbf{T}_A$ are axioms what proves that $\mathbf{T}_A$ is a proof of $\longrightarrow (A \Rightarrow \neg\neg A)$.

We don't need to construct other decomposition trees.

Solution of Part 2: in order to prove that

$$\nvdash_{\mathbf{LI}} \longrightarrow (\neg\neg A \Rightarrow A)$$

we have to construct all decomposition trees of

$$(\longrightarrow A \Rightarrow \neg\neg A)$$

and show that each of them has an non-axiom leaf.

Decomposition trees construction is as follows.

$$\mathbf{T1}_A$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$one\ of\ 2\ choices$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$A \longrightarrow$$

$$non - axiom$$

Another tree is:

$$\mathbf{T2}_A$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

]

$$\mid (contr \longrightarrow)$$

$$second\ of\ 2\ choices$$

$$\neg\neg A, \neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$first\ of\ 2\ choices$$

$$\neg\neg A, \neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$first\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$the\ only\ choice$$

$$A, \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$the\ only\ choice$$

$$\neg\neg A, A \longrightarrow$$

$$| \; (\longrightarrow \neg)$$

$$the \; only \; choice$$

$$A \longrightarrow \neg A$$

$$| \; (\longrightarrow \neg)$$

$$first \; of \; 2 \; choices$$

$$A, A \longrightarrow$$

$$non - axiom$$

We can see from the above decomposition trees that the "blind" construction of all possible trees only leads to more complicated trees.

This is due to the presence of structural rules.

Observe that the "blind" application of the rule $(contr \rightarrow)$ gives an infinite number of decomposition trees.

In order to decide that none of them will produce a proof we need some extra knowledge about patterns of their construction, or just simply about the number useful of application of structural rules within the proofs.

In this case we can just make an ”external” observation that the our first tree $\mathbf{T1}_A$ is in a sense a minimal one; that all other trees would only complicate this one in an inessential way, i.e. we will never produce a tree with all axioms leaves.

One can formulate a deterministic procedure giving a finite number of trees, but the proof of its correctness require some extra knowledge.

Within the scope of this book we accept the ”external” explanation as a sufficient solution, provided it is correct.

As we can see from the above examples structural rules and especially the $(contr \longrightarrow)$ rule complicates the proof searching task.

Both Gentzen type proof systems **RS** and **GL** from the previous chapter don't contain the structural rules.

They also are complete with respect to classical semantics.

The original Gentzen system **LK** which does contain the structural rules is also complete.

Hence, all three classical proof system **RS, GL, LK** are equivalent.

This proves that the structural rules can be eliminated from the system **LK**.

A natural question of elimination of structural rules from the intutionistic Gentzen system **LI** arizes.

The following example illustrates the negative answer.

Example 3 We know, by the theorem about the connection between classical and intuitionistic logic and corresponding Completeness Theorems that

For any formula $A \in \mathcal{F}$,

$$\models\ A \quad if\ and\ only\ if \quad \vdash_I \neg\neg A,$$

$\models\ A$ means that A is a classical tautology,

$\vdash_I$ means that A is intutionistically provable, i.e. is provable in any intuitionistically complete proof system.

The system LI is intuitionistically complete, so we have that for any formula A,

$$\models\ A \quad if\ and\ only\ if \quad \vdash_{\mathbf{LI}} \neg\neg A.$$

Obviously $\models (\neg\neg A \Rightarrow A)$, so we know that

$$\neg\neg(\neg\neg A \Rightarrow A)$$

must have a proof in **LI**.

We are going to prove the structural rule

$$(contr \longrightarrow)$$

is essential to the existence of its proof.

The formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in **LI** without the rule $(contr \longrightarrow)$.

The following decomposition tree is a proof of $A = \neg\neg(\neg\neg A \Rightarrow A)$ in **LI**.

