CSE 548: Analysis of Algorithms

Prerequisites Review 6
(Greedy Algorithms)

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An Activity-Selection Problem

Suppose:

- You are given a set $S = \{a_1, a_2, \dots, a_n\}$ of n proposed *activities* that wish to use a resource, such as a lecture hall, which can serve only one activity at a time.
- Each activity a_i has a *start time* s_i and *finish time* f_i, where $0 \leq s_i < f_i < \infty$. If selected, activity a_i takes place during the half-open time interval $[s_i, f_i)$.
- Activities a_i and a_j are *compatible* if the intervals $[s_i, f_i)$ and $[s_j, f_j)$ do not overlap. That is, a_i and a_j are compatible if $s_i \geq f_j$ or $s_j \geq f_i$.

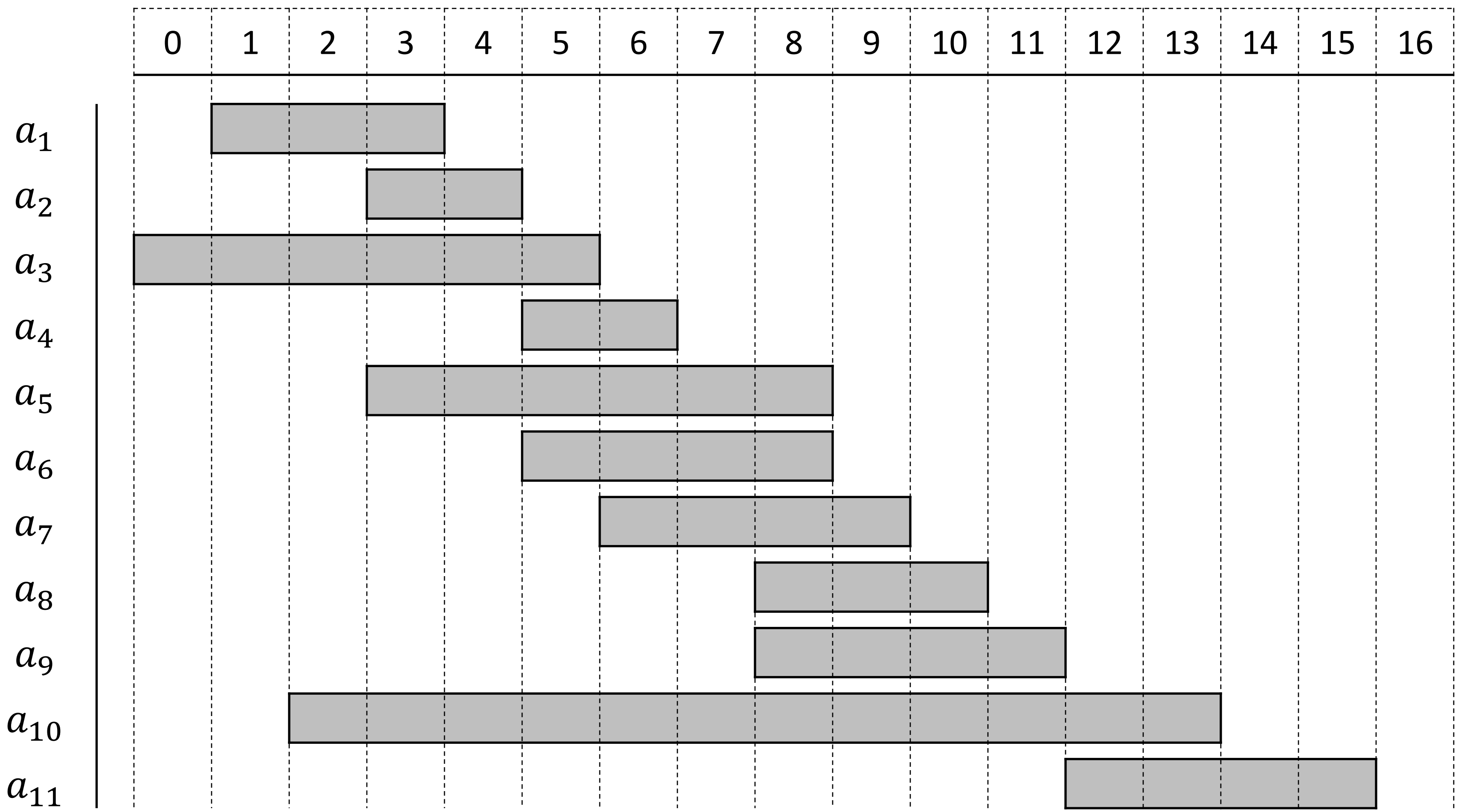
Goal: Select a maximum-size subset of mutually compatible activities.

Assume that the activities are sorted in monotonically increasing order of finish time: $f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_{n-1} \leq f_n$.

An Activity-Selection Problem

An example set S of activities

i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16



An Activity-Selection Problem

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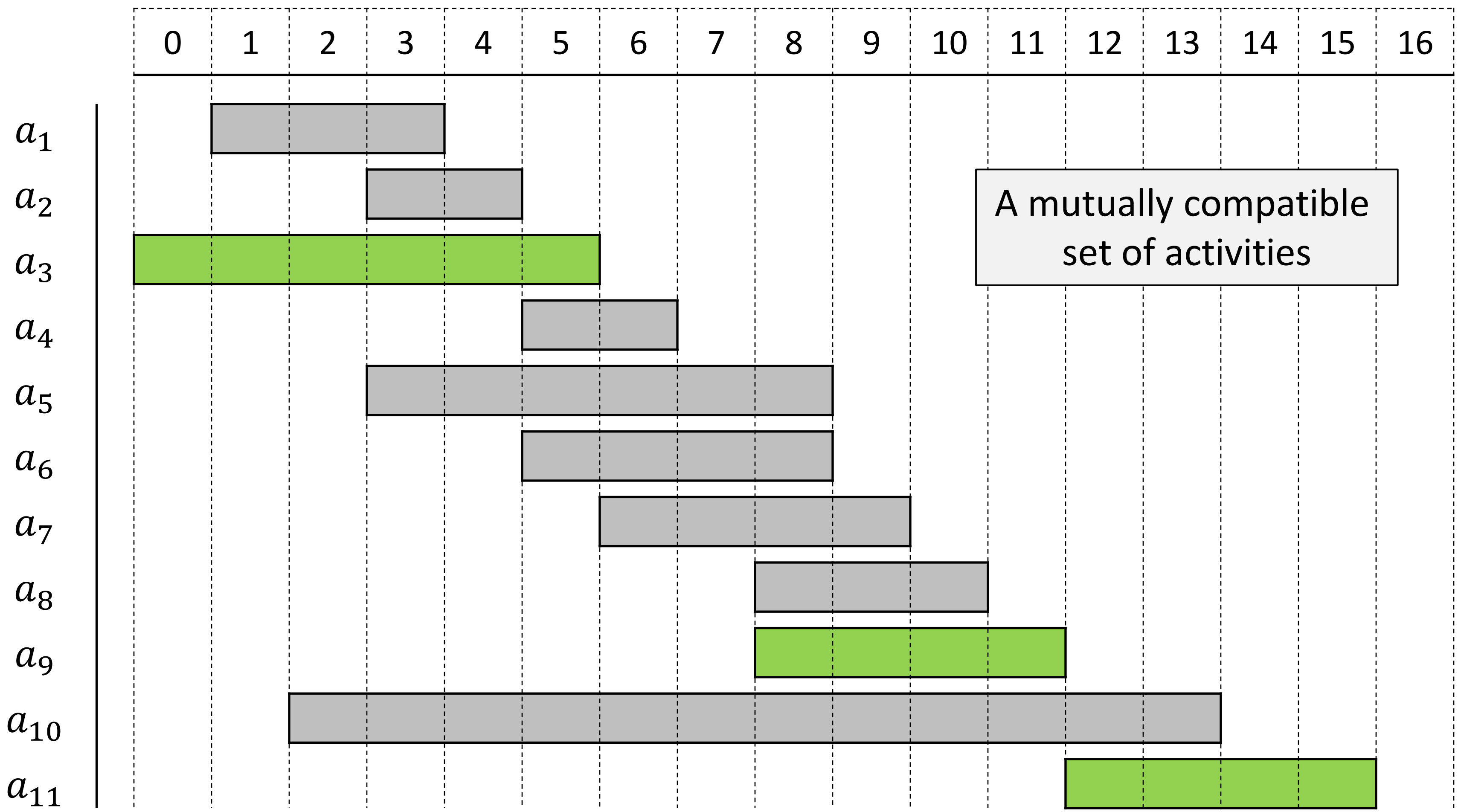

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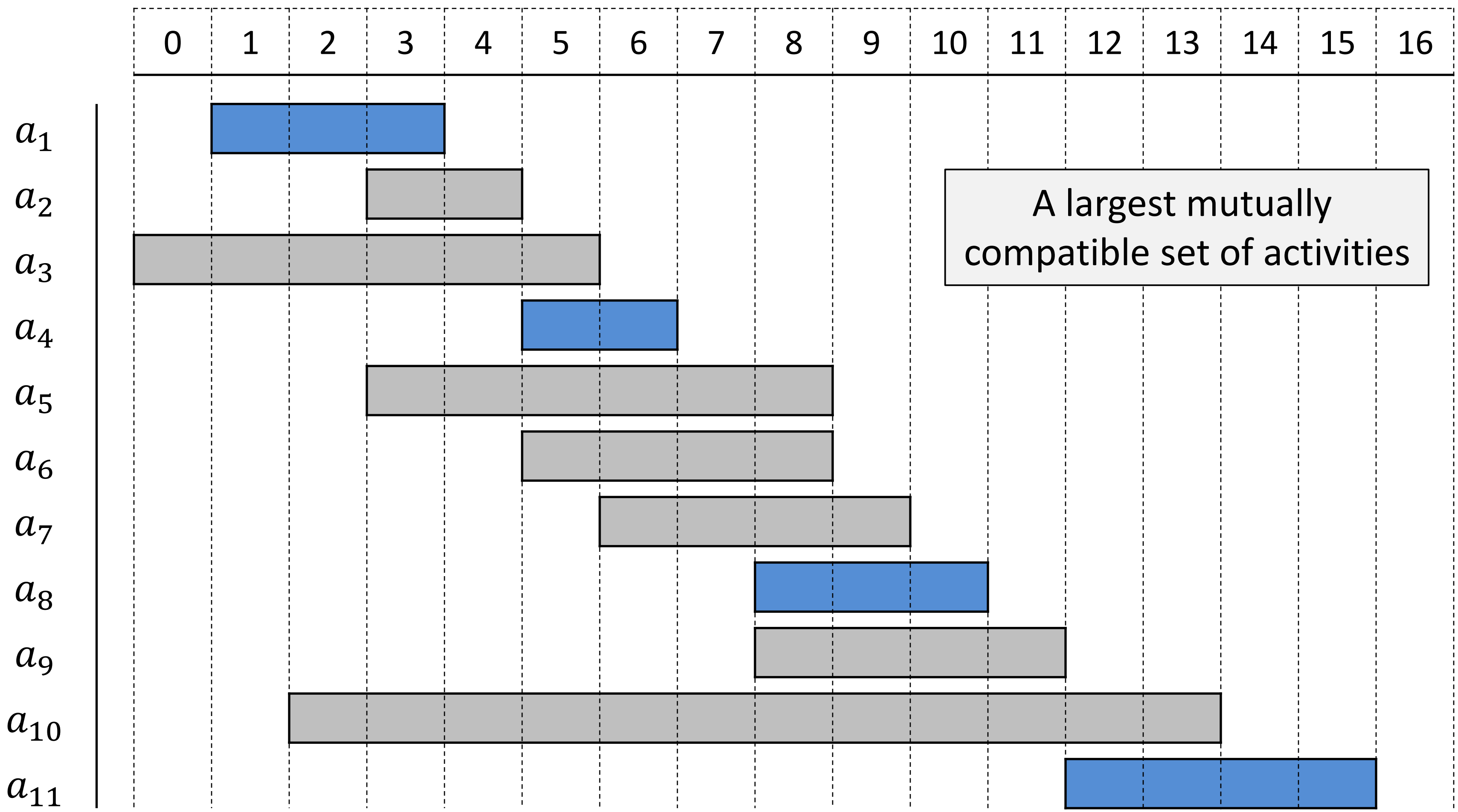

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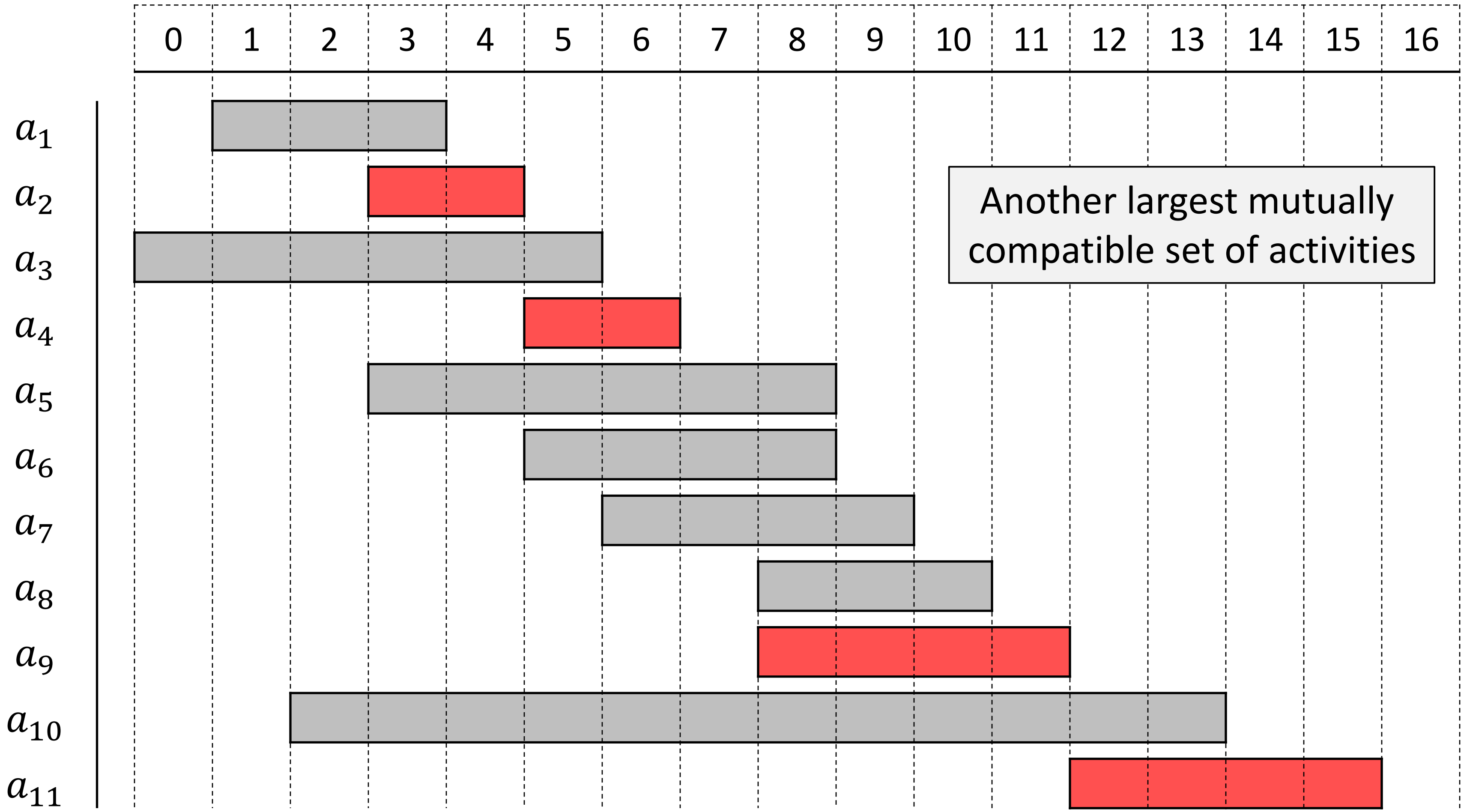

Activity-Selection: Optimal Substructure

Let S_{ij} = set of activities that start after a_i finishes and finishes before a_j starts

A_{ij} = a maximum set of mutually compatible activities in S_{ij}, which includes some activity a_k

Now by including a_k in an optimal solution we are left with the following two subproblems:

- finding mutually compatible activities in S_{ik}
- finding mutually compatible activities in S_{kj}

Let $A_{ik} = A_{ij} \cap S_{ik}$ and $A_{kj} = A_{ij} \cap S_{kj}$.

Then $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$ and $\left|A_{ij}\right| = |A_{ik}| + \left|A_{kj}\right| + 1$.

The cut-and-paste argument shows that the optimal solution A_{ij} must also include optimal solutions to subproblems for S_{ik} and S_{kj} .

Activity-Selection: Recurrence Relation

We have, $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$ and $|A_{ij}| = |A_{ik}| + |A_{kj}| + 1$.

Let $c[i, j] =$ size of an optimal solution for the set S_{ij}.

Then

$$c[i,j] = \begin{cases} 0, & if\ S_{ij} = \emptyset, \\ \max\limits_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\}, & if S_{ij} \neq \emptyset. \end{cases}$$

Hence, we can use either recursion with memorization or bottom-up dynamic programming to solve the problem in $\Theta(n^3)$ time.

Can we do better?

Activity-Selection: Improvement (Greedy Choice)

$$c[i,j] = \begin{cases} 0, & if\ S_{ij} = \emptyset, \\ \max\limits_{a_k \in S_{ij}} \{c[i,k] + c[k,j] + 1\}, & if S_{ij} \neq \emptyset. \end{cases}$$

Instead of iterating over all $a_k \in S_{ij}$ and checking solutions to subproblems for S_{ik} and S_{kj} to find the optimal a_k, can we find the optimal a_k without even solving the subproblems?

Observe that among the activities we choose for our solution, one must be the first one to finish. Intuitively, therefore, we should choose the activity in the input with the earliest finish time, since that would leave the resource available for as many of the activities that follow it as possible.

Activity-Selection: Improvement (Greedy Choice)

Let's consider choosing the activity in the input with the earliest finish time.

Since the activities set in the input $S = \{a_1, a_2, \dots, a_n\}$ sorted in monotonically increasing order of finish time, i.e., $f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_{n-1} \leq f_n$, we should choose a_1 to be in our solution.

Let $S_k = \{a_i \in S | s_i \geq f_k\}$, i.e., the set of activities that start after activity a_k finishes.

If we make the greedy choice of activity a_1, then S_1 remains as the only subproblem to solve.

Optimal substructure tells us that if a_1 is in the optimal solution, then an optimal solution to the original problem consists of activity a_1 and all the activities in an optimal solution to the subproblem S_1.

But is the intuition correct?

Activity-Selection: Improvement (Greedy Choice)

THEOREM: Consider any nonempty subproblem S_k, and let a_m be an activity in S_k with the earliest finish time. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k.

PROOF: Let $A_k =$ a maximum-size subset of mutually compatible activities in S_k.

Let a_j be the activity in A_k with the earliest finish time.

If $a_j = a_m$, we are done, since we have shown that a_m is in some maximum-size subset of mutually compatible activities of S_k.

If $a_j \neq a_m$, let $A'_k = A_k - \{a_j\} \cup \{a_m\}$.

The activities in A'_k are disjoint because the activities in A_k are disjoint, a_j is the first activity in A_k to finish, and $f_m \leq f_j$.

Since $|A'_k| = |A_k|$, we conclude that A'_k is a maximum-size subset of mutually compatible activities of S_k, and it includes a_m.

