CSE 548: Analysis of Algorithms

Lecture 6.5
(Linear Recurrences with Constant Coefficients)

Rezaul A. Chowdhury
Department of Computer Science
SUNY Stony Brook
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Linear Homogeneous Recurrence

A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real constants, and $c_k \neq 0$.

For constant r, $a_n = r^n$ is a solution of the recurrence relation iff:

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$
$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

The equation above is called the *characteristic equation* of the recurrence, and its roots are called *characteristic roots*.

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$,

Characteristic Equation: $r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0$

If the characteristic equation has k distinct roots $r_1, r_2, \ldots, r_k$,
then a sequence $\{a_n\}$ is a solution of the recurrence relation iff

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n \text{ for integers } n \geq 0,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Characteristic Equation: $r^2 - c_1 r - c_2 = 0$

$\boldsymbol{a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \Rightarrow \{a_n\}}$ is a solution to the recurrence:

$r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$

$$
\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1\left(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}\right) + c_2\left(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}\right) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n
\end{aligned}
$$

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Characteristic Equation: $r^2 - c_1 r - c_2 = 0$

$\{a_n\}$ is a solution to the recurrence $\Rightarrow a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

Assume initial conditions: $a_0 = C_0$ and $a_1 = C_1$

$$a_0 = C_0 = \alpha_1 + \alpha_2$$
$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$$

Solving: $\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$ and $\alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}$

Since the initial conditions uniquely determine the sequence, it follows that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Linear Homogeneous Recurrence

Recurrence for *Fibonacci numbers*:

$$f_n = \begin{cases} 0 & if\ n = 0, \\ 1 & if\ n = 1, \\ f_{n-1} + f_{n-2} & otherwise. \end{cases}$$

Characteristic equation: $r^2 - r - 1 = 0$

Characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$

Then for constants α_1 and α_2: $f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

Initial conditions: $f_0 = \alpha_1 + \alpha_2 = 0$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Constants: $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$

Solution: $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$,

Characteristic Equation: $r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0$

If the characteristic equation has t distinct roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$, respectively, so that all m_i's are positive and $\sum_{1 \le i \le t} m_i = k$, then a sequence $\{a_n\}$ is a solution of the recurrence relation iff

$$
\begin{aligned}
a_n = & \left(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}\right) r_1^n \\
& + \left(\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}\right) r_2^n \\
& + \cdots + \left(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}\right) r_t^n \quad \text{for integers } n \ge 0,
\end{aligned}
$$

where $\alpha_{i,j}$ are constants for $1 \le i \le t$ and $0 \le j \le m_i - 1$.

Linear Homogeneous Recurrence

$$a_n = \begin{cases} 1 & if\ n = 0, \\ 6 & if\ n = 1, \\ 6a_{n-1} - 9a_{n-2} & otherwise. \end{cases}$$

Characteristic equation: $r^2 - 6r + 9 = 0$

Characteristic root: $r = 3$

Then for constants α_1 and α_2: $a_n = \alpha_1 3^n + \alpha_2 n 3^n$

Initial conditions: $a_0 = \alpha_1 = 1$

$$a_1 = 3\alpha_1 + 3\alpha_2 = 6$$

Constants: $\alpha_1 = 1$ and $\alpha_2 = 1$

Solution: $a_n = 3^n(n + 1)$

Linear Homogeneous Recurrence

$$a_n = \begin{cases} 2 & \textit{if } n = 0, \\ 7 & \textit{if } n = 1, \\ a_{n-1} + 2a_{n-2} & \textit{otherwise}. \end{cases}$$

$$= 3 \cdot 2^n - (-1)^n$$

$$a_n = \begin{cases} 2 & \textit{if } n = 0, \\ 5 & \textit{if } n = 1, \\ 15 & \textit{if } n = 2, \\ 6a_{n-1} - 11a_{n-2} + 6a_{n-3} & \textit{otherwise}. \end{cases}$$

$$= 1 - 2^n + 2 \cdot 3^n$$

$$a_n = \begin{cases} 1 & \textit{if } n = 0, \\ -2 & \textit{if } n = 1, \\ -1 & \textit{if } n = 2, \\ -3a_{n-1} - 3a_{n-2} - a_{n-3} & \textit{otherwise}. \end{cases}$$

$$= \left(1 + 3n - 2n^2\right)(-1)^n$$

Linear Nonhomogeneous Recurrence

A *linear nonhomogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real constants, $c_k \neq 0$, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the *associated homogeneous recurrence relation*.

Linear Nonhomogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$,

Suppose $\left\{a_n^{(p)}\right\}$ is a particular solution of the recurrence above,

and $\left\{a_n^{(h)}\right\}$ is a solution of the associated homogeneous recurrence.

Then every solution of the given nonhomogeneous recurrence is of

the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$.

Linear Nonhomogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$,

Suppose $F(n) = \left(b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0\right) s^n$,

where $b_0, b_1, \dots, b_t$ and s are real numbers.

If s is not a solution of the characteristic equation of the associated homogeneous recurrence, then there is an $a_n^{(p)}$ of the form:

$$\left(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0\right) s^n.$$

If s is a solution of the characteristic equation and its multiplicity is m, then there is an $a_n^{(p)}$ of the form:

$$n^m \left(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0\right) s^n.$$

Linear Nonhomogeneous Recurrence

$$a_n = \begin{cases} 3 & if\ n = 1, \\ 3a_{n-1} + 2n & otherwise. \end{cases}$$

Associated homogeneous equation: $a_n = 3a_{n-1}$

Homogeneous solution: $a_n^{(h)} = \alpha 3^n$

Particular solution of nonhomogeneous recurrence: $a_n^{(p)} = p_1 n + p_0$

Then $p_1 n + p_0 = 3(p_1(n-1) + p_0) + 2n$

$$\Rightarrow (2 + 2p_1)n + (2p_0 - 3p_1) = 0 \Rightarrow p_1 = -1, p_0 = -\tfrac{3}{2}$$

Solution: $a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n$

$$a_1 = 3 \Rightarrow \alpha = \frac{11}{6}$$

Hence $a_n = -n - \frac{3}{2} + \frac{11}{6} \cdot 3^n$