$\mathbf{T}_A$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (contr \longrightarrow)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A), \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow\Rightarrow)$$

one of 3 *choices*

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

one of 2 *choices*

$$\neg(\neg\neg A \Rightarrow A), \neg\neg A \longrightarrow$$

$$\mid (exch \longrightarrow)$$

one of 3 *choices*

$$\neg\neg A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$one\ of\ 3\ choices$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow \neg A$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 3\ choices$$

$$A, \neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (exch \longrightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg(\neg\neg A \Rightarrow A), A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$one\ of\ 3\ choices$$

$$A \longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$one\ of\ 3\ choices$$

$$\neg\neg A, A \longrightarrow A$$

$$axiom$$

Assume now that the rule $(contr \longrightarrow)$ is not available.

All possible decomposition trees are as follows.

$$\mathbf{T1}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

$$one\ of\ 2\ choices$$

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$only\ one\ choice$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow \neg A$$

$$|\ (\longrightarrow \neg)$$

$$one\ of\ 2\ choices$$

$$A \longrightarrow$$

$$non - axiom$$

Next one is

$$\mathbf{T2}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non - axiom$$

And the next is

$$\mathbf{T3}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non - axiom$$

And the last one is

$$\mathbf{T4}_A$$

$$\longrightarrow \neg\neg(\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \neg)$$

$$\neg(\neg\neg A \Rightarrow A) \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$\longrightarrow (\neg\neg A \Rightarrow A)$$

$$\mid (\longrightarrow \Rightarrow)$$

]

$$\neg\neg A \longrightarrow A$$

$$\mid (\longrightarrow weak)$$

$$only\ one\ choice$$

$$\neg\neg A \longrightarrow$$

$$\mid (\neg \longrightarrow)$$

$$only\ one\ choice$$

$$\longrightarrow \neg A$$

$$\mid (\longrightarrow weak)$$

$$second\ of\ 2\ choices$$

$$\longrightarrow$$

$$non-axiom$$

This proves that the formula $\neg\neg(\neg\neg A \Rightarrow A)$ is not provable in $\mathbf{LI}$ without $(contr \longrightarrow)$ rule, i.e. that this rule can't be eliminated.

PART 3: Proof Search Heuristics

Before we define a heuristic method of searching for proof in **LI** let's make some observations.

Observation 1 : the logical rules of **LI** are similar to those in Gentzen type classical formalizations we examined in previous chapters in a sense that each of them introduces a logical connective.

Observation 2 : The process of searching for a proof is hence a decomposition process in which we use the inverse of logical and structural rules as decomposition rules.

For example the implication rule:

$$(\rightarrow\Rightarrow)\quad \frac{A, \Gamma \ \longrightarrow \ B}{\Gamma \ \longrightarrow \ (A \Rightarrow B)}$$

becomes an implication decomposition rule (we use the same name $(\rightarrow\Rightarrow)$ in both cases)

$$(\rightarrow\Rightarrow)\quad \frac{\Gamma \ \longrightarrow \ (A \Rightarrow B)}{A, \Gamma \ \longrightarrow \ B}.$$

Observation 3 : we write our proofs in a form of trees, instead of sequences of expressions, so the proof search process is a process of building a decomposition tree.

To facilitate the process we write the decomposition rules in a "tree " form. For example the the above implication decomposition rule is written as follows

$$\Gamma \longrightarrow (A \Rightarrow B)$$

$$\mid (\rightarrow\Rightarrow)$$

$$A, \Gamma \longrightarrow B.$$

The two premisses implication rule $(\Rightarrow\rightarrow)$ written as the tree decomposition rule becomes

$$(A \Rightarrow B), \Gamma \longrightarrow$$

$$\bigwedge(\Rightarrow\rightarrow)$$

$$\Gamma \longrightarrow A \qquad B, \Gamma \longrightarrow$$

Observation 4 : we stop the decomposition process when we obtain an axiom or indecomposable leaf. The indecomposable leaf is a sequent built from indecomposable formulas only, i.e. formulas that do not contain logical connectives (positive literals).

Observation 5 : Our goal while constructing the decomposition tree is to obtain axiom or indecomposable leaves. With respect to this goal the use logical decomposition rules has a priority over the use of the structural rules and we use this information while describing the proof search heuristic.

Observation 6 : all logical decomposition rules ($\circ \rightarrow$), where $\circ$ denotes any connective, must have a formula we want to decompose as the first formula at the decomposition node.