Activity-Selection: Recursive Algorithm

RECURSIVE-ACTIVITY-SELECTOR (s, f, k, n)

$m \quad k + 1$

while $m \leq n$ *and* $s[m] < f[k]$

$\quad m \leftarrow m + 1$

if $m \leq n$ *then*

$\quad$ *return* $\{a_m\} \cup$ RECURSIVE-ACTIVITY-SELECTOR (s, f, m, n)

else return $\emptyset$

Greedy Activity Selection

An example set S of activities

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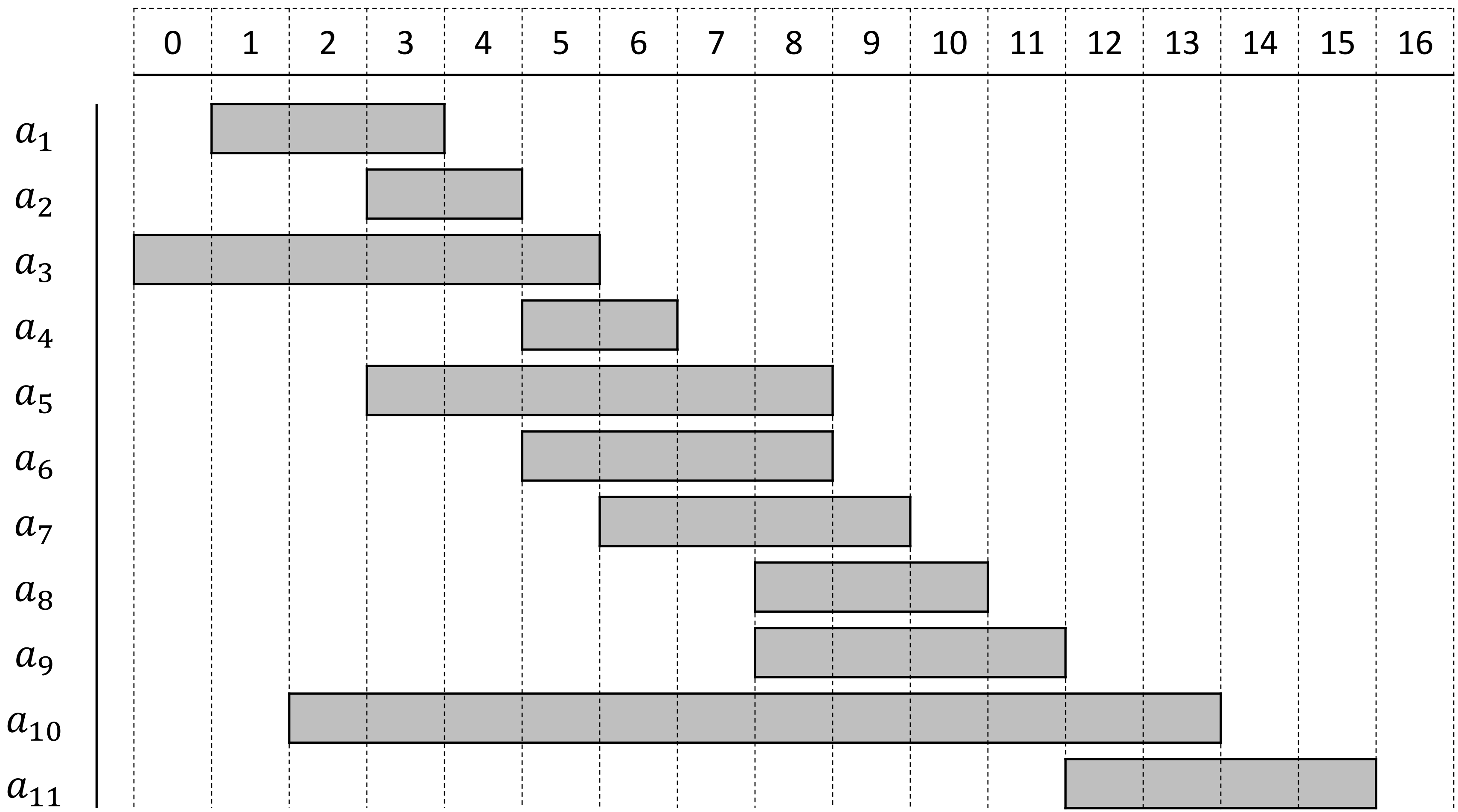

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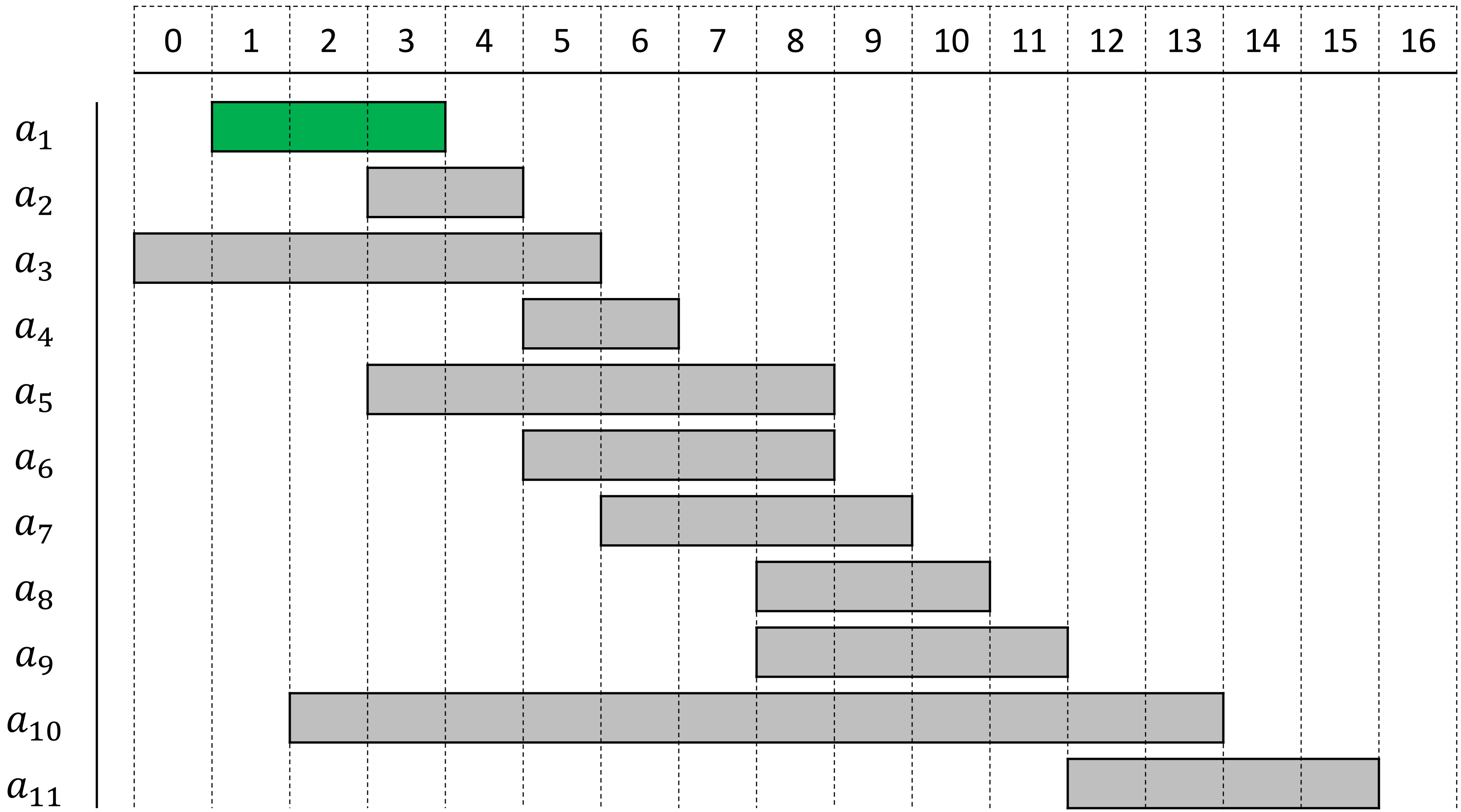

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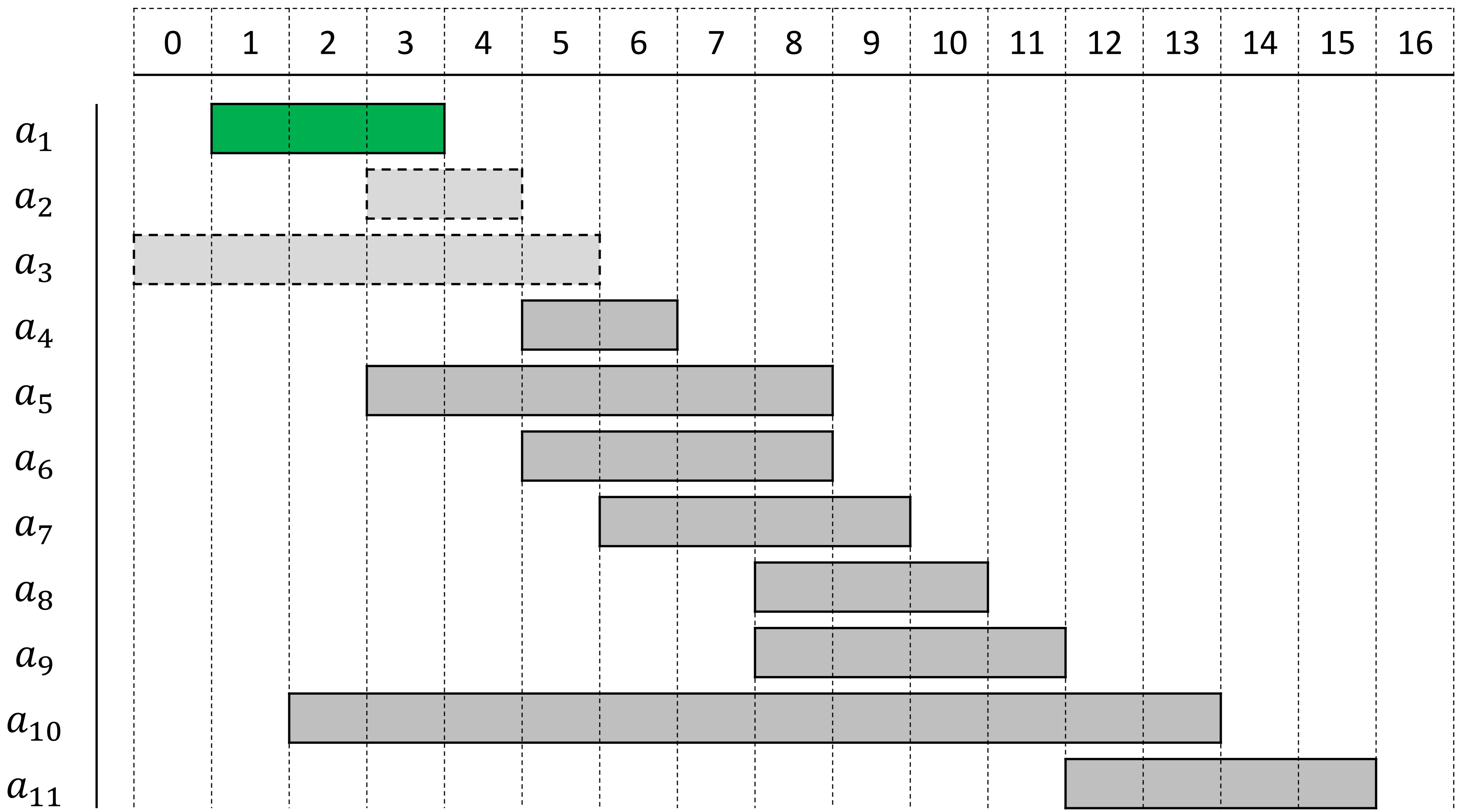

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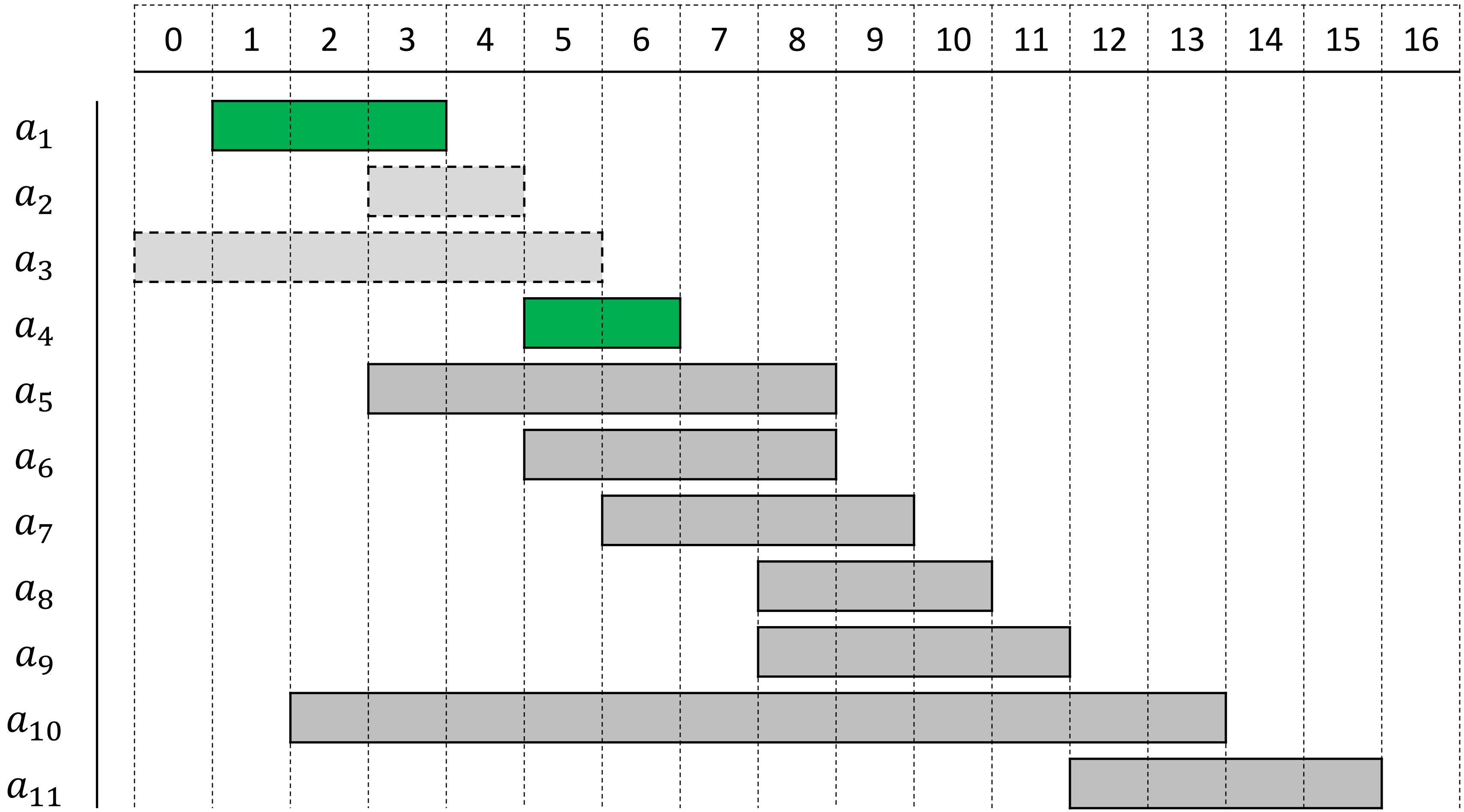

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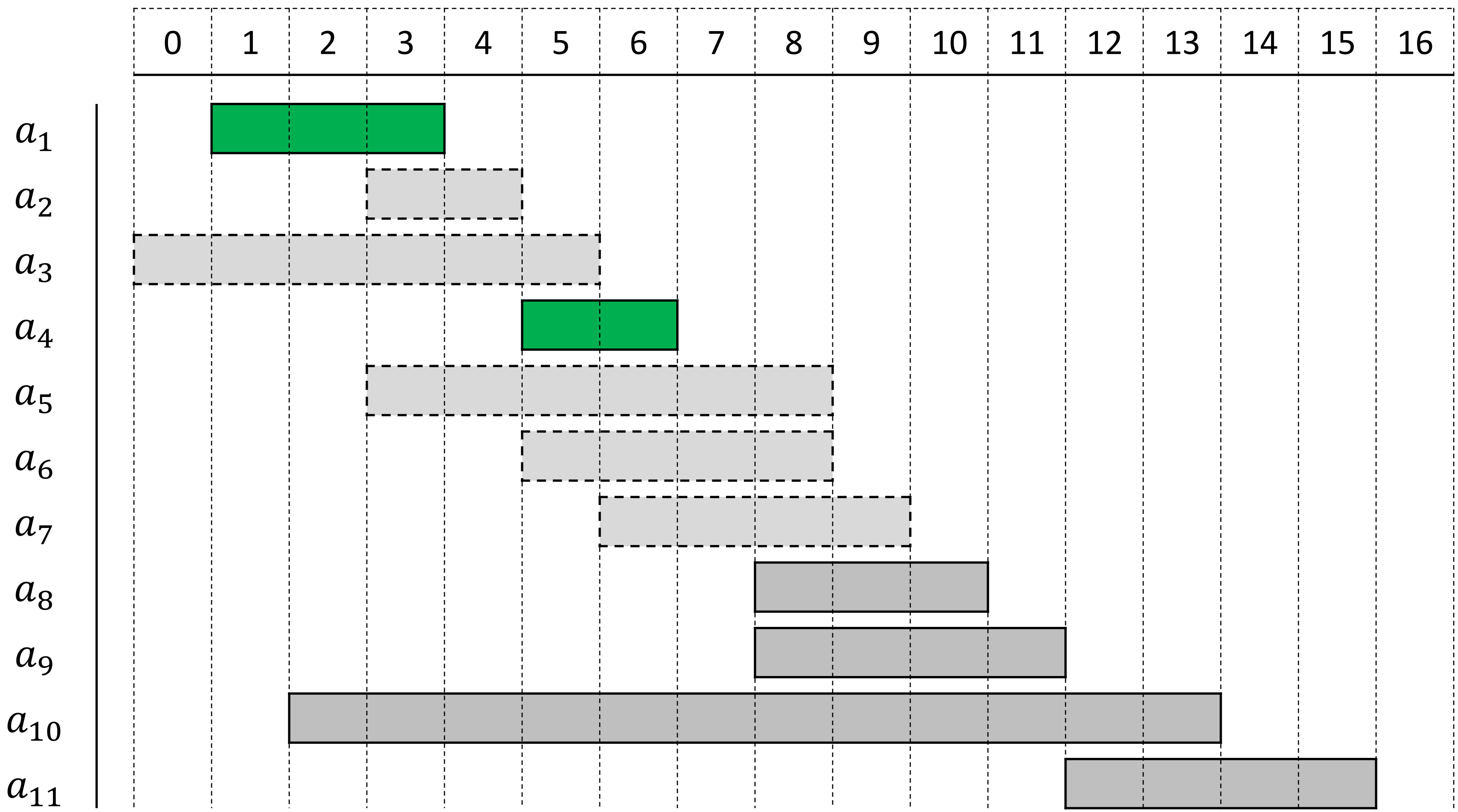

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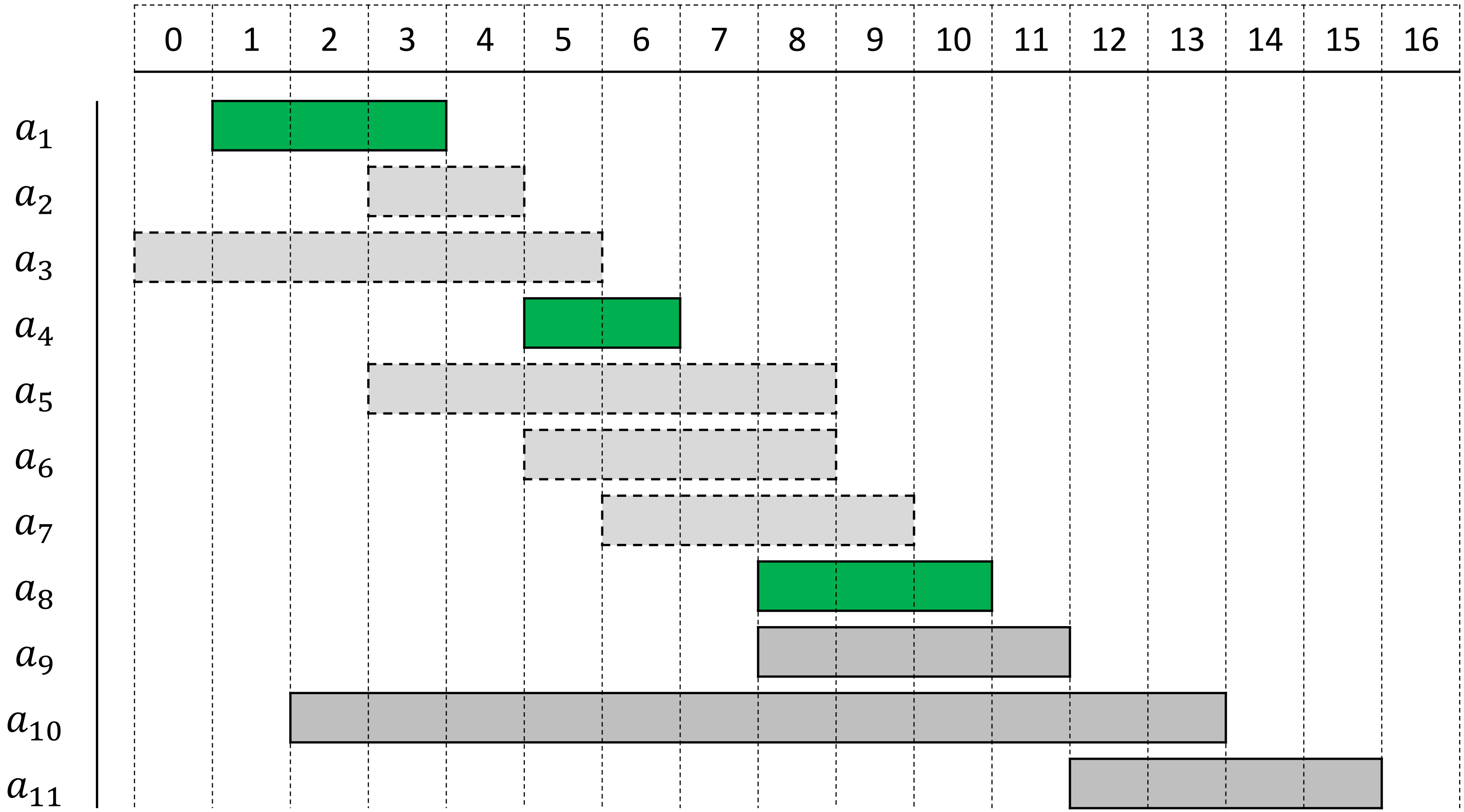

