CSE 548: Analysis of Algorithms

Lectures 9 & 10
(Generating Functions)

Rezaul A. Chowdhury

Department of Computer Science
SUNY Stony Brook
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An Impossible Counting Problem

Suppose you went to a grocery store to buy some fruits. There are some constraints though:

A. The store has only two **apples** left: one red and one green. So you cannot take more than 2 apples.

B. All but 3 **bananas** are rotten. You do not like rotten bananas.

F. **Figs** are sold 6 per pack. You can take as many packs as you want.

M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.

P. They sell 4 **peaches** per pack. Take as many packs as you want.

Now the question is: in how many ways can you buy n fruits from the store?

Generating Functions

Generating functions represent sequences by coding the terms of a sequence as coefficients of powers of a variable in a formal power series.

For example, one can represent a sequence $s_0, s_1, s_2, \ldots$ as:

$$S(z) = s_0 + s_1 z + s_2 z^2 + s_3 z^3 + \cdots + s_n z^n + \ \ldots$$

So s_n is the coefficient of z^n in $S(z)$.

An Impossible Counting Problem

A. The store has only two **apples** left: one red and one green.
So you cannot take more than 2 apples.

$$A(z) = 1 + 2z + z^2 = (1 + z)^2$$

B. All but 3 **bananas** are rotten. You do not like rotten bananas.

$$B(z) = 1 + z + z^2 + z^3 = \frac{1 - z^4}{1 - z}$$

F. **Figs** are sold 6 per pack. You can take as many packs as you want.

$$F(z) = 1 + z^6 + z^{12} + z^{18} + \cdots = \frac{1}{1 - z^6}$$

M. **Mangoes** are sold in pairs. But you must not take more than a pair of pairs.

$$M(z) = 1 + z^2 + z^4 = \frac{1 - z^6}{1 - z^2}$$

P. They sell 4 **peaches** per pack. Take as many packs as you want.

$$P(z) = 1 + z^4 + z^8 + z^{12} + \cdots = \frac{1}{1 - z^4}$$

An Impossible Counting Problem

Suppose you can choose n fruits in s_n different ways.

Then the generating function for s_n is:

$$S(z) = A(z)B(z)F(z)M(z)P(z) = (1+z)^2 \times \frac{1-z^4}{1-z} \times \frac{1}{1-z^6} \times \frac{1-z^6}{1-z^2} \times \frac{1}{1-z^4}$$

$$= \frac{1+z}{(1-z)^2}$$

$$= (1+z)\sum_{n=0}^{\infty}(n+1)z^n$$

$$= \sum_{n=0}^{\infty}(2n+1)z^n$$

Equating the coefficients of z^n from both sides:

$$s_n = 2n+1$$

Fibonacci Numbers

Recurrence for *Fibonacci numbers*:

$$f_n = \begin{cases} 0 & if\ n = 0, \\ 1 & if\ n = 1, \\ f_{n-1} + f_{n-2} & otherwise. \end{cases}$$

$$\Rightarrow f_n = f_{n-1} + f_{n-2} + [n = 1]$$

Generating function: $F(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + \ \ldots$

$$F(z) = \sum_n f_n z^n = \sum_n f_{n-1} z^n + \sum_n f_{n-2} z^n + \sum_n [n = 1] z^n$$

$$= \sum_n f_n z^{n+1} + \sum_n f_n z^{n+2} + z$$

$$= zF(z) + z^2 F(z) + z$$

Fibonacci Numbers

$$F(z) = zF(z) + z^2F(z) + z$$

$$\Rightarrow F(z) = \frac{z}{1 - z - z^2}$$

$$= \frac{z}{(1 - \varphi z)(1 - \hat{\varphi} z)}, \text{ where } \varphi = \frac{1 + \sqrt{5}}{2} \ \& \ \hat{\varphi} = \frac{1 - \sqrt{5}}{2}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1}{1 - \varphi z} - \frac{1}{1 - \hat{\varphi} z}\right)$$