When we decompose a formula $\circ A$, the node must have a form $\circ A, \Gamma \longrightarrow \Delta$. Sometimes it is necessary to decompose a formula within the sequence Γ first in order to find a proof.

For example, consider two nodes

$$n_1 = \neg\neg A, (A \cap B) \longrightarrow B$$

and

$$n_2 = (A \cap B), \neg\neg A \longrightarrow B.$$

We are going to see that the results of decomposing n_1 and n_2 differ dramatically.

We decompose the node n_1.

Observe that the only way to be able to decompose the formula $\neg\neg A$ is to use the rule $(\rightarrow weak)$ first.

The two possible decomposition trees that starts at the node n_1 are as follows.

$$\mathbf{T1}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$
$$\mid (\rightarrow weak)$$
$$\neg\neg A, (A \cap B) \longrightarrow$$
$$\mid (\neg \rightarrow)$$
$$(A \cap B) \longrightarrow \neg A$$
$$\mid (\cap \rightarrow)$$
$$A, B \longrightarrow \neg A$$
$$\mid (\rightarrow \neg)$$
$$A, A, B \longrightarrow$$
$$non - axiom$$

Next tree is

$$\mathbf{T2}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (\rightarrow weak)$$

$$\neg\neg A, (A \cap B) \longrightarrow$$

$$\mid (\neg \rightarrow)$$

$$(A \cap B) \longrightarrow \neg A$$

$$\mid (\rightarrow \neg)$$

$$A, (A \cap B) \longrightarrow$$

$$\mid (\cap \rightarrow)$$

$$A, A, B \longrightarrow$$

$$non - axiom$$

Now we decompose the node n_2.

Observe that following Observation 5 we start by decomposing the formula $(A \cap B)$ by the use of the rule $(\cap \rightarrow)$ first.

A decomposition tree that starts at the node n_2 is as follows.

$$\mathbf{T}_{n_2}$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

This proves that the node n_2 is provable in **LI**, i.e.

$$\vdash_{\mathbf{LI}} \quad (A \cap B), \neg\neg A \longrightarrow B.$$

Of course, we have also that the node n_1 is also provable in **LI**, as one can obtain the node n_2 from it by the use of the rule $(exch \rightarrow)$.

Observation 7: the use of structural rules are important and necessary while we search for proofs. Nevertheless we have to use them on the "must" basis and set up some guidelines and priorities for their use.

For example, use of weakining rule discharges the weakening formula, and hence an information that may be essential to the proof.

We should use it only when it is absolutely necessary for the next decomposition steps.

Hence, the use of weakining rule ($\rightarrow weak$) can, and should be restricted to the cases when it leads to possibility of the use of the negation rule ($\neg \rightarrow$).

In the case of the decomposition tree $\mathbf{T1}_{n_1}$ we used it as an necessary step, but still it discharged too much information and we didn't get a proof, when proof of the node existed.

In fact, the first rule in our search should have been the exchange rule, followed by the conjunction rule (no information discharge!) not the weakening (discharge of information) followed by negation rule.

The full proof of the node n_1 is the following.

$$\mathbf{T3}_{n_1}$$

$$\neg\neg A, (A \cap B) \longrightarrow B$$

$$\mid (exch \longrightarrow)$$

$$\mathbf{T2}_A$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$(A \cap B), \neg\neg A \longrightarrow B$$

$$\mid (\cap \rightarrow)$$

$$A, B, \neg\neg A \longrightarrow B$$

$$axiom$$

As a result of the observations 1- 7 we adopt the following heuristics for proof search in **LI**.

Decomposition Tree Generation rules.

1. Use first logical decomposition rules where applicable without the use of $(\rightarrow weak)$.

2. Use $(exch \rightarrow)$ rule to decompose as many formulas on the left side of $\longrightarrow$ as possible.

3. Use $(\rightarrow weak)$ only on a "must" basis in connection with $(\neg \rightarrow)$.

4. Use $(contr \rightarrow)$ rule as the last recourse and only to formulas that contain $\neg$ or $\Rightarrow$ as connectives.

5. Within the process use $(contr \rightarrow)$ rule only a finite number of times, no more times that number of all sub-formulas of the formula you are building the tree for.