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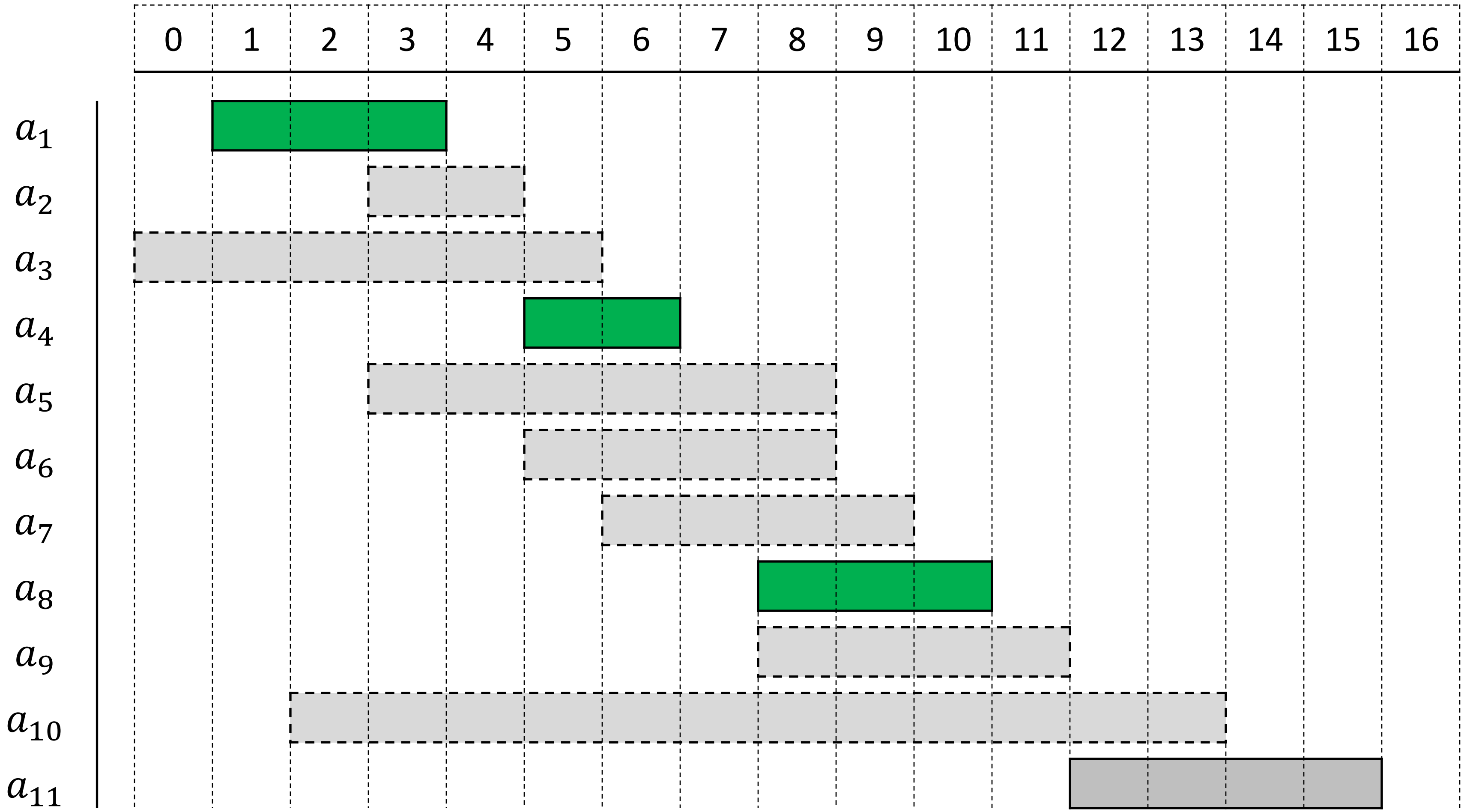

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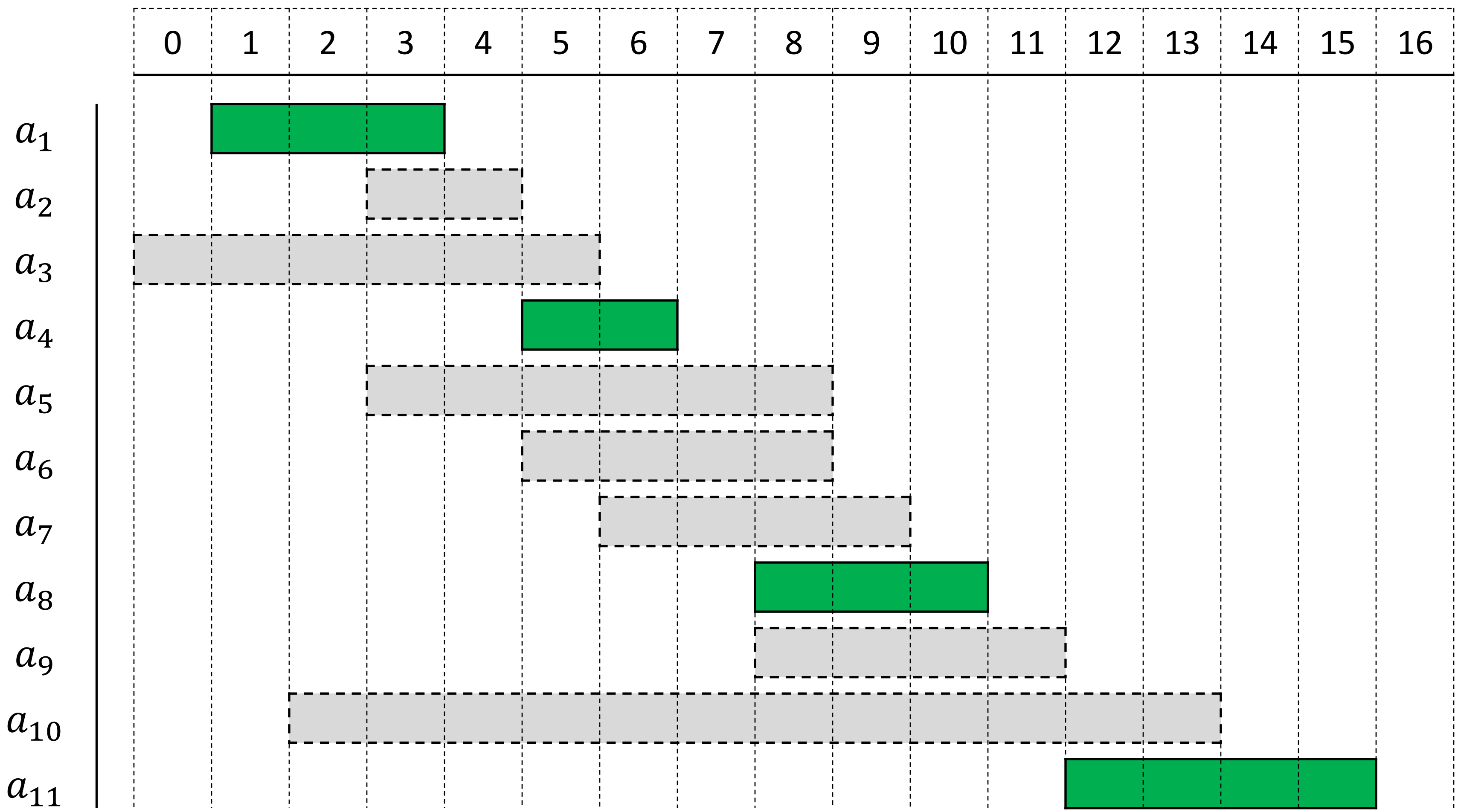


Activity-Selection: Iterative Algorithm

```
GREEDY-ACTIVITY-SELECTOR ( s, f )
    n  s.length
    A    {a_1}
    k ← 1
    for m    2 to n do
        if s[m] ≥ f[k] then
            A    A ∪ {a_m}
            k ← m
    return A
```

Running time $= \Theta(n)$

The Minimum Spanning Tree (MST) Problem

We are given a weighted connected undirected graph $G = (V, E)$ with vertex set V and edge set E, and a weight function w such that for each edge $(u, v) \in E$, $w(u, v)$ represents its weight.

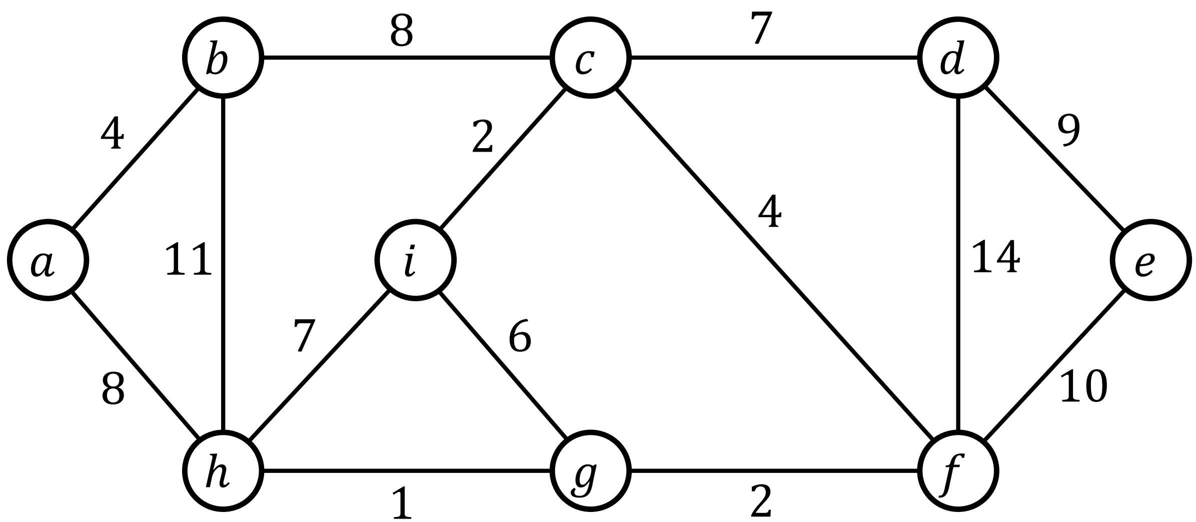
Our goal is to find an acyclic subset $T \subseteq E$ that connects all vertices of V and whose total weight $w(T) = \sum_{(u,v)\in T} w(u, v)$ is minimized.

Since T is acyclic and connects all of the vertices, it must form a tree, which we call a ***spanning tree*** since it "spans" the graph G.

We call the problem of determining the tree T the ***minimum-spanning-tree problem***.

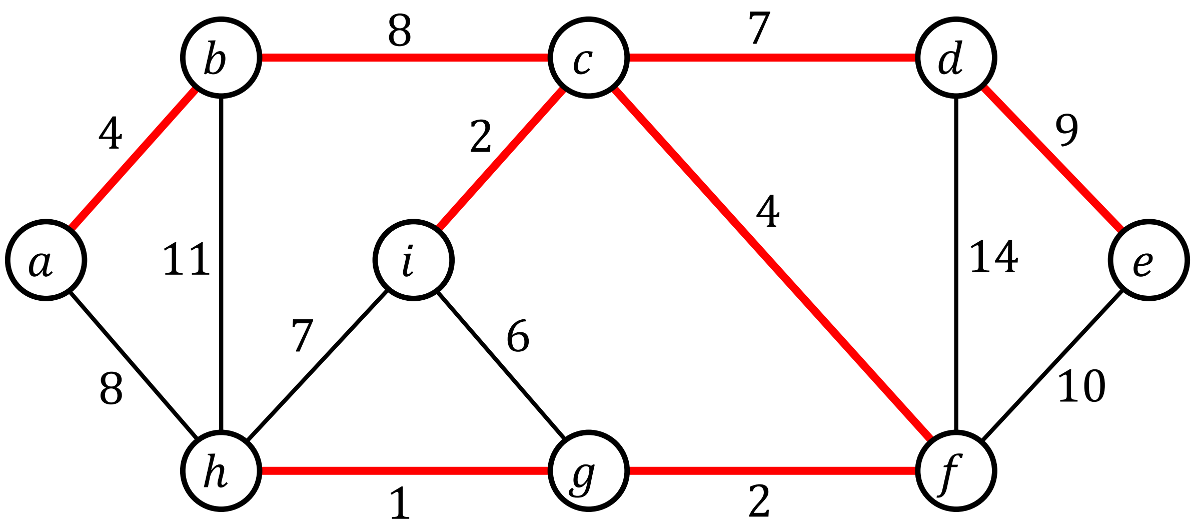
The Minimum Spanning Tree (MST) Problem

A weighted undirected graph



The Minimum Spanning Tree (MST) Problem

A weighted undirected graph



Its MST (in red) of total weight 37

MST: Greedy Strategy for Growing an MST

We are given a weighted connected undirected graph $G = (V, E)$ with vertex set V and edge set E, and a weight function w such that for each edge $(u, v) \in E$, $w(u, v)$ represents its weight.

Suppose set $A \subset E$ is a subset of some MST of G.

Now if edge $(u, v) \in E$ but edge $(u, v) \notin A$, we call (u, v) a ***safe edge*** provided $A \cup \{u, v\}$ is also a subset of an MST of G.

MST: Greedy Strategy for Growing an MST

Generic-MST ($G = (V, E),\ w$)

1. $A \leftarrow \emptyset$
2. *while* A does not form a spanning tree of G *do*
3. find an edge $(u, v) \in E$ that is safe for A
4. $A \leftarrow A \cup \{(u, v)\}$
5. *return* A

MST: Finding Safe Edges

A ***cut*** $(S, V \setminus S)$ of an undirected graph $G = (V, E)$ is a partition of V.

We say that an edge $(u, v) \in E$ ***crosses*** the cut $(S, V \setminus S)$ if one of its endpoints is in S and the other is in $V \setminus S$.

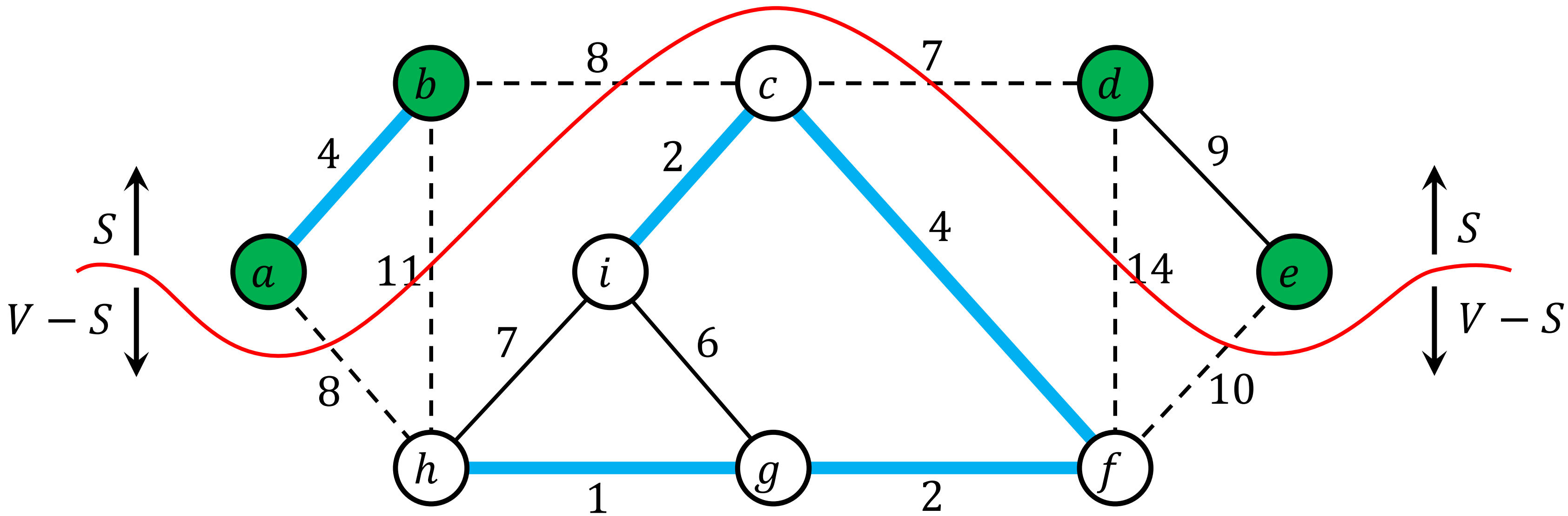
We say that a cut ***respects*** a set A of edges if no edge in A crosses the cut.

An edge is a ***light edge*** crossing a cut if its weight is the minimum of any edge crossing the cut.

Note that there can be more than one light edge crossing a cut in the case of ties.

More generally, we say that an edge is a *light edge* satisfying a given property if its weight is the minimum of any edge satisfying the property.

MST: Finding Safe Edges



Green vertices belong to set S, i.e., $S = \{a, b, d, e\}$.

White vertices belong to set $V - S$, i.e., $V - S = \{c, f, g, h, i\}$.

The red line represent the cut $(S, V - S)$.

Dotted edges are the cut edges, i.e., they cross the red line.

Blue thick edges form set A, i.e.,

$$A = \{(a, b), (c, f), (c, i), (f, g), (g, h)\}.$$

MST: Finding Safe Edges

THEOREM: Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $(S, V \setminus S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V \setminus S)$. Then, edge (u, v) is safe for A.

COROLLARY: Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, and let $C = (V_C, E_C)$ be a connected component (tree) in the forest $G_A = (V, A)$ If (u, v) is a light edge crossing C to some other component of G_A, then edge (u, v) is safe for A.

A Disjoint-Set Data Structure (for Kruskal's MST Algorithm)

A *disjoint-set data structure* maintains a collection of disjoint dynamic sets. Each set is identified by a *representative* which must be a member of the set.

The collection is maintained under the following operations:

MAKE-SET(x): create a new set $\{x\}$ containing only element x. Element x becomes the representative of the set.

FIND(x): returns a pointer to the representative of the set containing x

UNION(x, y): replace the dynamic sets S_x and S_y containing x and y, respectively, with the set $S_x \cup S_y$

A Disjoint-Set Data Structure (for Kruskal's MST Algorithm)

Make-Set (x)

1. $\pi(x) \leftarrow x$
2. $rank(x) \leftarrow 0$

Link (x, y)

1. *if* $rank(x) > rank(y)$ *then* $\pi(y) \leftarrow x$
2. *else* $\pi(x) \leftarrow y$
3. *if* $rank(x) = rank(y)$ *then* $rank(y) \leftarrow rank(y) + 1$

Union (x, y)

1. *Link* (*Find* (x), *Find* (y))

Find (x)

1. *if* $x \neq \pi(x)$ *then* $\pi(x) \leftarrow$ *Find* ($\pi(x)$)
2. *return* $\pi(x)$

A Disjoint-Set Data Structure (for Kruskal's MST Algorithm)

THEOREM: A sequence of N MAKE-SET, UNION and FIND operations of which exactly n $(\leq N)$ are MAKE-SET operations takes $\mathrm{O}\big(N\alpha(n)\big)$ times to execute, where $\alpha(n)$ is the extremely slowly growing *Inverse Ackermann Function* which has a value no larger than 3 for all practical values of n.