$$= \frac{1}{\sqrt{5}}\sum_{n}(\phi^n - \hat{\phi}^n)z^n$$

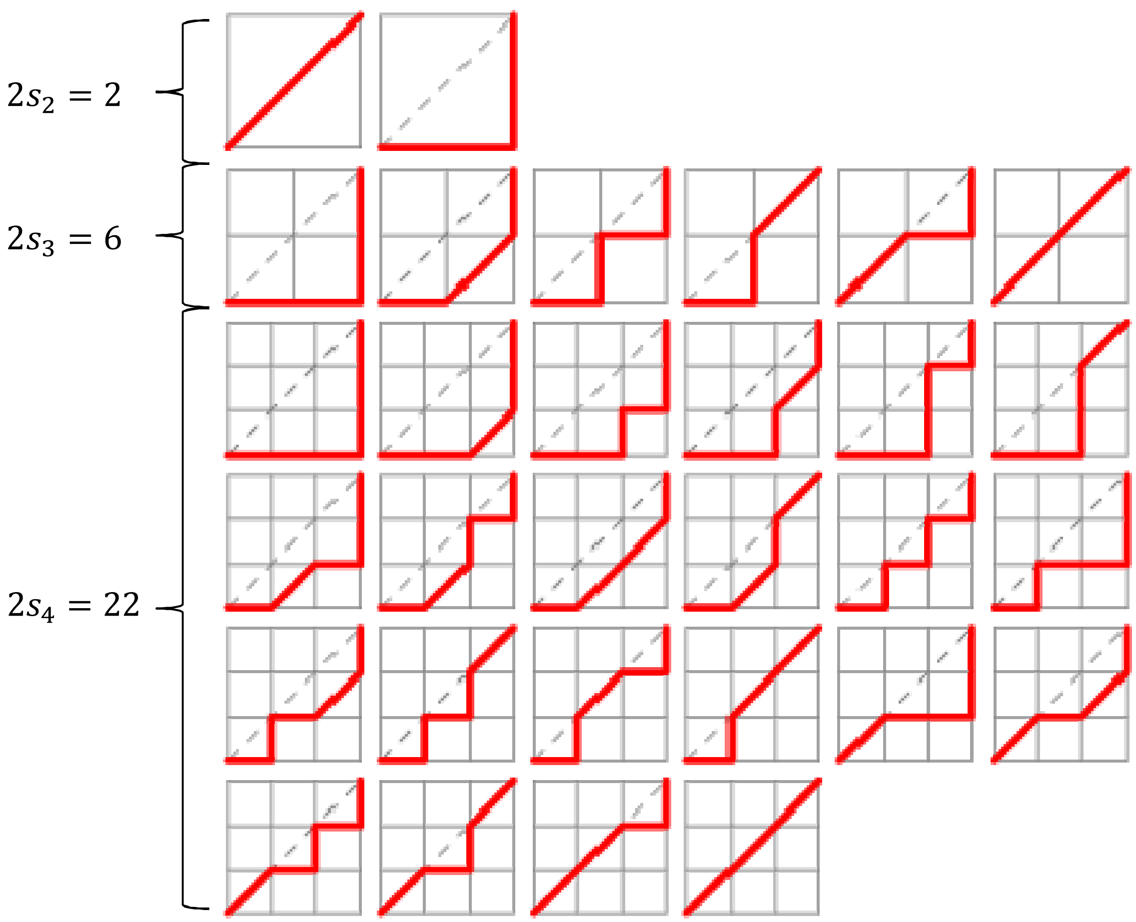
Equating the coefficients of z^n from both sides:

$$f_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n) = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^n$$

Schröder Numbers

For positive integer n:

$$s_n = \begin{cases} 1 & if\ n \leq 2, \\ (6s_{n-1} - s_{n-2}) - \frac{3}{n}(3s_{n-1} - s_{n-2}) & otherwise. \end{cases}$$



For $n \geq 2$, $2s_n$ is the number of lattice paths in the Cartesian plane that start at $(1,1)$, end at (n,n), contain no points above the line $y = x$, and are composed only of steps $(0,1)$, $(1,0)$ and $(1,1)$.

