CSE 548: Analysis of Algorithms

Lecture 8 (Amortized Analysis)

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<u>A Binary Counter</u>

counter		#bit	#bit resets	#bit sets
value	counter	flips	($1 \rightarrow 0$)	($0 \rightarrow 1$)
0	0000000			
1	000000001	1	0	1
2	000000010	2	1	1
3	0000011	1	0	1
4	000000100	3	2	1
5	00000101	1	0	1
6	00000110	2	1	1
7	00000111	1	0	1
8	000001000	4	3	1
9	000001001	1	0	1
10	000001010	2	1	1
11	00001011	1	0	1
12	000011000	3	2	1
13	00001101	1	0	1
14	0000011100	2	1	1
15	0000011111	1	0	1
16	00010000	5	4	1

A Binary Counter

Consider a k-bit counter initialized to 0 (i.e., all bits are 0's). Suppose we increment the counter n times. and cost of an increment = #bits flipped

Question: What is the worst-case total cost of *n* increments?

Worst-case cost of a single increment:

#bit sets (0 \rightarrow 1), $b_1 \leq 1$ #bit resets (1 \rightarrow 0), $b_0 \leq k - b_1$ #bit flips $= b_1 + b_0 \leq k$

Worst-case cost of *n* increments:

#bit flips $\leq nk$

This turns out to be a very loose upper bound!

Aggregate Analysis

A better upper bound can be obtained as follows.

Each increment sets ($0 \rightarrow 1$) at most one bit, i.e., $b_1 \leq 1$ So, total number of bits set by n increments, $B_1 = b_1 n \leq n$

Since at most n bits are set, there cannot be more than n bit resets ($1 \rightarrow 0$), i.e., $B_0 \leq B_1 \leq n$

So, total number of bit flips $= B_1 + B_0 \le n + n = 2n$

Thus worst-case cost of a sequence of n increments, $T(n) \leq 2n$

Hence, in the worst case, average cost of an increment $=\frac{T(n)}{n} \le 2$

This *worst-case average cost* is called the *amortized cost* of an increment in a sequence of *n* increments.

<u>A Binary Counter</u>

counter		#bit	#bit resets		total
value	counter	flips	$(1 \rightarrow 0)$	$(0 \rightarrow 1)$	#bit flips
0	00000000000				
1	000000000	1	0	1	1
2	0000000010	2	1	1	3
3	000000011	1	0	1	4
4	000000100	3	2	1	7
5	000000101	1	0	1	8
6	00000110	2	1	1	10
7	00000111	1	0	1	11
8	00001000	4	3	1	15
9	00001001	1	0	1	16
10	00001010	2	1	1	18
11	00001011	1	0	1	19
12	000011000	3	2	1	22
13	00001101	1	0	1	23
14	000011100	2	1	1	25
15	000011111	1	0	1	26
16	000100000	5	4	1	31

Amortized Analysis

- often obtains a tighter worst-case upper bound on the cost of a sequence of operations on a data structure by reasoning about the interactions among those operations
- the actual cost of any given operation may be very high, but that operation may change the state of the data structure in such a way that similar high-cost operations cannot appear for a while
- tries to show that there must be enough low-cost operations in the sequence to average out the impact of high-cost operations
- unlike average case analysis proves a worst-case upper bound on the total cost of the sequence of operations
- unlike expected case analysis no probabilities are involved

Accounting Method (Banker's View)

Consider a k-bit counter initialized to 0 (i.e., all bits are 0's).

Worst-case cost of a single increment:

#bit sets (0
$$\rightarrow$$
 1), $b_1 \leq 1$
#bit resets (1 \rightarrow 0), $b_0 \leq k - b_1$
#bit flips
$$= b_1 + b_0 \leq k$$

Thus each increment is paying for the bit it sets (fair).

But also paying for resetting bits set by prior increments (unfair)!

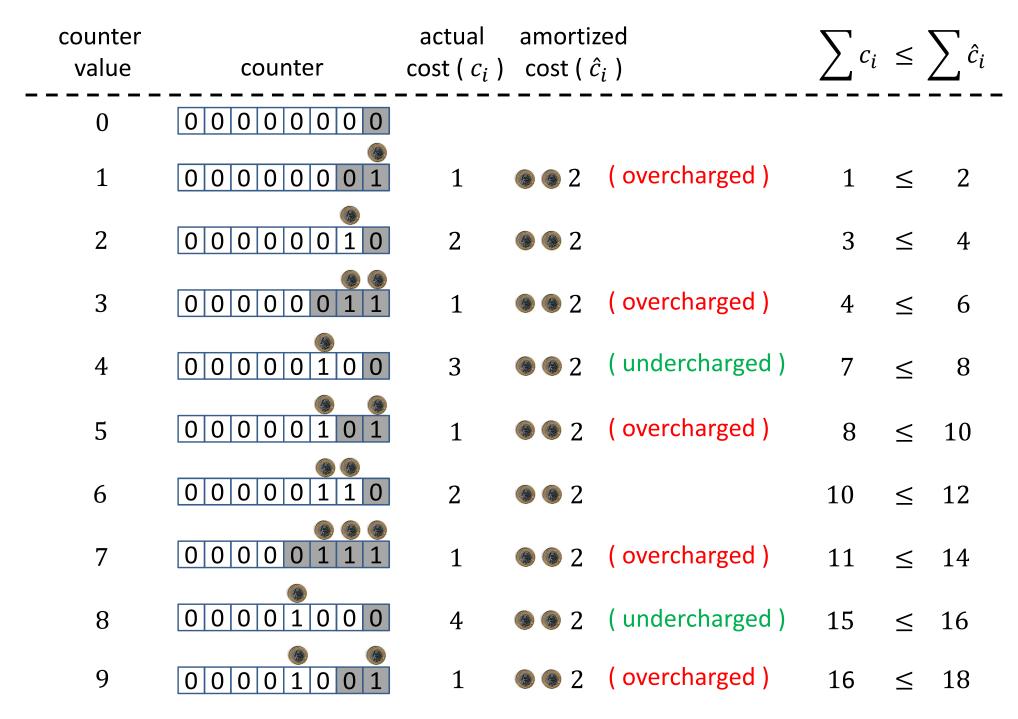
A fairer cost accounting for each increment:

(1) Pay for the bit it sets.

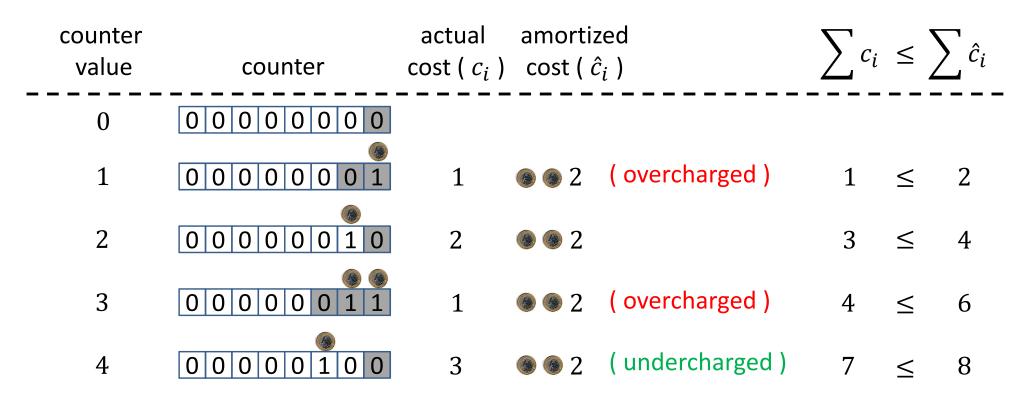
(2) Pay in advance for resetting this bit (by some other increment) in the future. Store this advanced payment as a *credit* associated with that bit position.

(3) When resetting a bit use the credit stored in that bit position.

Accounting Method (Banker's View)



Accounting Method (Banker's View)



Total credits remaining after *n* increments, $\Delta_n = \sum_{i=1}^n \hat{c}_i - \sum_{i=1}^n c_i$

We must make sure that for all $n, \Delta_n \ge 0$

$$\Rightarrow \sum_{i=1}^{n} \hat{c}_i \ge \sum_{i=1}^{n} c_i$$

This will ensure that *the total amortized cost is always an upper bound on the total actual cost*.

Potential Method (Physicist's View)

Banker's View: Store prepaid work as credit with specific objects in the data structure.

Physicist's View: Represent total remaining credit in the data structure as a single potential function.

Suppose: state of the initial data structure = D_0 state of the data structure after the *i*-th operation = D_i potential associated with D_i is = $\Phi(D_i)$

Then amortized cost of the *i*-th operation,

 \hat{c}_i = actual cost + potential change due to that operation = $c_i + \Phi(D_i) - \Phi(D_{i-1})$

Potential Method (Physicist's View)

Then amortized cost of the *i*-th operation,

$$\hat{c}_i$$
 = actual cost + potential change due to that operation
= $c_i + \Phi(D_i) - \Phi(D_{i-1})$

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} (c_i + \Phi(D_i) - \Phi(D_{i-1})) = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$$

Since we do not know n in advance, if we make sure that for all n, $\Phi(D_n) \ge \Phi(D_0)$, we ensure that always $\sum_{i=1}^n \hat{c}_i \ge \sum_{i=1}^n c_i$.

In other words, in that case, the total amortized cost will always be an upper bound on the total actual cost.

One way of achieving that is to find a Φ such that $\Phi(D_0) = 0$ and for all $n, \Phi(D_n) \ge 0$.

Potential Method (Physicist's View)

For the binary counter,

 $\Phi(D_i)$ = number of set bits (i.e., 1 bits) after the *i*-th operation

counter value	counter	actual cost (c_i)	$\Phi(D_i)$	amortized cost (\hat{c}_i)	$\sum c_i$	\leq	$\sum \hat{c}_i$
0	000000000	<u> </u>	\int_{0}^{0}				
1	000000001	1		2 (overcharged)	1	\leq	2
2	0000000010	2	1	3 2	3	\leq	4
3	00000011	1	2	2 (overcharged)	4	\leq	6
4	000000100	3	1		7	\leq	8
5	00000101	1	2	2 (overcharged)	8	\leq	10
6	00000110	2	2	3 2	10	\leq	12
7	00000111	1	3	2 (overcharged)	11	\leq	14
8	00001000	4	$\int 1$		15	\leq	16