PROOF: We will prove this later in the semester.

MST: Kruskal's Algorithm

MST-Kruskal ($G = (V, E)$, w)

$A \leftarrow \emptyset$

for each vertex $v \in G.V$ *do*

    MAKE-SET(v)

sort the edges of $G.E$ into nondecreasing order by weight w

for each edge $(u, v) \in G.E$ taken in nondecreasing order by weight *do*

    if FIND-SET(u) $\neq$ *FIND-SET*(v) *then*

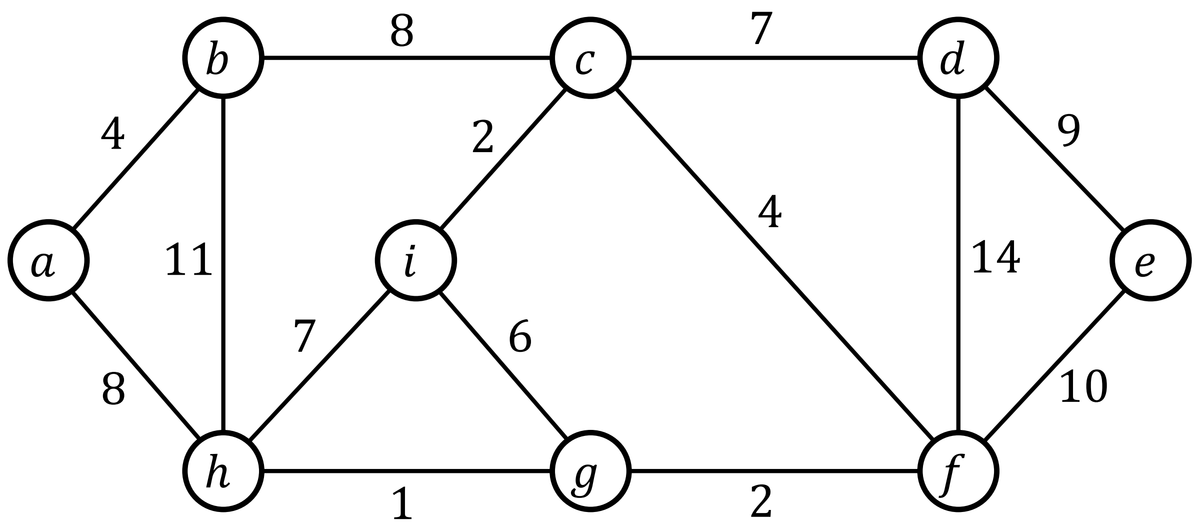
        $A \leftarrow A \cup \{(u, v)\}$

        UNION(u, v)

return A

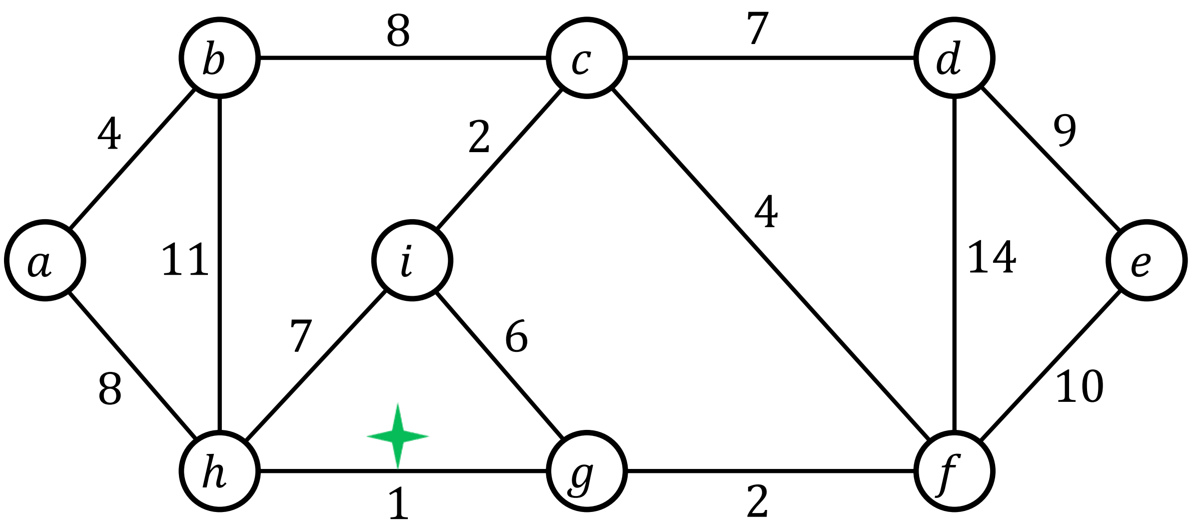
MST: Kruskal's Algorithm

Initial State



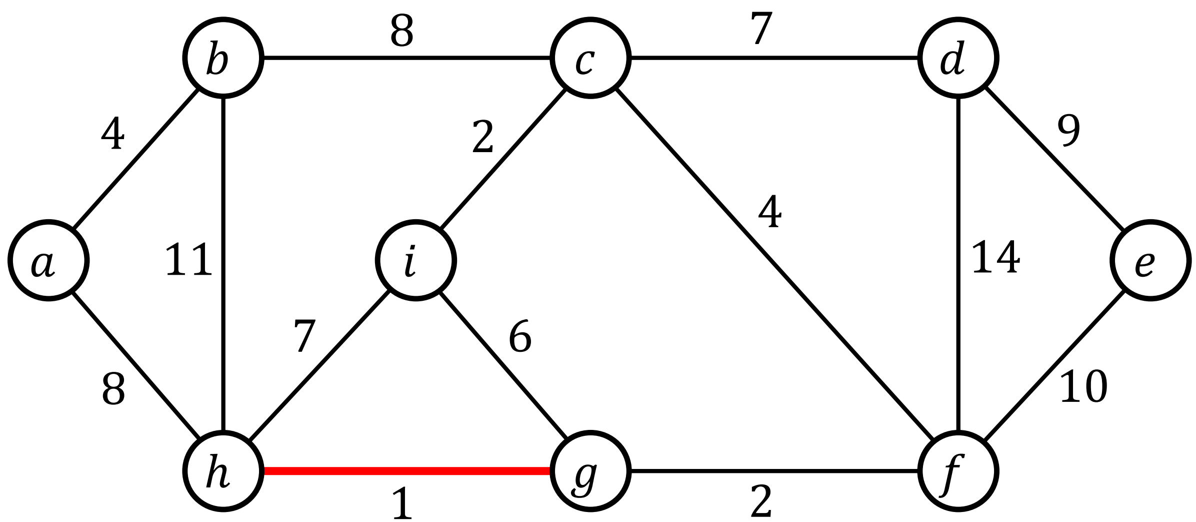
MST: Kruskal's Algorithm

(1) edge (h, g)



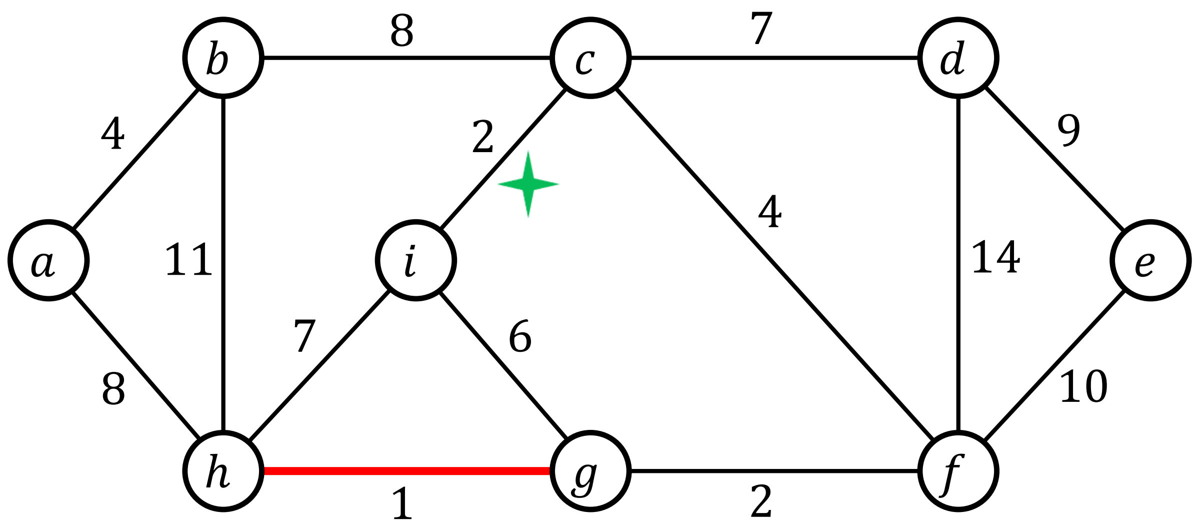
MST: Kruskal's Algorithm

(1) edge $(\boldsymbol{h}, \boldsymbol{g})$



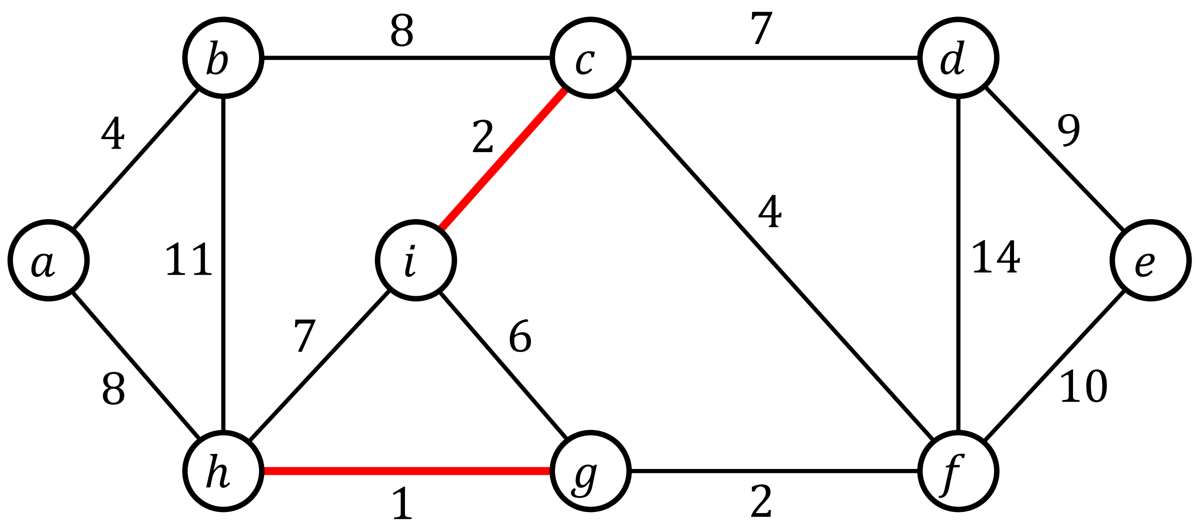
MST: Kruskal's Algorithm

(2) edge (i, c)



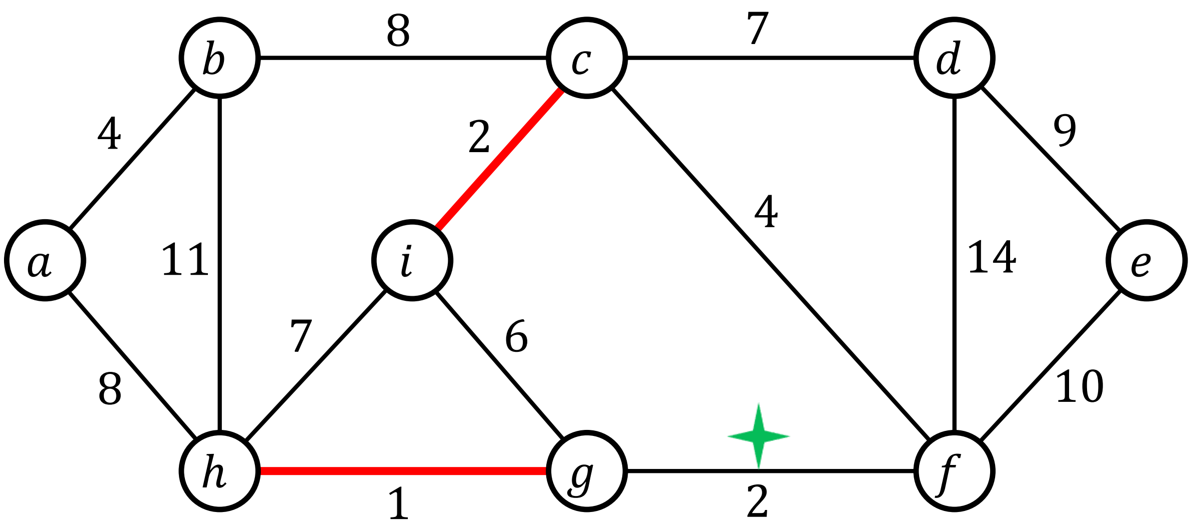
MST: Kruskal’s Algorithm

(2) edge $(\boldsymbol{i}, \boldsymbol{c})$



MST: Kruskal's Algorithm

(3) edge (g, f)



MST: Kruskal's Algorithm

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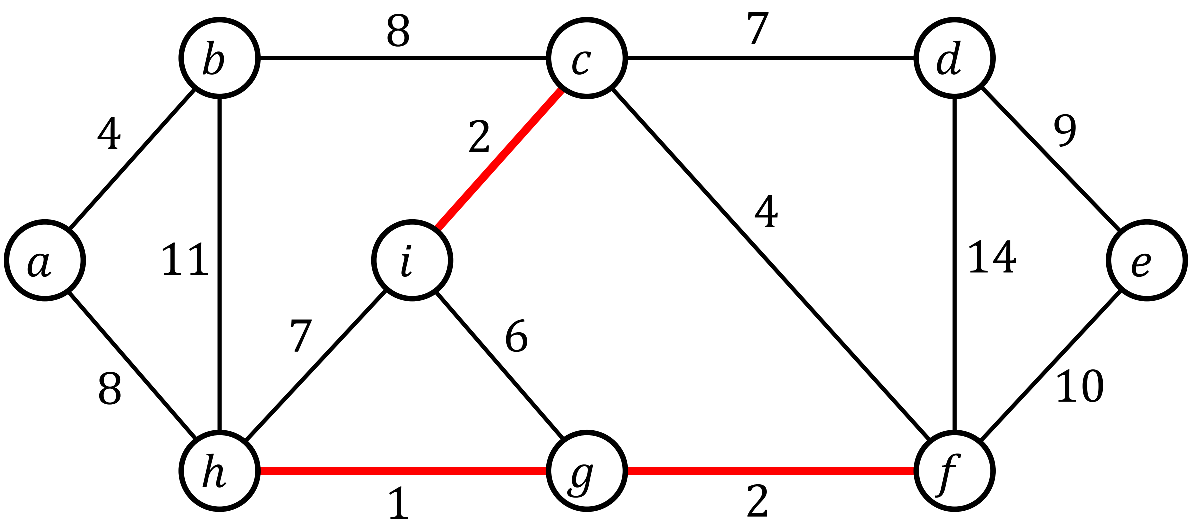
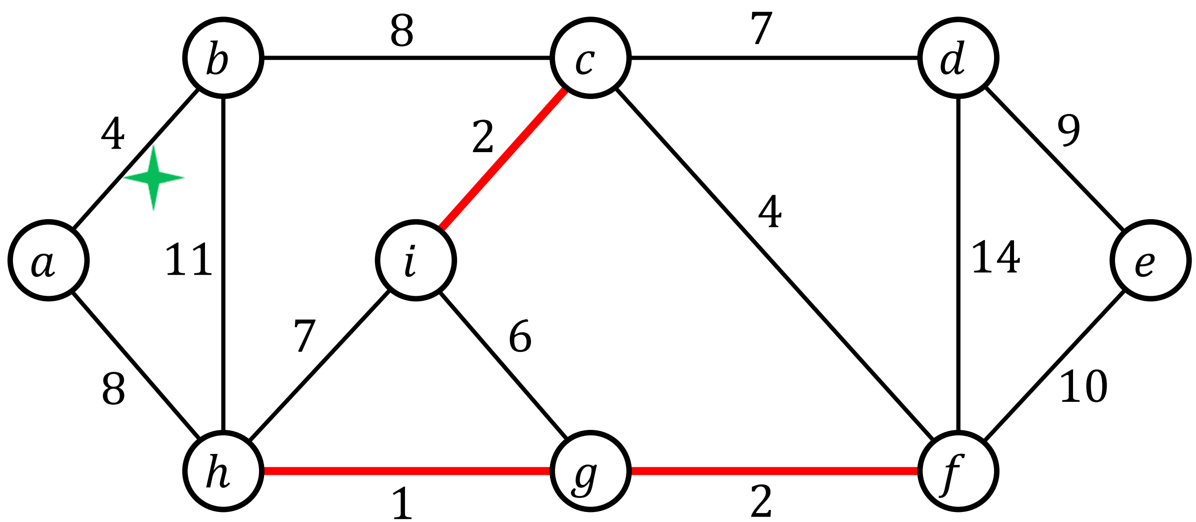


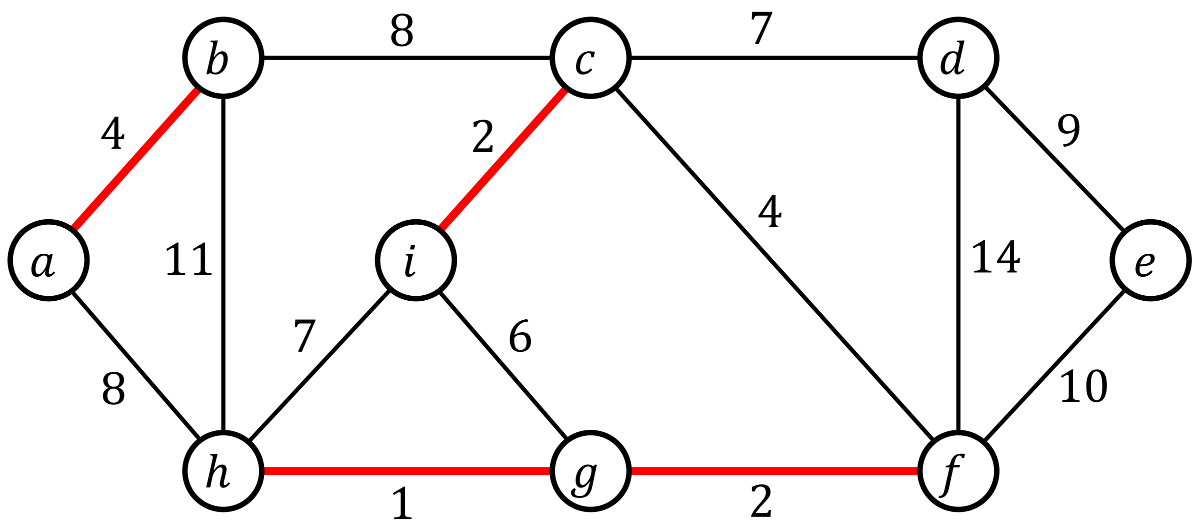
MST: Kruskal's Algorithm

(4) edge (a, b)