Schröder Numbers

We have: $s_1 = s_2 = 1$

and for $n > 2$: $ns_n - 6ns_{n-1} + 9s_{n-1} + ns_{n-2} - 3s_{n-2} = 0$

$$\Rightarrow 3(3s_{n-1} - s_{n-2}) + n(s_{n-2} - 6s_{n-1} + s_n) = 0$$

Generating function: $S(z) = s_1 z + s_2 z^2 + s_3 z^3 + s_4 z^4 + \ \ldots$

$$\Rightarrow S'(z) = s_1 + 2s_2 z + 3s_3 z^2 + 4s_4 z^3 + \ \ldots$$

$$3zS(z) - z^2S(z) + z^3S'(z) - 6z^2S'(z) + zS'(z)$$

$$= s_1 z - (3s_1 - 2s_2)z^2 + \big(3(3s_2 - s_1) + 3(s_1 - 6s_2 + s_3)\big)z^3$$

$$+ \big(3(3s_3 - s_2) + 4(s_2 - 6s_3 + s_4)\big)z^4 + \cdots$$

$$+ (3(3s_{n-1} - s_{n-2}) + n(s_{n-2} - 6s_{n-1} + s_n))z^n + \cdots$$

$$\Rightarrow 3zS(z) - z^2S(z) + z^3S'(z) - 6z^2S'(z) + zS'(z)$$

$$= s_1 z - (3s_1 - 2s_2)z^2$$

$$\Rightarrow (3 - z)S(z) + (z^2 - 6z + 1)S'(z) + (z - 1) = 0$$

Schröder Numbers

$$(3-z)S(z)+(z^2-6z+1)S'(z)+(z-1)=0$$

$$\Rightarrow \frac{3-z}{(z^2-6z+1)^{\frac{3}{2}}}S(z)+\frac{z^2-6z+1}{(z^2-6z+1)^{\frac{3}{2}}}S'(z)+\frac{z-1}{(z^2-6z+1)^{\frac{3}{2}}}=0$$

$$\Rightarrow \frac{3-z}{(z^2-6z+1)^{\frac{3}{2}}}S(z)+\frac{1}{(z^2-6z+1)^{\frac{1}{2}}}S'(z)=\frac{1-z}{(z^2-6z+1)^{\frac{3}{2}}}$$

$$\Rightarrow \frac{d}{dz}\left(\frac{S(z)}{(z^2-6z+1)^{\frac{1}{2}}}\right)=\frac{1-z}{(z^2-6z+1)^{\frac{3}{2}}}$$

$$\Rightarrow \int_0^z \frac{d}{dz}\left(\frac{S(z)}{(z^2-6z+1)^{\frac{1}{2}}}\right)dz=\int_0^z \frac{1-z}{(z^2-6z+1)^{\frac{3}{2}}}dz$$

$$\Rightarrow \frac{S(z)}{(z^2-6z+1)^{\frac{1}{2}}}=\left[\frac{z+1}{4(z^2-6z+1)^{\frac{1}{2}}}\right]_0^z$$

$$\Rightarrow S(z)=\frac{1}{4}\left(z+1-\sqrt{z^2-6z+1}\right)$$

Schröder Numbers

$$S(z) = \frac{1}{4}\left(z + 1 - \sqrt{z^2 - 6z + 1}\right)$$

$$= z + z^2 + 3z^3 + 11z^4 + 45z^5 + \cdots$$

Equating the coefficients of z^n from both sides:

$$s_n = \sum_{k=0}^{\infty} \frac{(n-1)_{k,-1}(n+2)_{k,+1}}{k!\,(k+1)!},$$

where, $(a)_{k,l} = a(a+l)(a+2l)\dots(a+(k-1)l)$.