Predicate Logic

CSE 595 – Semantic Web

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The alphabet of predicate logic

- Variables
- Constants (identifiers, numbers etc.)
- Functors (identifiers with arity > 0; e.g. **`date/3`**).
- Predicate symbols (identifiers with arity >= 0; e.g. **`append/3`**)
- Connectives:
  - ∧ (conjunction),
  - ∨ (disjunction),
  - ¬ (negation),
  - ↔ (logical equivalence),
  - → (implication).
- Quantifiers: ∀ (universal), ∃ (existential).
- Auxiliary symbols such as parentheses and comma.

Predicate Logic Formulas

- Terms (T) over an alphabet A is the smallest set such that:
  - Every constant **`c`** ∈ A is also **`c`** ∈ T .
  - Every variable **`X`** ∈ A is also **`X`** ∈ T .
  - If **`f/n`** ∈ A and $\mathbf{t_1}$, $\mathbf{t_2}$, …, $\mathbf{t_n}$ ∈ T then $\mathbf{f(t_1, t_2, \ldots, t_n)}$ ∈ T .
- Well-formed formulas (*wffs*, denoted by ***F***) over alphabet A is the smallest set such that:
  - If **`p/n`** is a predicate symbol in **`A`** and $\mathbf{t_1}$, $\mathbf{t_2}$,…, $\mathbf{t_n}$ ∈ T then $\mathbf{p(t_1, t_2, \ldots, t_n)}$ ∈ ***F***.
  - If F,G ∈ ***F*** then so are (¬F), (F ∧ G), (F ∨ G), (F → G) and (F ↔ G)
  - If F ∈ ***F*** and **`X`** is a variable in A then (∀**`X`** F) and (∃**`X`** F) ∈ ***F***.

Bound and Free Variables

- A variable $\mathbf{X}$ is *bound* in formula F if $(\forall \mathbf{X}\ G)$ or $(\exists \mathbf{X}\ G)$ is a sub-formula of F.
- A variable that occurs in F, but is not bound in F is said to be *free* in F.
- A formula F is *closed* if it has no free variables.
- Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ be all the free variables in F. Then
  - $(\forall \mathbf{X}_1\ (\ldots(\forall \mathbf{X}_n\ F)\ldots))$ is the *universal closure* of F, and is denoted by $\forall$F.
  - $(\exists \mathbf{X}_1\ (\ldots(\exists \mathbf{X}_n\ F)\ldots))$ is the *existential closure* of F, and is denoted by $\exists$F.

Interpretation

- An ***interpretation*** I of an alphabet is
  - a non-empty domain D, and
  - a mapping that associates:
    - each constant **`c`** ∈ A with an element $\mathtt{c}_{\mathtt{I}}$ ∈ D
    - each n-ary functor **`f`** ∈ A with an function $\mathtt{f}_{\mathtt{I}} : D^n \rightarrow D$
    - each n-ary predicate symbol **`p`** ∈ A with an relation $\mathtt{p}_{\mathtt{I}} \subseteq D^n$
- For instance, one interpretation of the symbols in our "relations" program is that '*bob*', '*pam*' et. al. are people in some set, and **`parent/2`** is the parent-of relation, etc.
- Another interpretation could be that '*bob*', '*pam*' etc are natural numbers, **`parent/2`** is the greater-than relation, etc.

Valuation

- Given an interpretation I, the semantics of a variable-free (a.k.a. ***ground***) term is clear from I itself:

$$I(f(t_1, t_2, \ldots, t_n)) = f_I(I(t_1), I(t_2), \ldots, I(t_n))$$

- But to attach a meaning to terms with variables, we must first give a meaning to its variables!
  - This is done by a ***valuation***: which is a mapping from variables to the domain D of an interpretation.

$$\varphi = \{X_1 \rightarrow d_1, X_2 \rightarrow d_2, \ \ldots, \ X_n \rightarrow d_n\}$$

$\varphi[\mathbf{X} \longrightarrow d]$ is identical to φ except that it maps $\mathbf{X}$ to d

Semantics of terms

- Terms are given a meaning with respect to a ***valuation:***
  - Given an interpretation I and valuation φ, the meaning of a term $\mathtt{t}$, denoted by $\varphi_{\mathtt{I}}(\mathtt{t})$ is defined as:
    - if $\mathtt{t}$ is a constant $\mathtt{c}$ then $\varphi_{\mathtt{I}}(\mathtt{t}) = \mathtt{c}_{\mathtt{I}}$
    - if $\mathtt{t}$ is a variable $\mathtt{X}$ then $\varphi_{\mathtt{I}}(\mathtt{t}) = \varphi\mathtt{X}$
    - if $\mathtt{t}$ is a structure $\mathtt{f(t_1,\ t_2,...,\ t_n)}$ then $\varphi_{\mathtt{I}}(\mathtt{t}) = f_{\mathtt{I}}(\varphi_{\mathtt{I}}(\mathtt{t}_1), \varphi_{\mathtt{I}}(\mathtt{t}_2), \ldots, \varphi_{\mathtt{I}}(\mathtt{t}_n))$