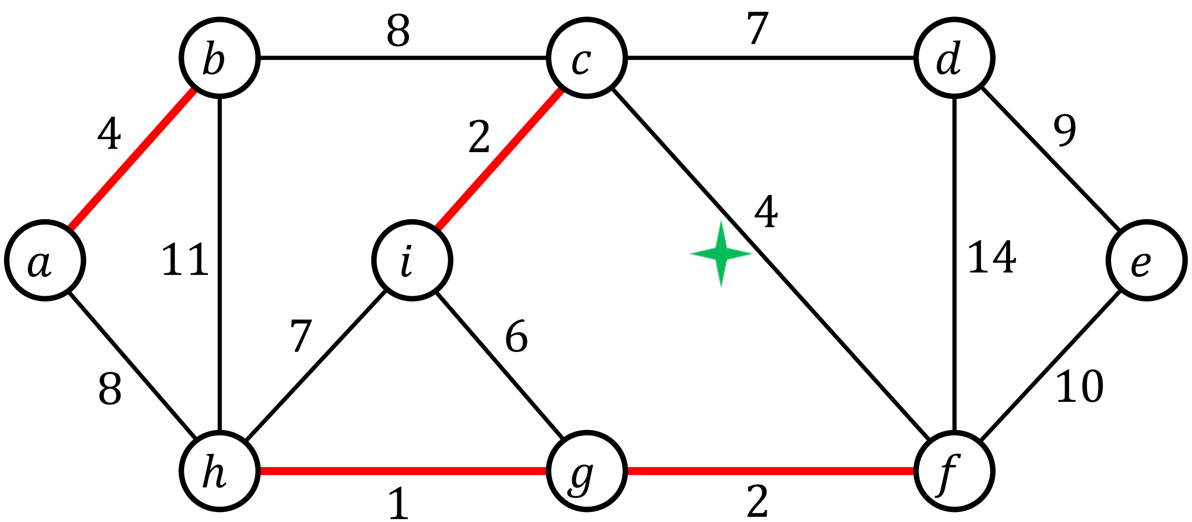
MST: Kruskal's Algorithm

(4) edge (a, b)



MST: Kruskal's Algorithm

(5) edge (c, f)



MST: Kruskal's Algorithm

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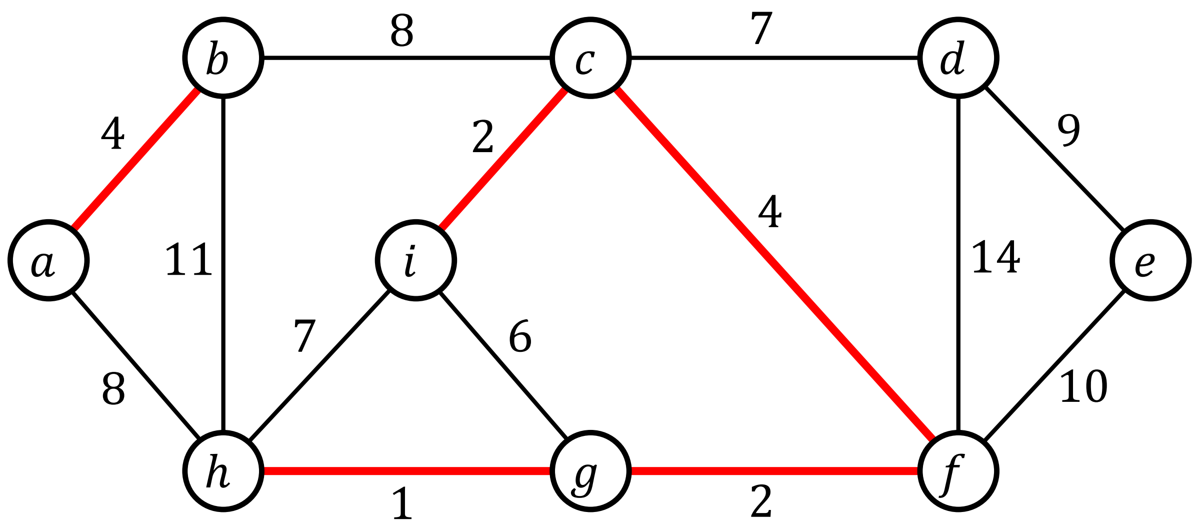
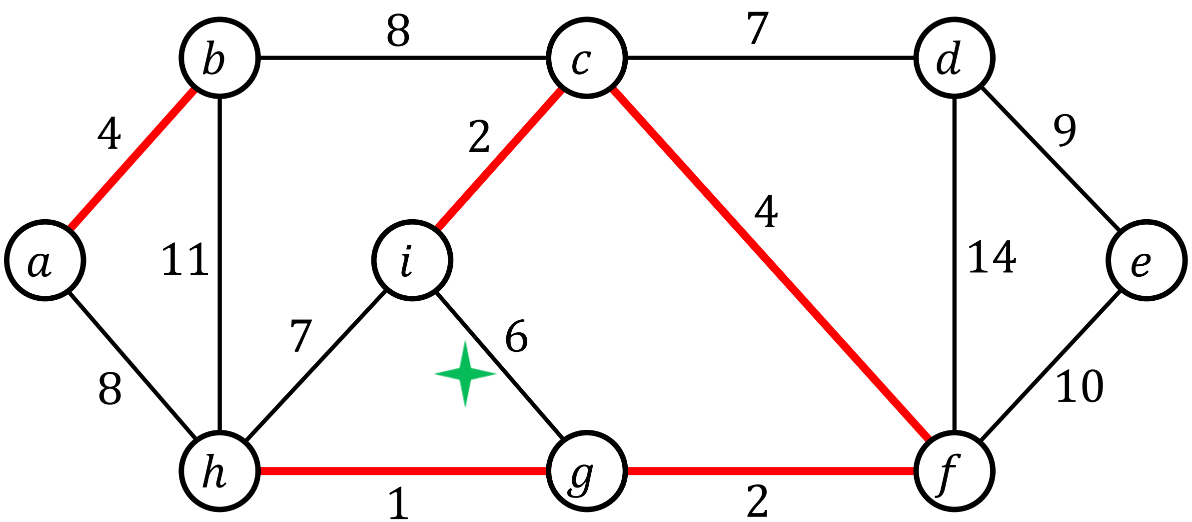


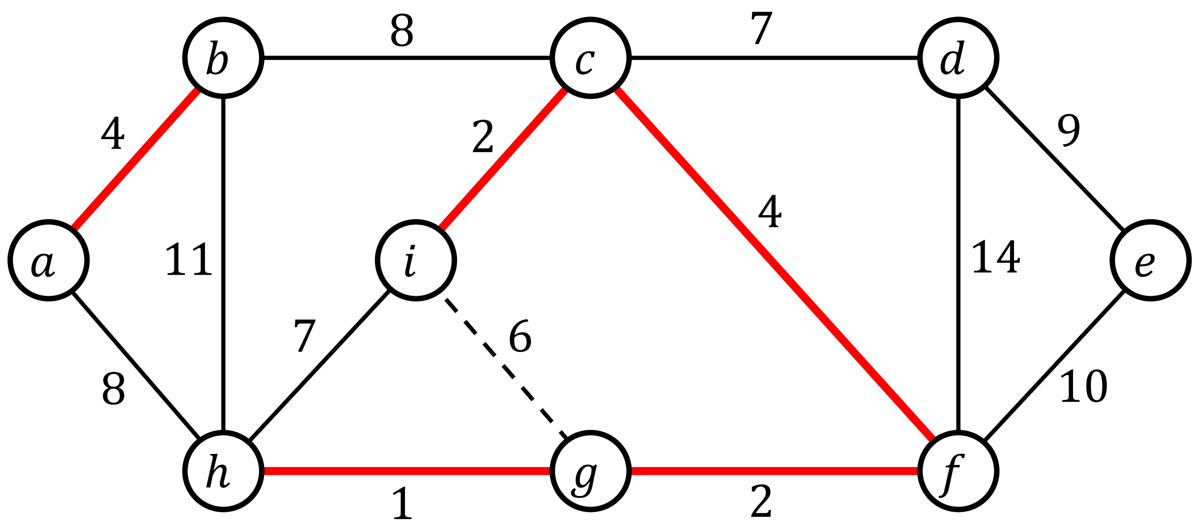
MST: Kruskal's Algorithm

(6) edge $(\boldsymbol{i}, \boldsymbol{g})$



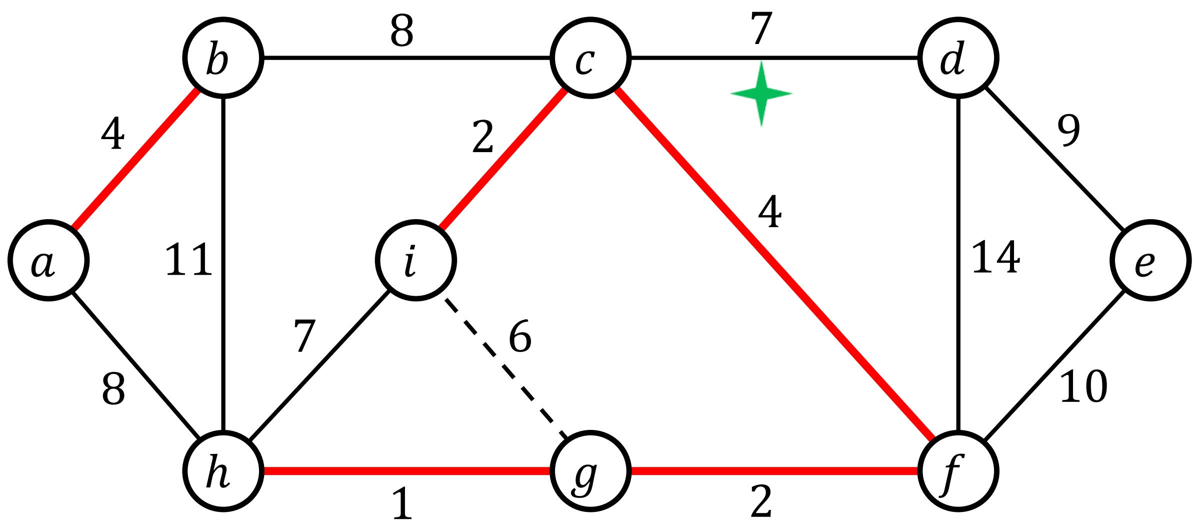
MST: Kruskal's Algorithm

(6) edge $(\boldsymbol{i}, \boldsymbol{g})$



MST: Kruskal's Algorithm

(7) edge (c, d)



MST: Kruskal's Algorithm

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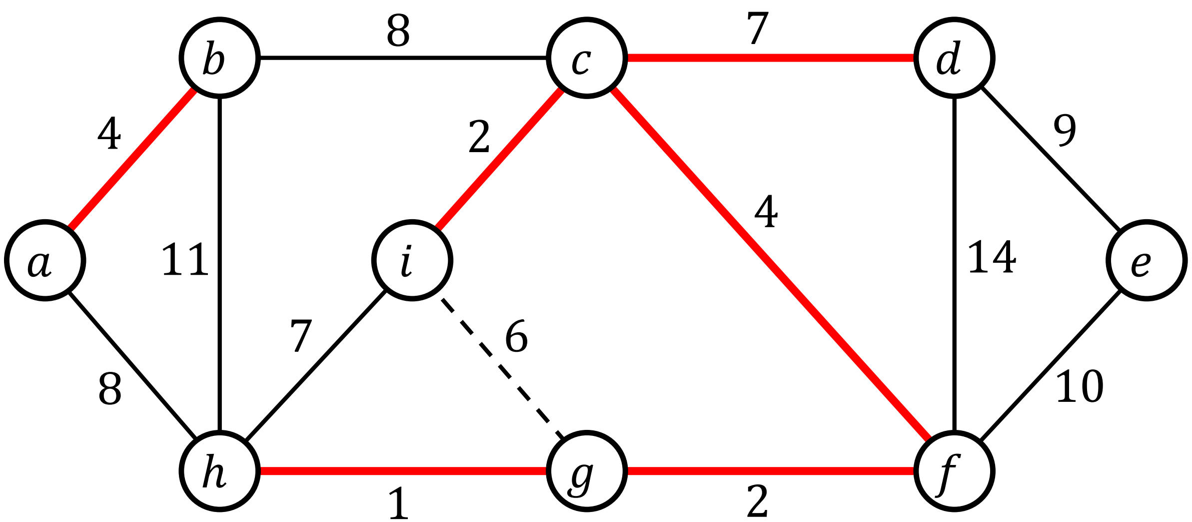
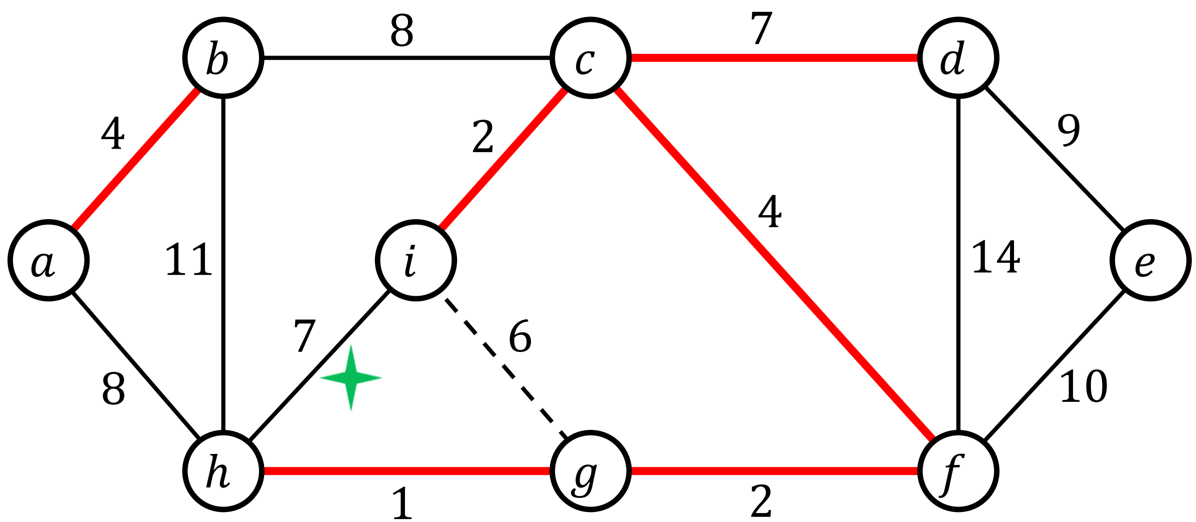


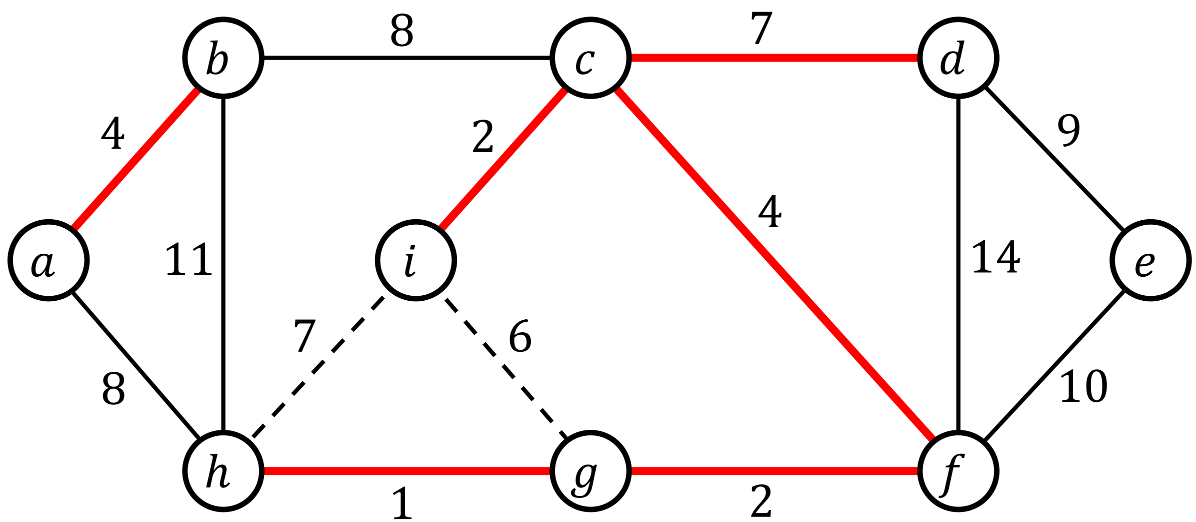
MST: Kruskal's Algorithm

(8) edge $(\boldsymbol{i}, \boldsymbol{h})$



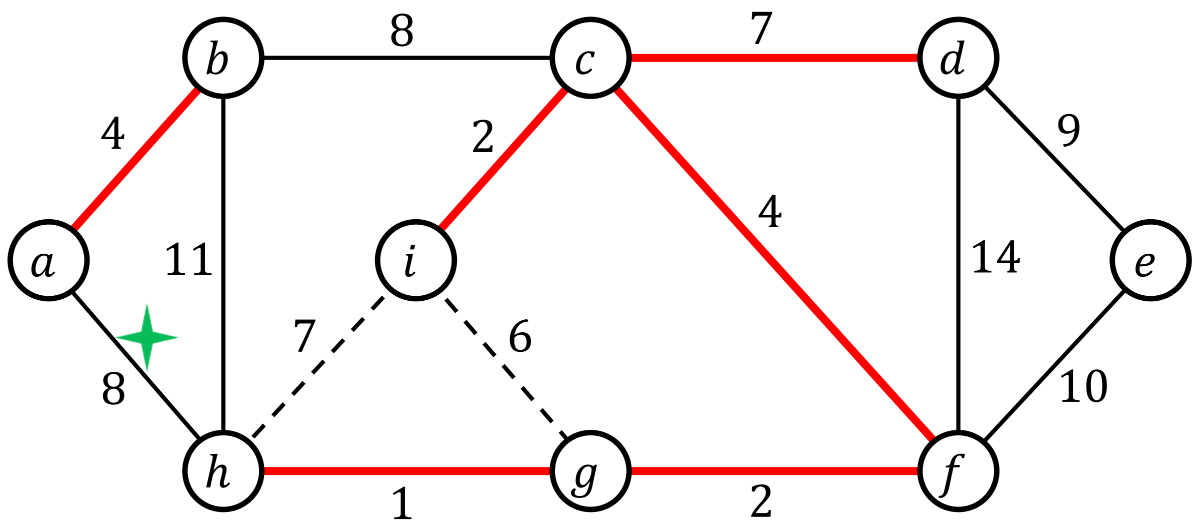
MST: Kruskal's Algorithm

(8) edge $(\boldsymbol{i}, \boldsymbol{h})$



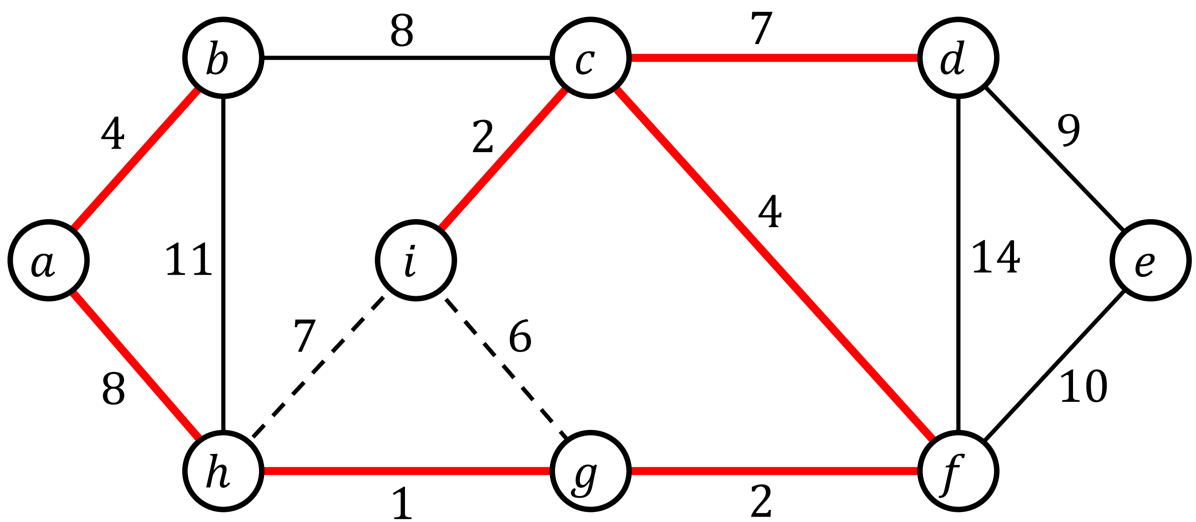
MST: Kruskal's Algorithm

(9) edge $(\boldsymbol{a}, \boldsymbol{h})$



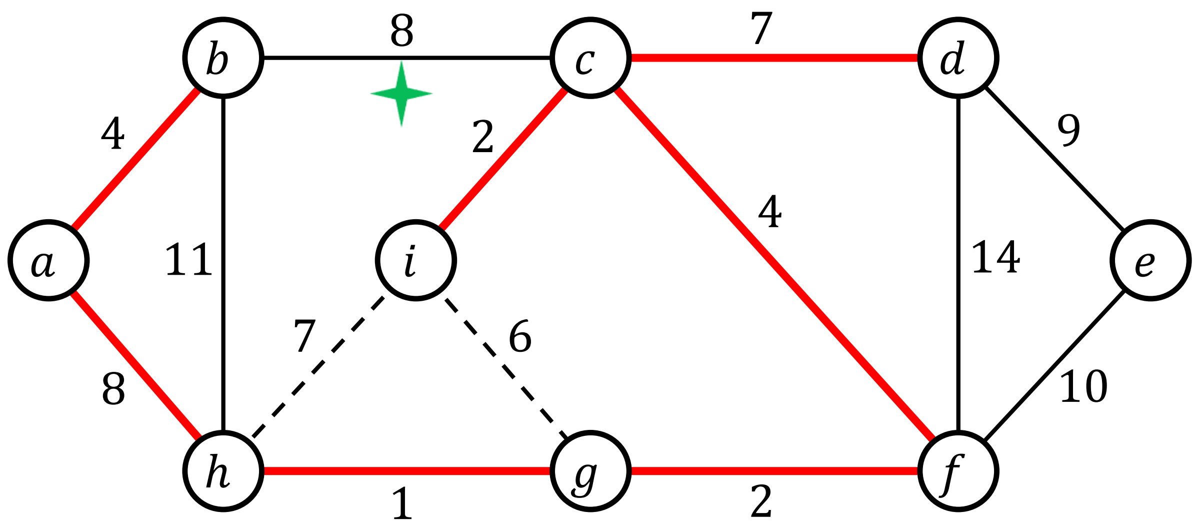
MST: Kruskal's Algorithm

(9) edge (a, h)



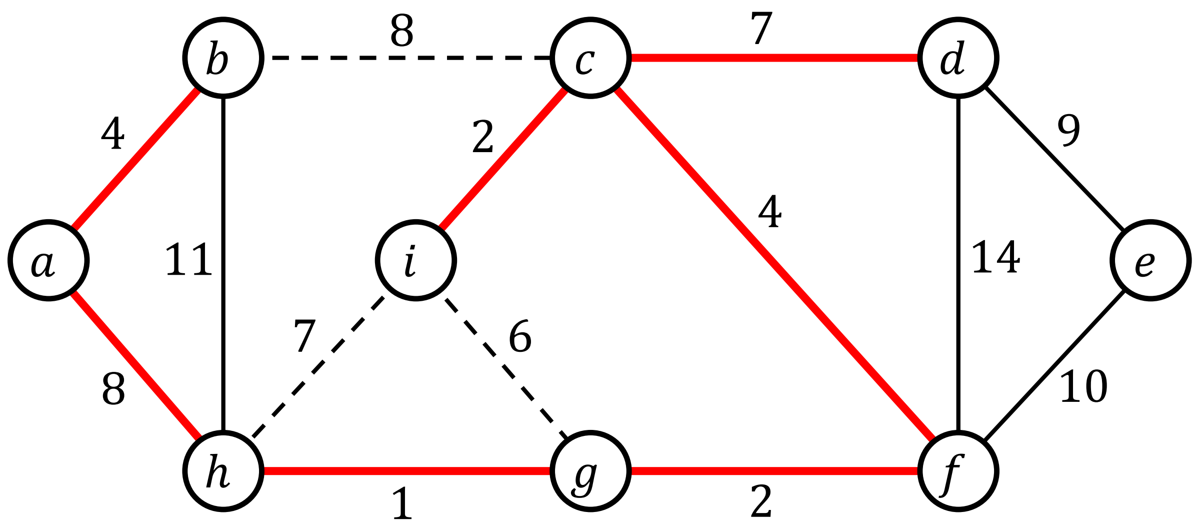
MST: Kruskal's Algorithm

(10) edge (b, c)



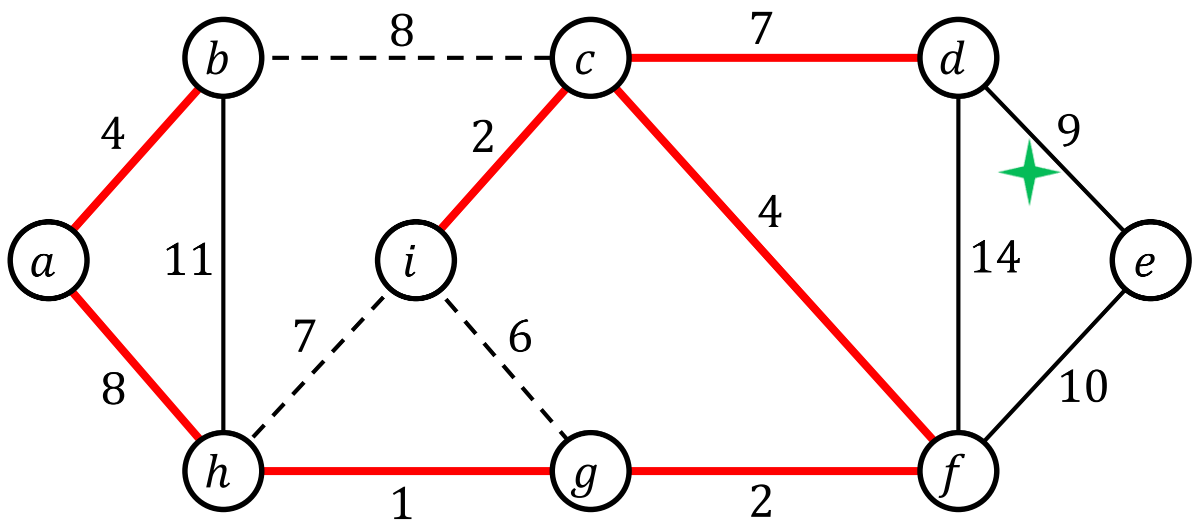
MST: Kruskal's Algorithm

(10) edge (b, c)



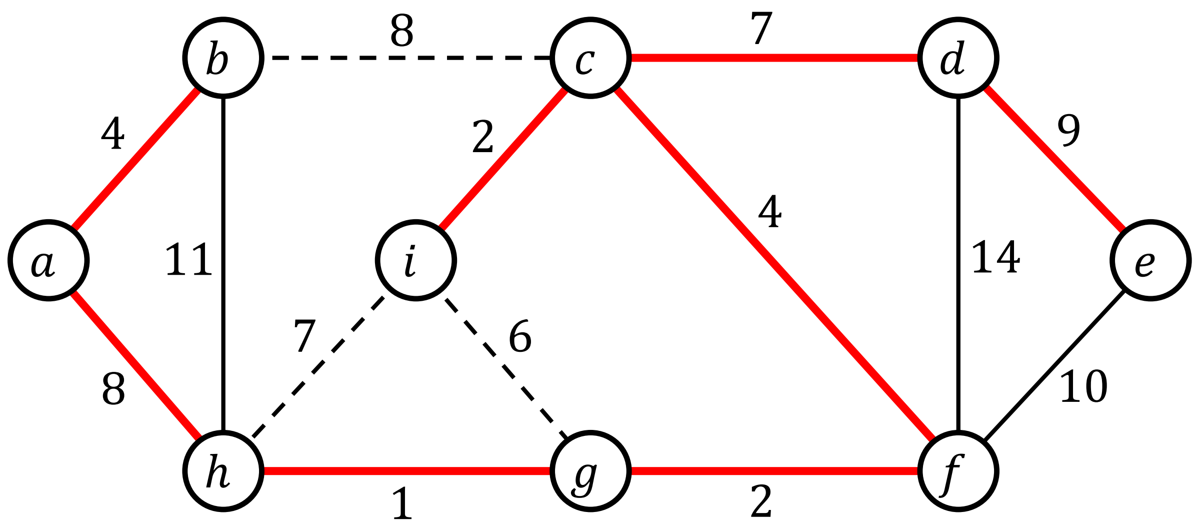
MST: Kruskal's Algorithm

(11) edge (d, e)



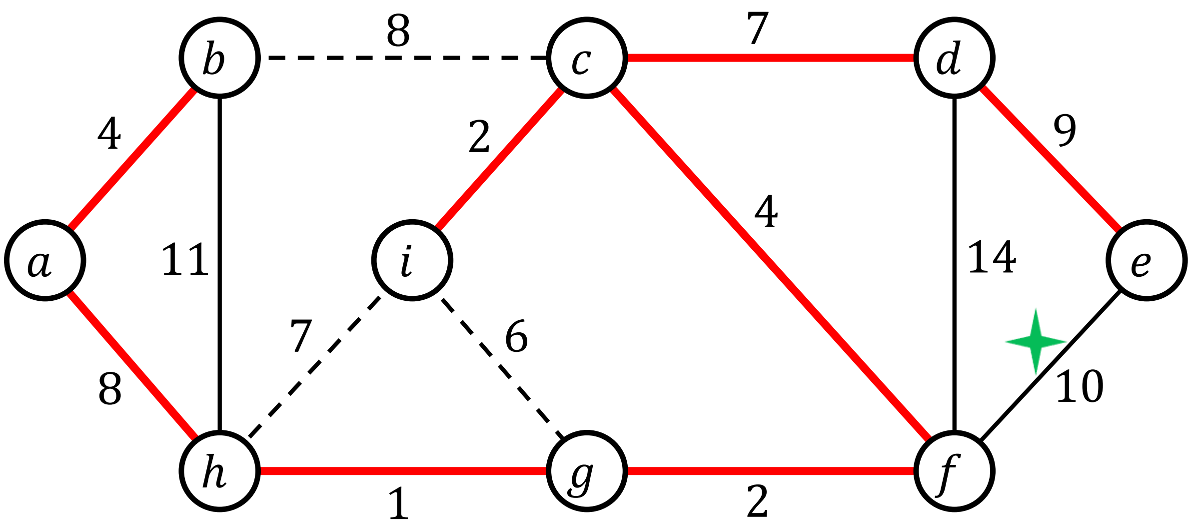
MST: Kruskal's Algorithm

(11) edge (d, e)



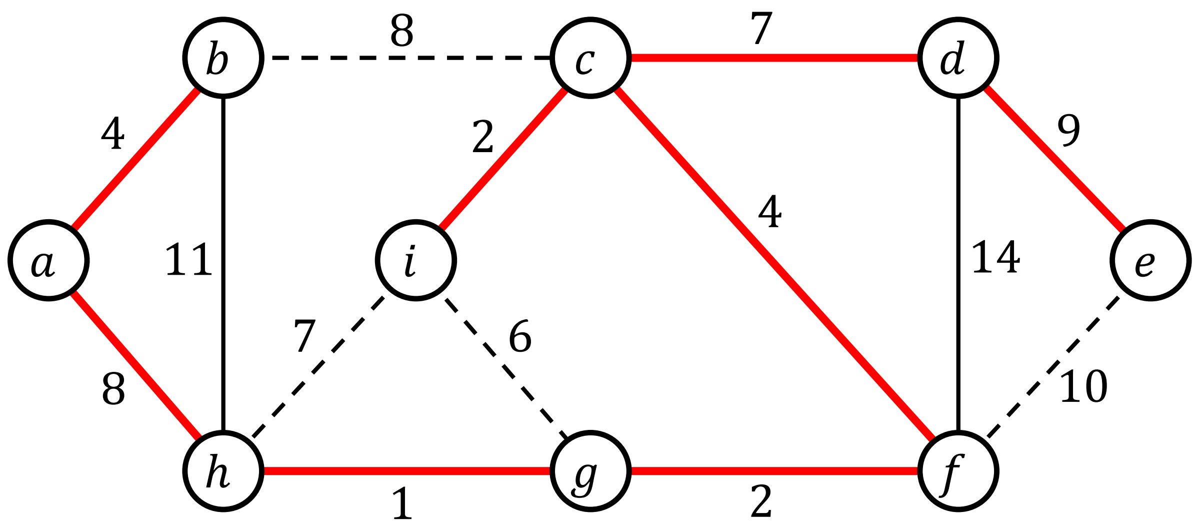
MST: Kruskal's Algorithm

(12) edge (e, f)



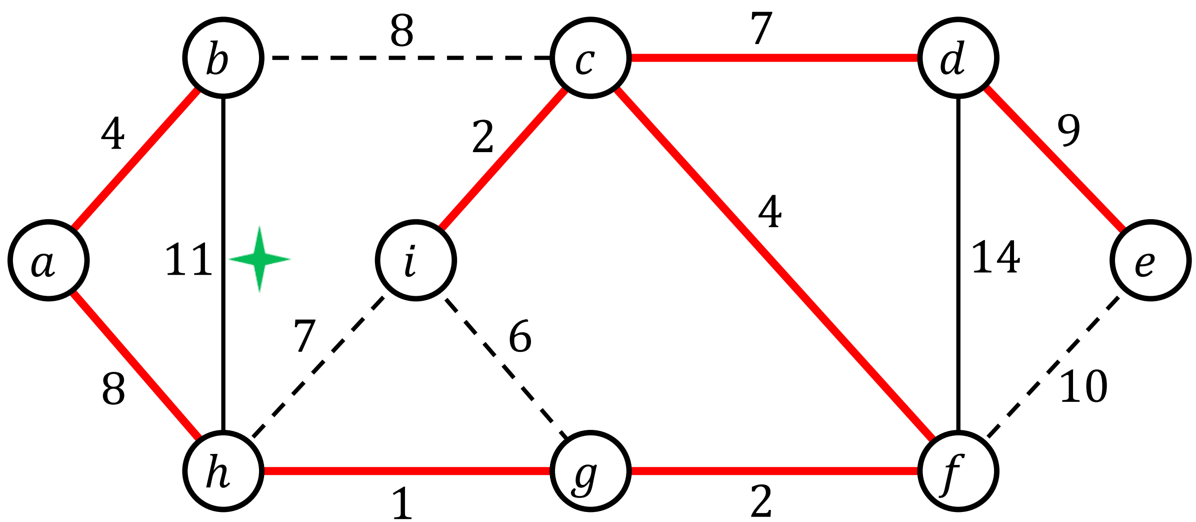
MST: Kruskal's Algorithm

(12) edge (e, f)



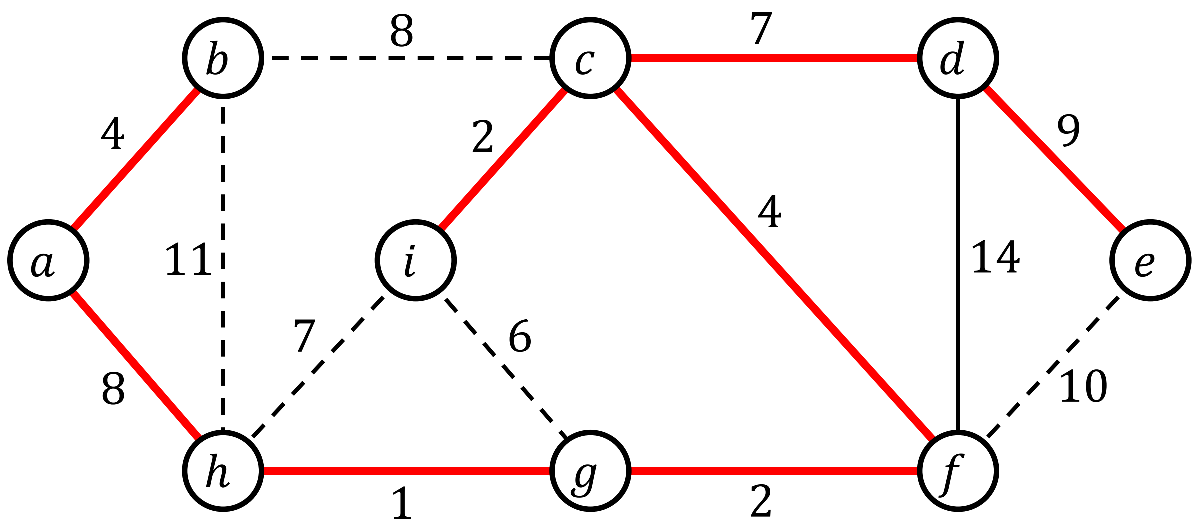
MST: Kruskal's Algorithm

(13) edge (b, h)



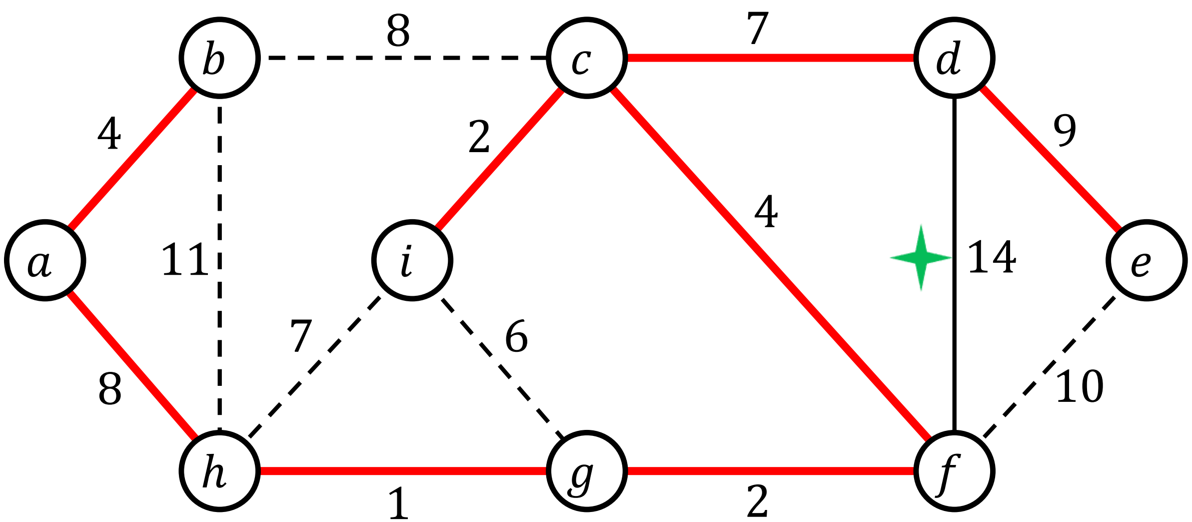
MST: Kruskal's Algorithm

(13) edge (b, h)



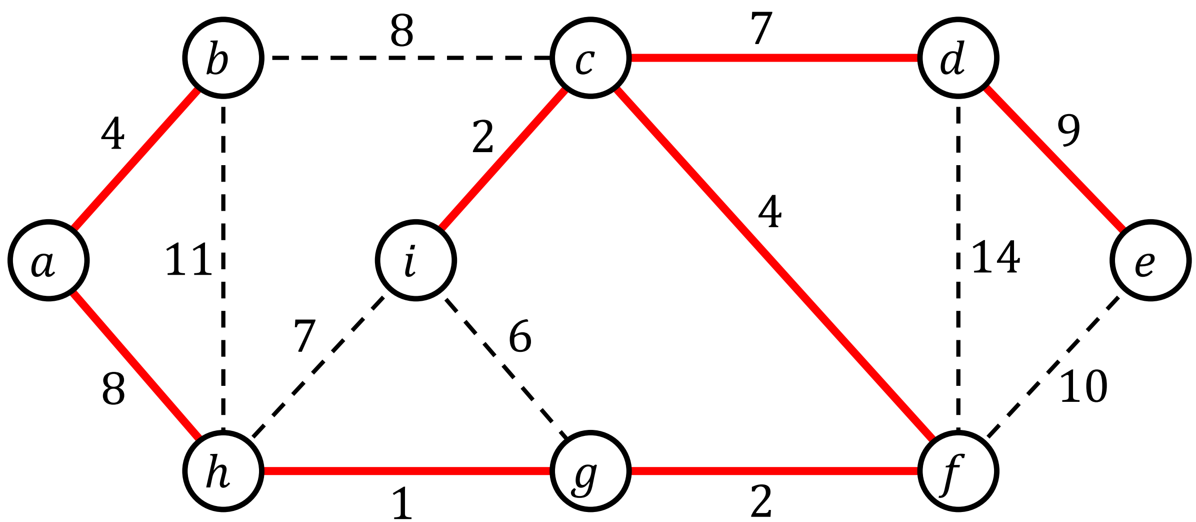
MST: Kruskal's Algorithm

(14) edge (d, f)



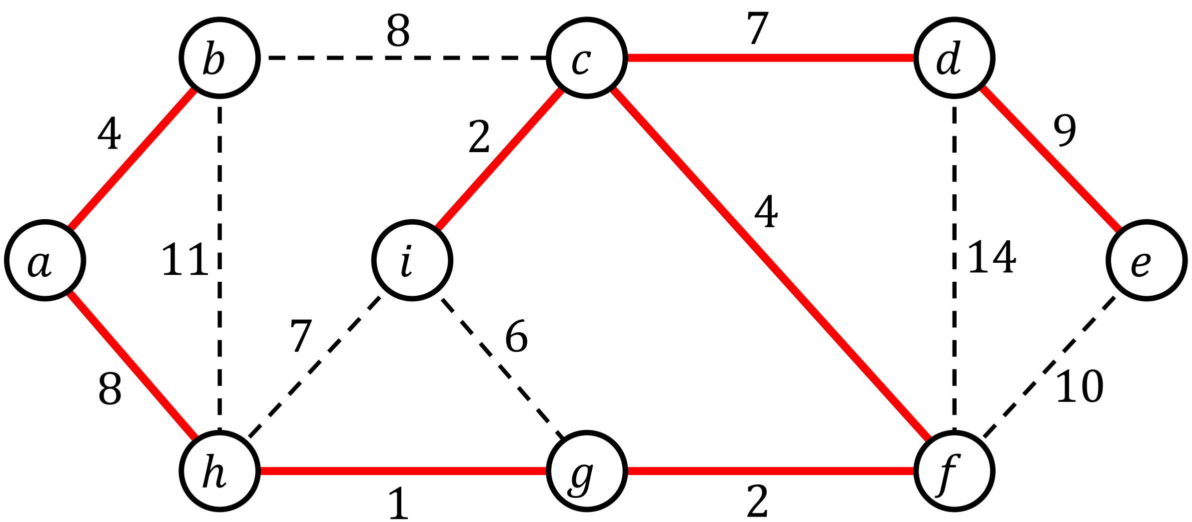
MST: Kruskal’s Algorithm

(14) edge (d, f)



MST: Kruskal's Algorithm

(14) edge (d, f)



Total weight $= 37$

MST: Kruskal's Algorithm

```
MST-Kruskal ( G = (V, E), w )

1.    A ← ∅
2.    for each vertex v ∈ G.V do
3.        MAKE-SET( v )
4.    sort the edges of G.E into nondecreasing order by weight w
5.    for each edge (u, v) ∈ G.E taken in nondecreasing order by weight do
6.        if FIND-SET( u ) ≠ FIND-SET( v ) then
7.            A ← A ∪ {(u, v)}
8.            UNION( u, v )
9.    return A
```

Let $n = |V|$ and $m = |E|$. Since G is connected, we have $m \geq n - 1$.