Example

- Let A be an alphabet containing constant zero, a unary functor **`s`** and a binary functor **`plus`**.
- I, defined as follows, is an interpretation with N (the set of natural numbers) as its domain:
  - $\texttt{zero}_I = 0$
  - $\texttt{s}_I(\texttt{x}) = 1 + \texttt{x}$
  - $\texttt{plus}_I(\texttt{x}, \texttt{y}) = \texttt{x} + \texttt{y}$
- Now, if $\varphi = \{\texttt{X} \rightarrow \texttt{1}\}$, then

$$\varphi_I(\texttt{plus(s(zero),X)}) = \varphi_I(\texttt{s(zero)}) + \varphi_I(\texttt{X})$$

$$= (\texttt{1} + \varphi_I(\texttt{zero})) + \varphi(\texttt{X})$$

$$= (\texttt{1} + \texttt{0}) + \texttt{1} = \texttt{2}$$

Semantics of Well-Formed Formulae

- A formula's meaning is given w.r.t. an interpretation I and valuation φ
  - $I \models \varphi\ p(t_1, t_2, \ldots, t_n)$ iff $(\varphi I(t_1), \varphi I(t_2), \ldots, \varphi I(t_n)) \in p_I$
  - $I \models \varphi\ \neg F$ iff $I \not\models \varphi\ F$
  - $I \models \varphi\ F \wedge G$ iff $I \models \varphi\ F$ and $I \models \varphi\ G$
  - $I \models \varphi\ F \vee G$ iff $I \models \varphi\ F$ or $I \models \varphi\ G$ (or both)
  - $I \models \varphi\ F \rightarrow G$ iff $I \models \varphi\ G$ whenever $I \models \varphi\ F$
  - $I \models \varphi\ F \leftrightarrow G$ iff $I \models \varphi\ F \rightarrow G$ and $I \models \varphi\ G \rightarrow F$
  - $I \models \varphi\ \forall X\ F$ iff $I[X \rightarrow d] \models \varphi\ F$ for every $d \in |\ I\ |$ (domain D of I)
  - $I \models \varphi\ \exists X\ F$ iff $I[X \rightarrow d] \models \varphi\ F$ for some $d \in |\ I\ |$

Semantics of Well-Formed Formulae

- Given a set of closed formulas P, an interpretation I is said to be a *model* of P iff every formula of P is true in I.

Example 1.

- Consider the language with zero as the lone constant, s/1 as the only functor symbol, and a predicate symbol p/1.
- Consider an interpretation I with | I | = N, the set of natural numbers, $zero_I = 0$ and $s_I(x) = 1 + x$
- Now consider the formula:

$$F_1 = p(zero) \wedge (\forall X\ p(s(s(X))) \leftrightarrow p(X))$$

- Find an interpretation for p/1 such that $I \models F_1$.
  - $p_{I1} = \{0, 2, 4, 6, 8, 10, \ldots\}$
  - $p_{I2} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

Example 2.

- Recall Example 1:

$$F_1 = p(zero) \land (\forall X \; p(s(s(X))) \boxed{\leftrightarrow} p(X))$$

- Consider extending the previous example with another predicate symbol q/1, and consider the formula:

$$F_2 = q(s(zero)) \land (\forall X \; q(s(s(X))) \boxed{\leftrightarrow} q(X))$$

- Now extend the previous interpretation such that

$$I \models F_1 \land F_2$$

Example 2.

- $p_{I1} = \{0, 2, 4, 6, 8, 10, \ldots\}$
- $q_{I1} = \{1, 3, 5, 7, 9, 11, \ldots\}$

- $p_{I2} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
- $q_{I2} = \{1, 3, 5, 7, 9, 11, \ldots\}$

- $p_{I3} = \{0, 2, 4, 6, 8, 10, \ldots\}$
- $q_{I3} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

- $p_{I4} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
- $q_{I4} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

Example 3.

- Recall Example 2:

$$F_1 = p(zero) \land (\forall X\ p(s(s(X))) \leftrightarrow p(X))$$

$$F_2 = q(s(zero)) \land (\forall X\ q(s(s(X))) \leftrightarrow q(X))$$

- In the previous example, consider a new formula:

$$F_3 = (\forall X\ q(s(X)) \leftrightarrow p(X))$$

- Now extend the previous interpretation such that

$$I \models F_1 \land F_2 \land F_3$$

Example 3.

- $p_{I1} = \{0, 2, 4, 6, 8, 10, \ldots\}$
- $q_{I1} = \{1, 3, 5, 7, 9, 11, \ldots\}$

- $p_{I4} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
- $q_{I4} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

Interpretations and Consequences

- Is there any interpretation I such that

  $I \models F_1 \land F_2$, but $I \not\models F_3$?

  - Yes:
    - $p_{I2} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$
    - $q_{I2} = \{1, 3, 5, 7, 9, 11, \ldots\}$

    - $p_{I3} = \{0, 2, 4, 6, 8, 10, \ldots\}$
    - $q_{I3} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

Logical Consequence

- Let P and F be closed formulas.
  - F is a *logical consequence* of P (denoted by $P \models F$) iff
  - F is true in every model of P.

Logical Consequence: An Example

1) $(\forall X\ (\forall Y\ (mother\ (X) \land child(Y\ ,X)) \rightarrow loves(X,Y)))$
2) $mother\ (mary) \land child(tom, mary)$

- Is loves(mary, tom) a logical consequence of the above two statements?
  - Yes. Proof:
  - For 1) to be true in some interpretation I:

    $$I \vDash_{\varphi} (mother\ (X) \land child(Y\ ,X)) \rightarrow loves(X,Y)$$

    must hold for any valuation φ.
  - Specifically, for $\varphi = [X \rightarrow mary,\ Y \rightarrow tom]$

    $$I \vDash_{\varphi} (mother(mary) \land child(tom,mary)) \rightarrow loves(mary,tom)$$
  - Hence loves(mary,tom) is true in I if 2) above is true in I.