Then the sorting in step 4 can be done in $\mathrm{O}(m \log m)$ time.

#disjoint-set operations performed, $N = 2m + 2n - 1$, of which

#MAKE-SET: n, #FIND-SET: $2m$, #UNION: $n - 1$

So, total time taken by disjoint-set operations $= \mathrm{O}\big((n + m)\alpha(n)\big)$

Hence, MST-Kruskal's running time $= \mathrm{O}(m \log m)$

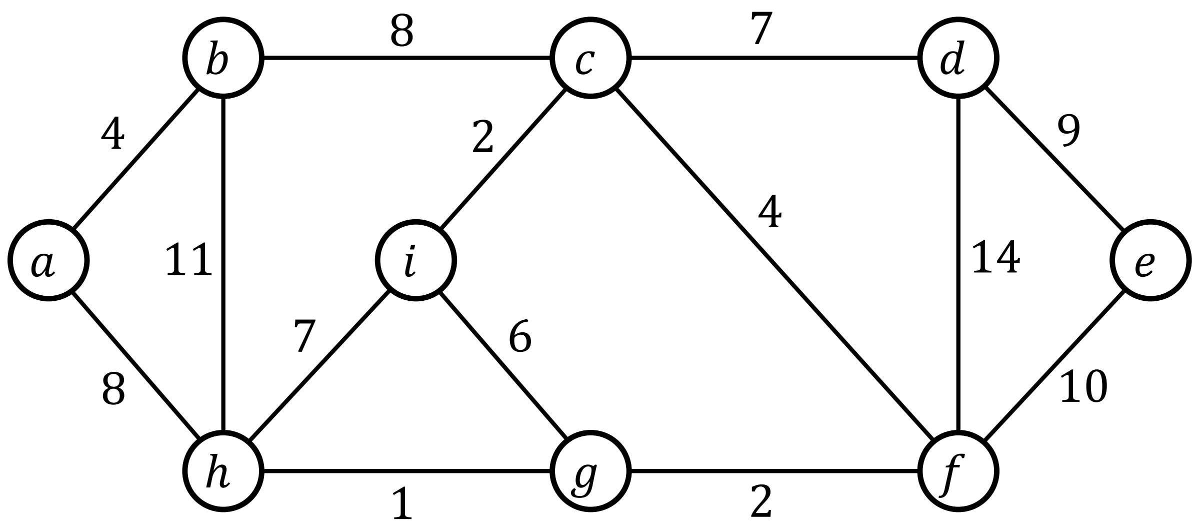
MST: Prim's Algorithm

```
MST-Prim ( G = (V,E), w, r )

    for each vertex v ∈ G.V do
        v.d ← ∞
        v.π ← NIL
    r.d ← 0
    Min-Heap Q ← ∅
    for each vertex v ∈ G.V do
        INSERT( Q, v )
    while Q ≠ ∅ do
        u ← EXTRACT-MIN( Q )
        for each (u, v) ∈ G.E do
            if v ∈ Q and w(u, v) < v.d then
                v.d ← w(u, v)
                v.π ← u
                DECREASE-KEY( Q, v, w(u, v) )
```

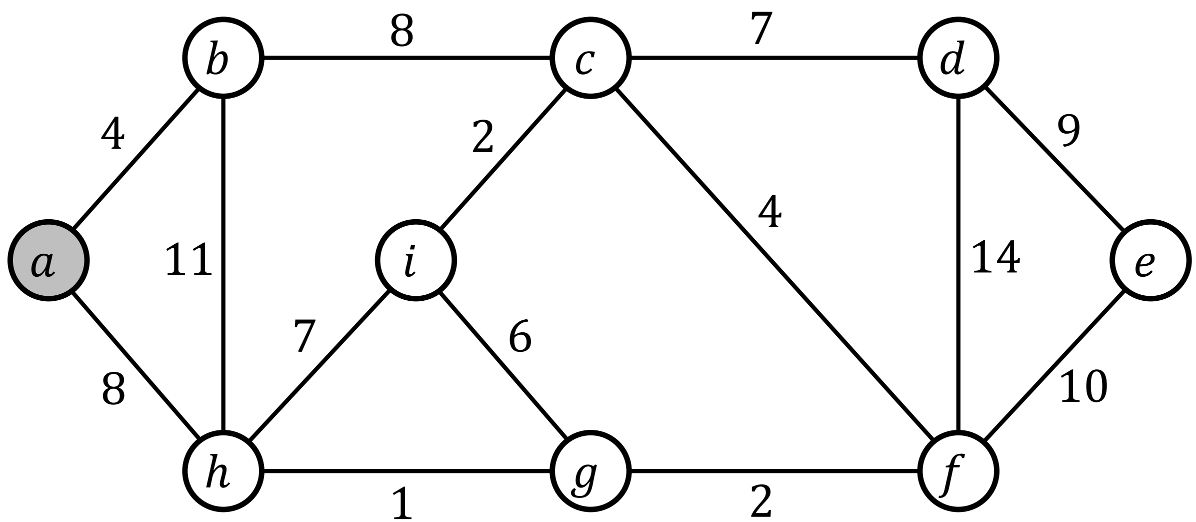
MST: Prim’s Algorithm

Initial State



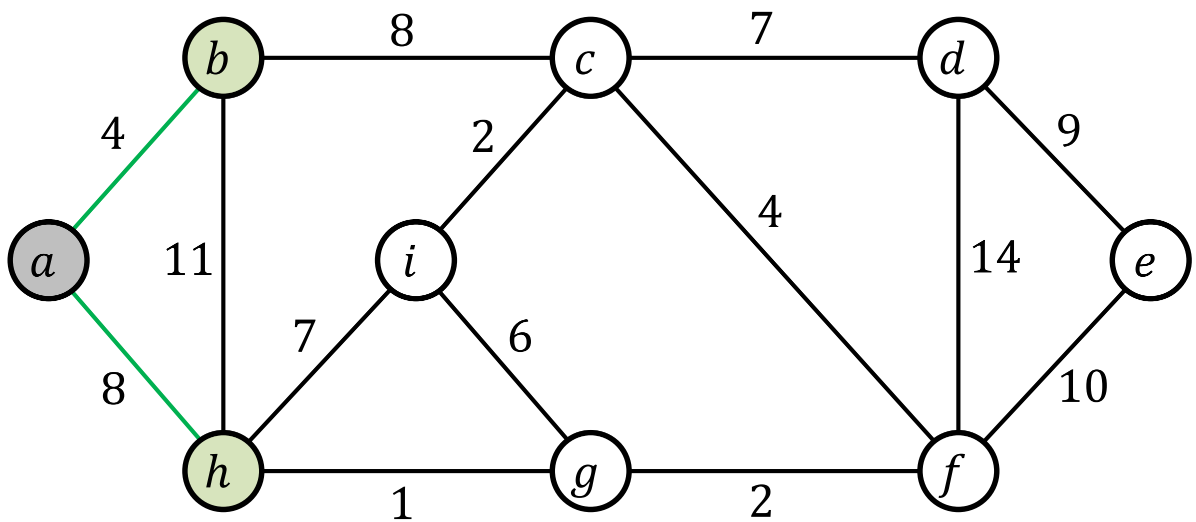
MST: Prim's Algorithm

Step 1: add vertex a to MST



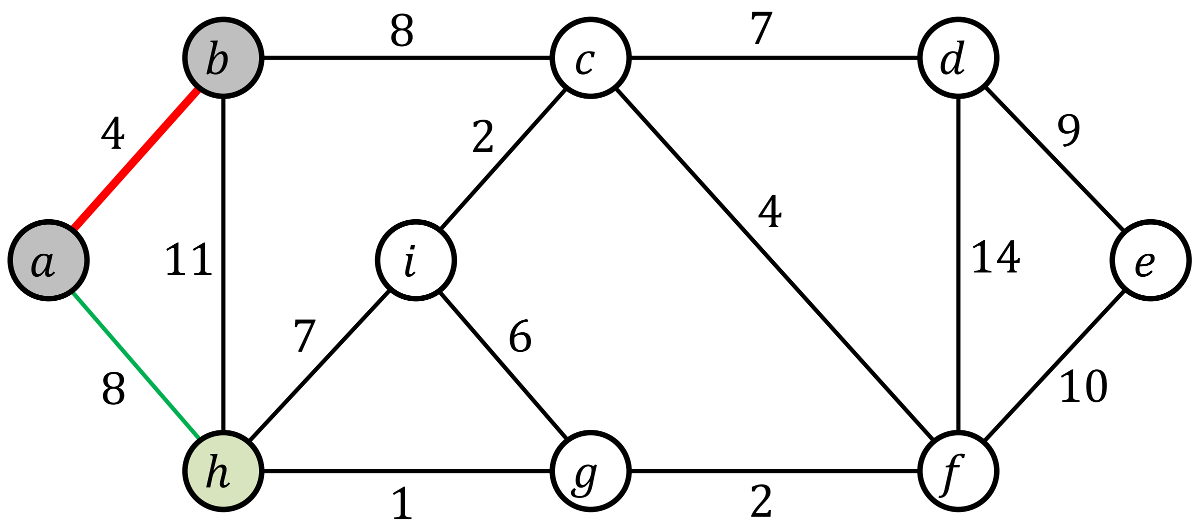
MST: Prim's Algorithm

Step $1'$: update neighbors of a



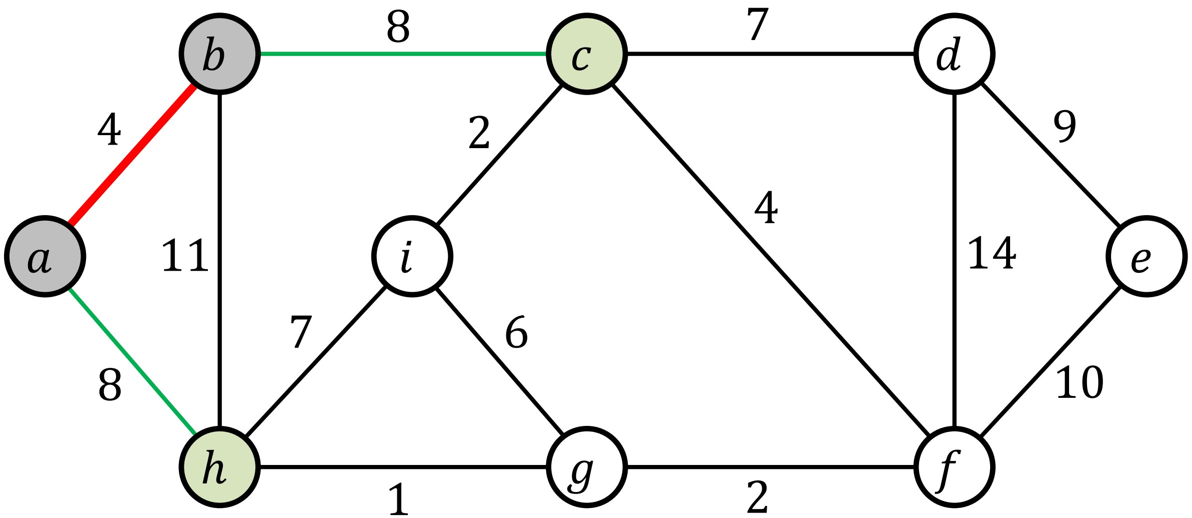
MST: Prim's Algorithm

Step 2: add vertex b through edge (a, b)



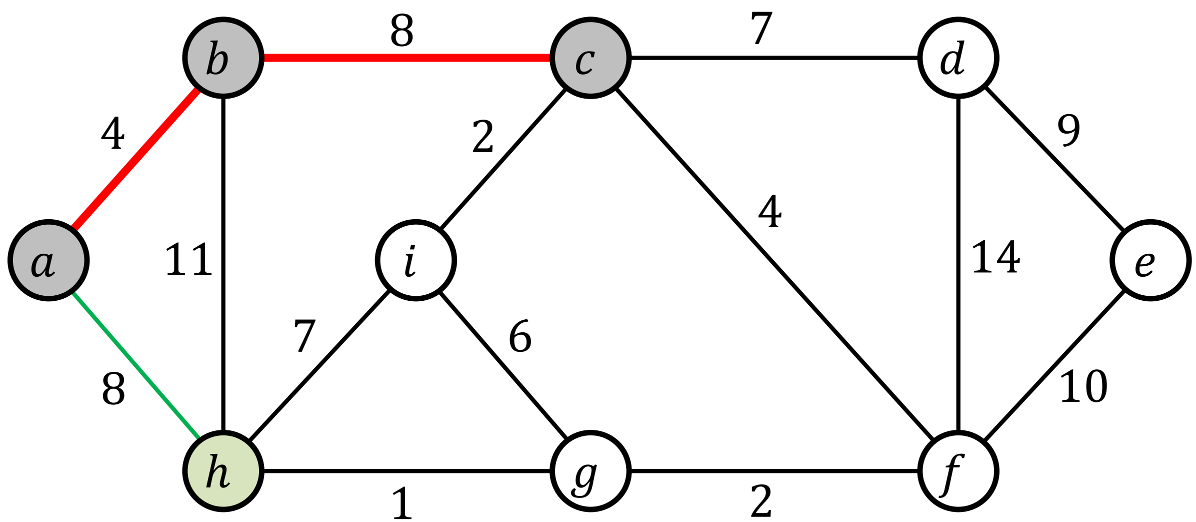
MST: Prim's Algorithm

Step $2'$: update neighbors of b



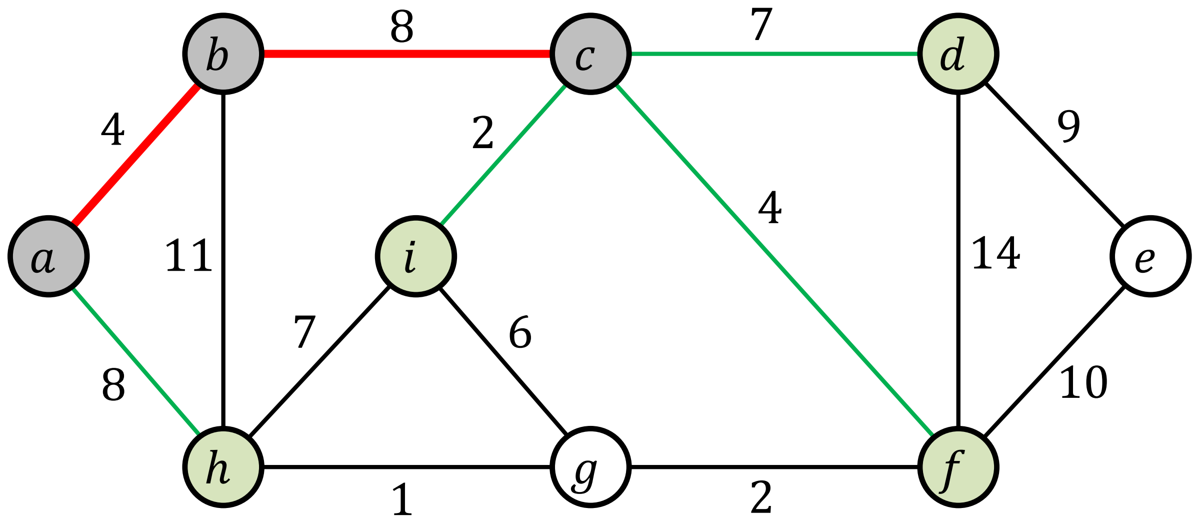
MST: Prim's Algorithm

Step 3: add vertex c through edge (b, c)



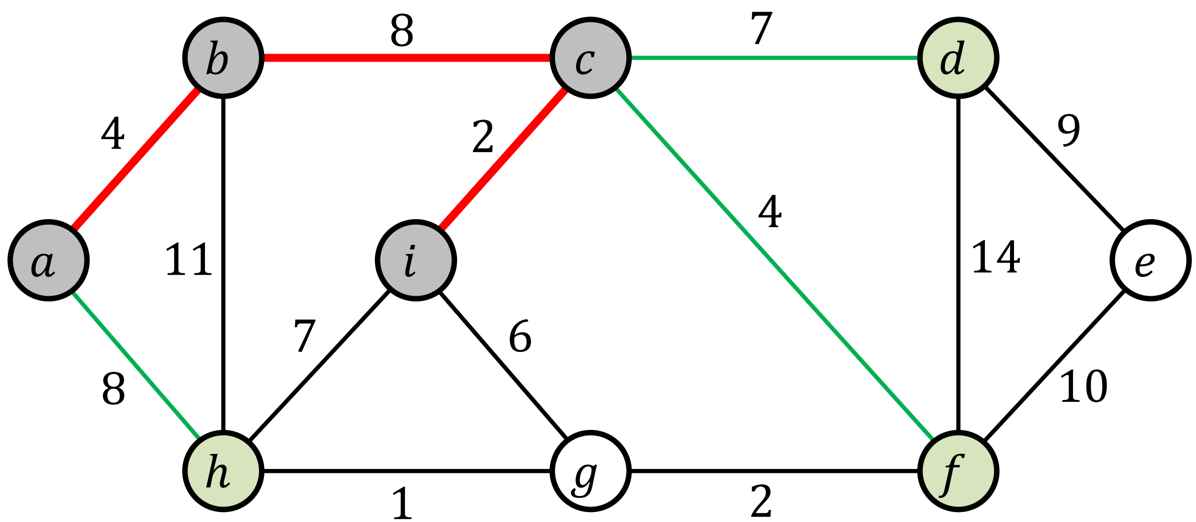
MST: Prim's Algorithm

Step $3'$: update neighbors of c



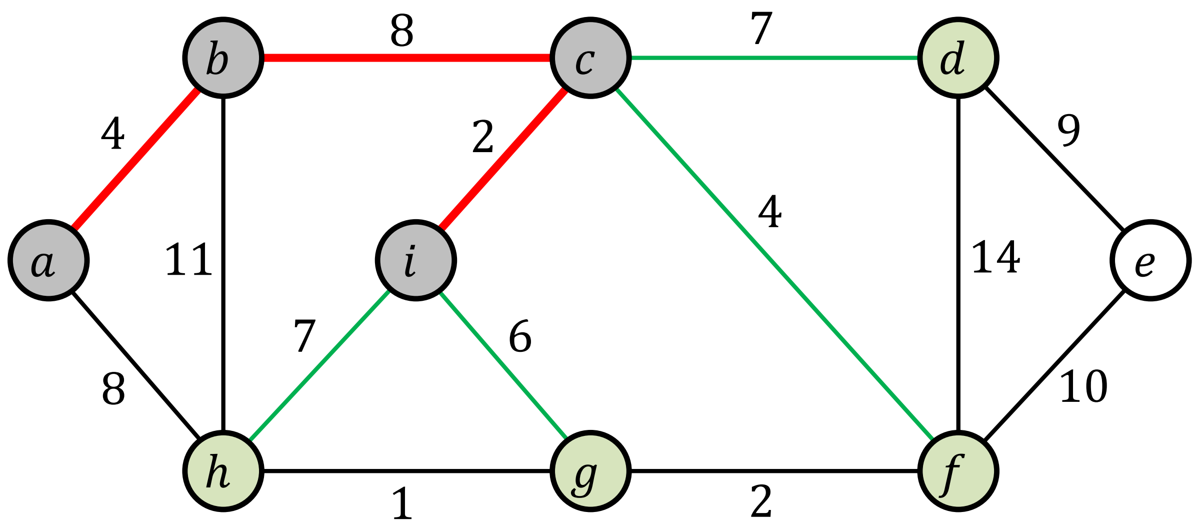
MST: Prim's Algorithm

Step 4: add vertex i through edge (c, i)



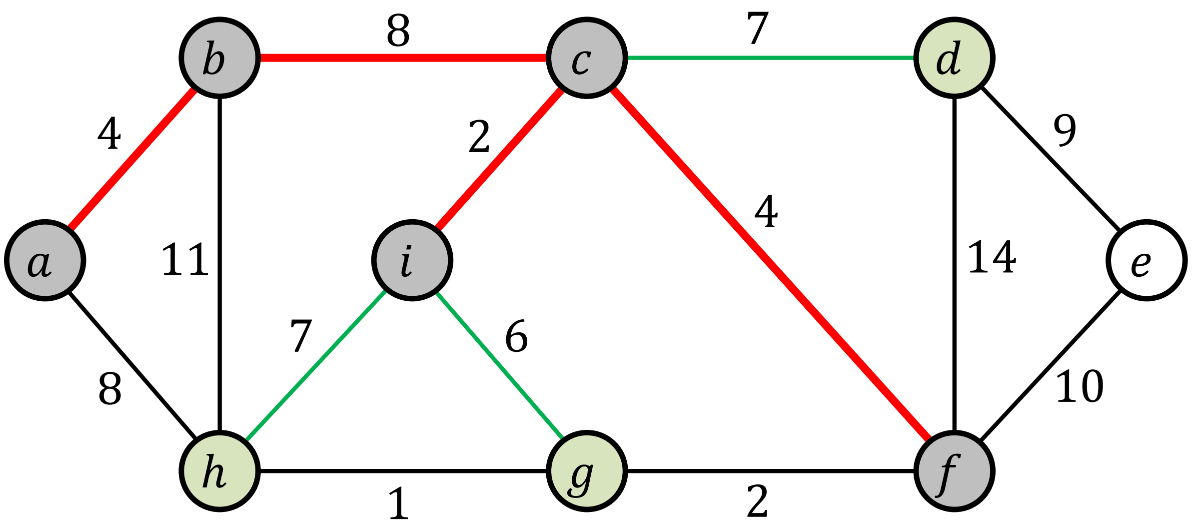
MST: Prim's Algorithm

Step $4'$: update neighbors of $\boldsymbol{i}$



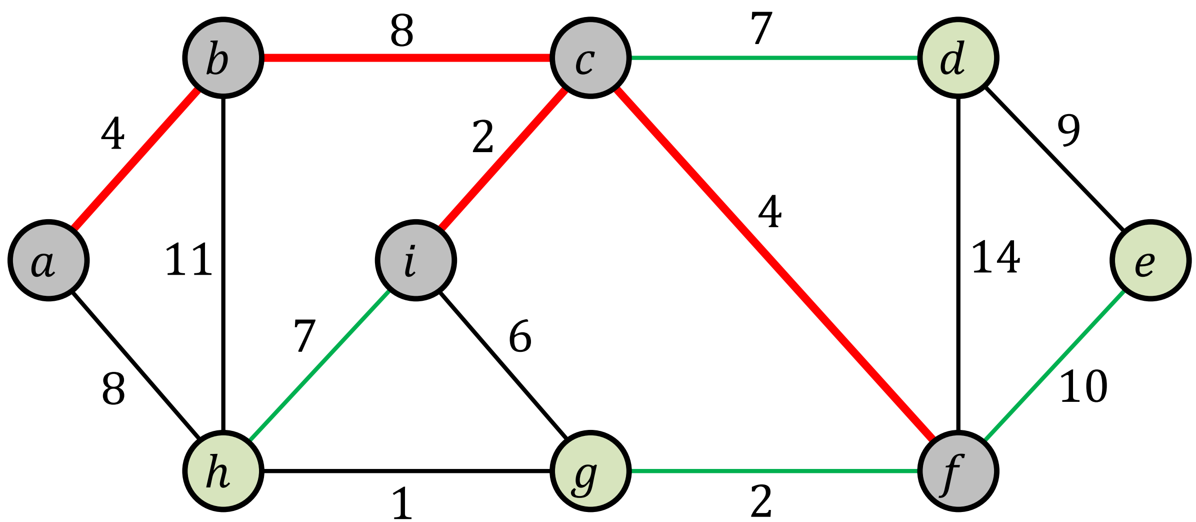
MST: Prim's Algorithm

Step 5: add vertex f through edge (c, f)



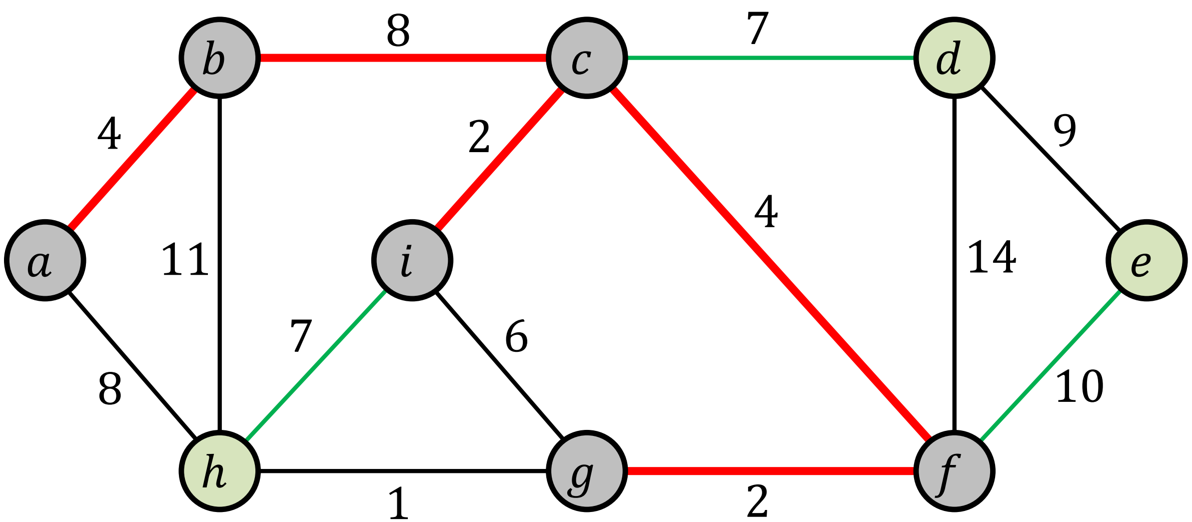
MST: Prim's Algorithm

Step $5'$: update neighbors of f



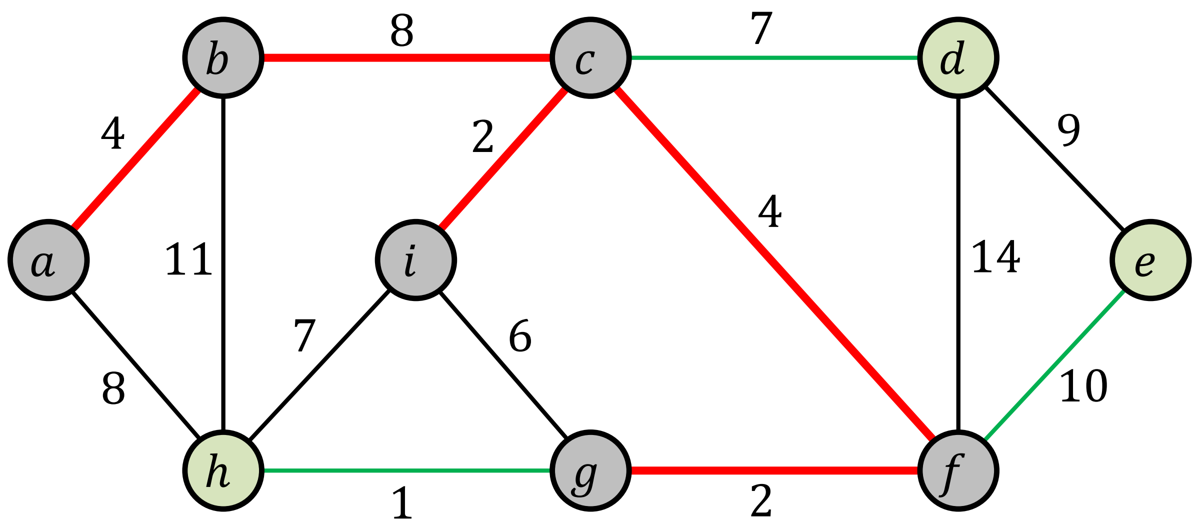
MST: Prim's Algorithm

Step 6: add vertex g through edge (f, g)



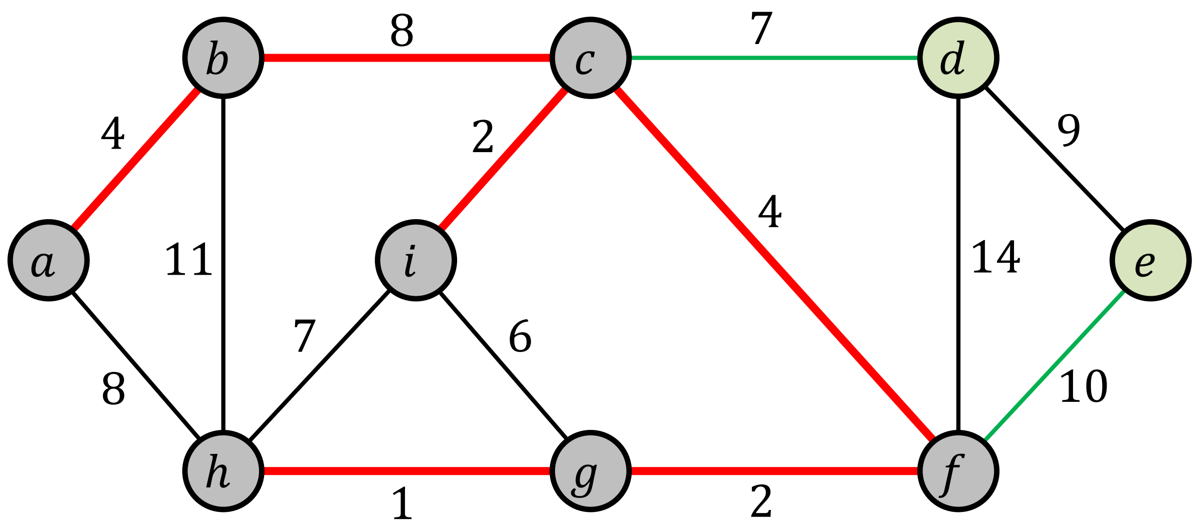
MST: Prim's Algorithm

Step $6'$: update neighbors of g



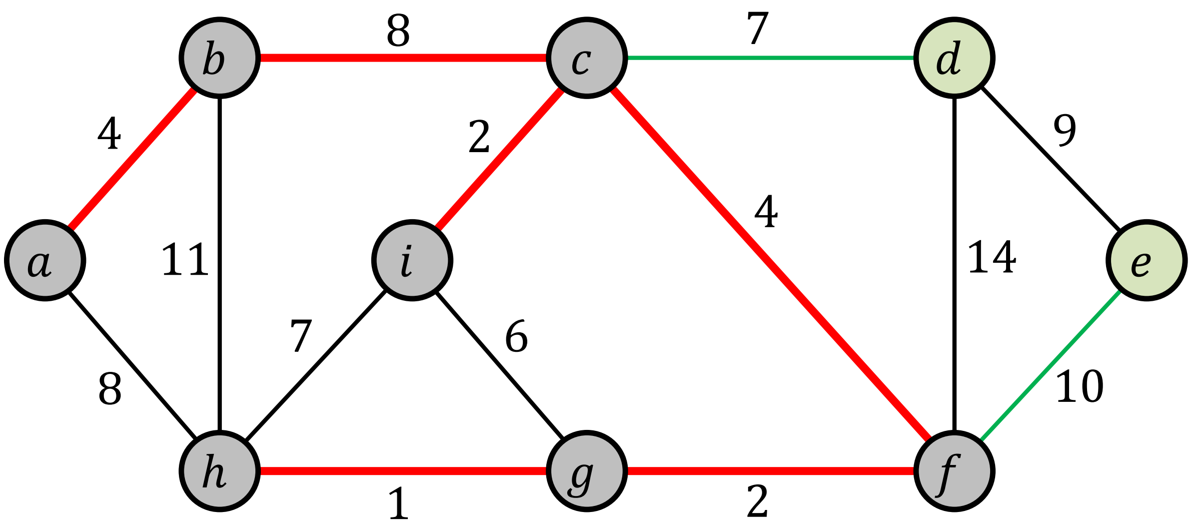
MST: Prim's Algorithm

Step 7: add vertex $\boldsymbol{h}$ through edge $(\boldsymbol{g}, \boldsymbol{h})$



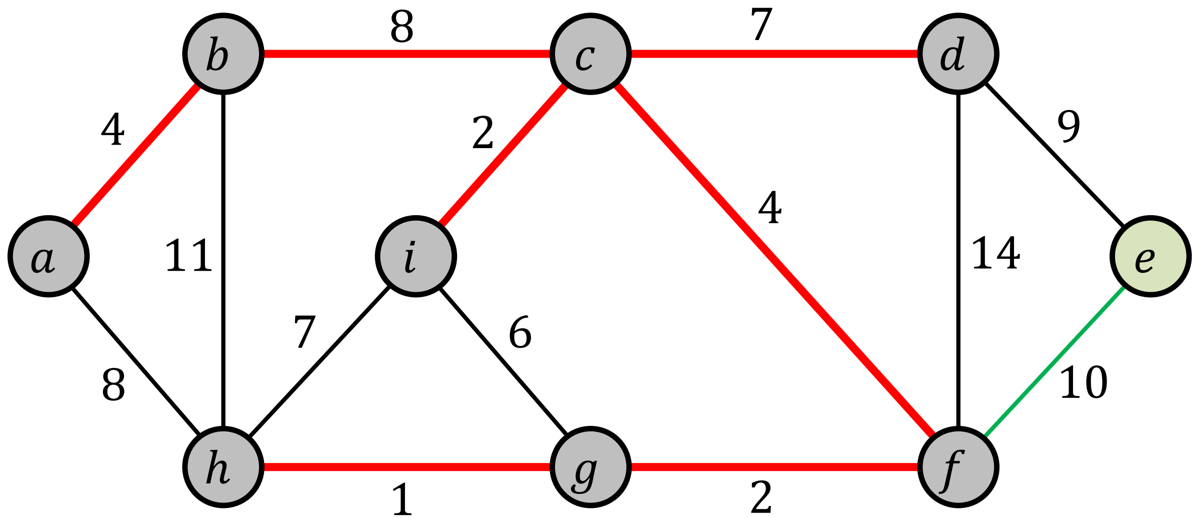
MST: Prim's Algorithm

Step $7'$: update neighbors of h



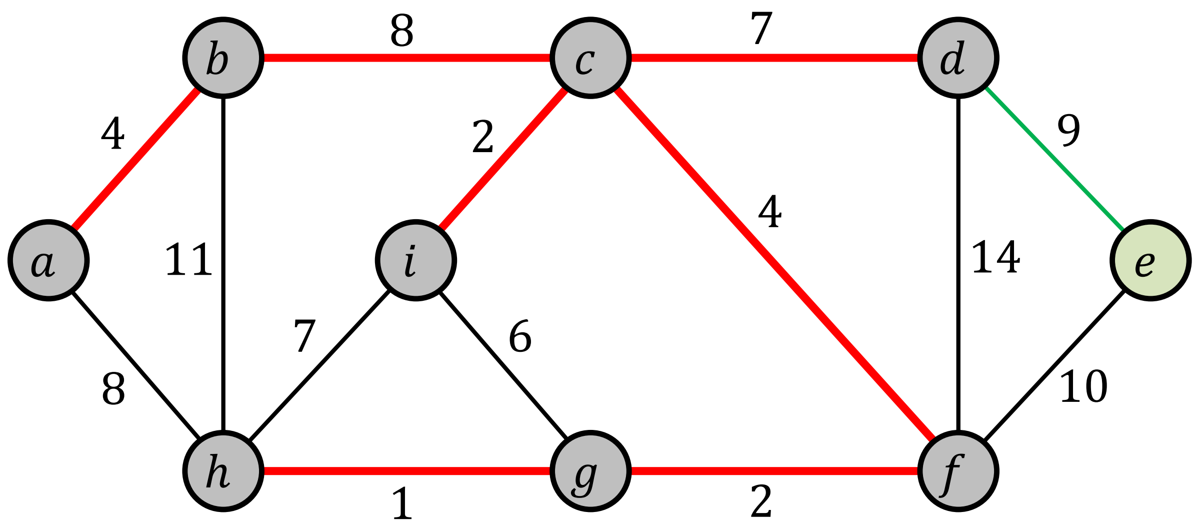
MST: Prim's Algorithm

Step 8: add vertex d through edge (c, d)



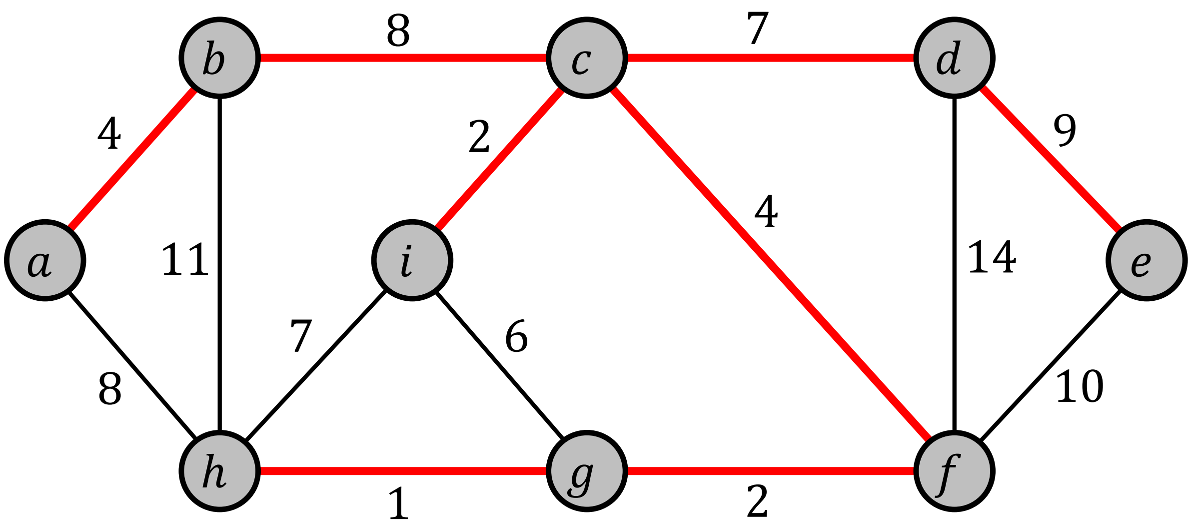
MST: Prim's Algorithm

Step $8'$: update neighbors of d



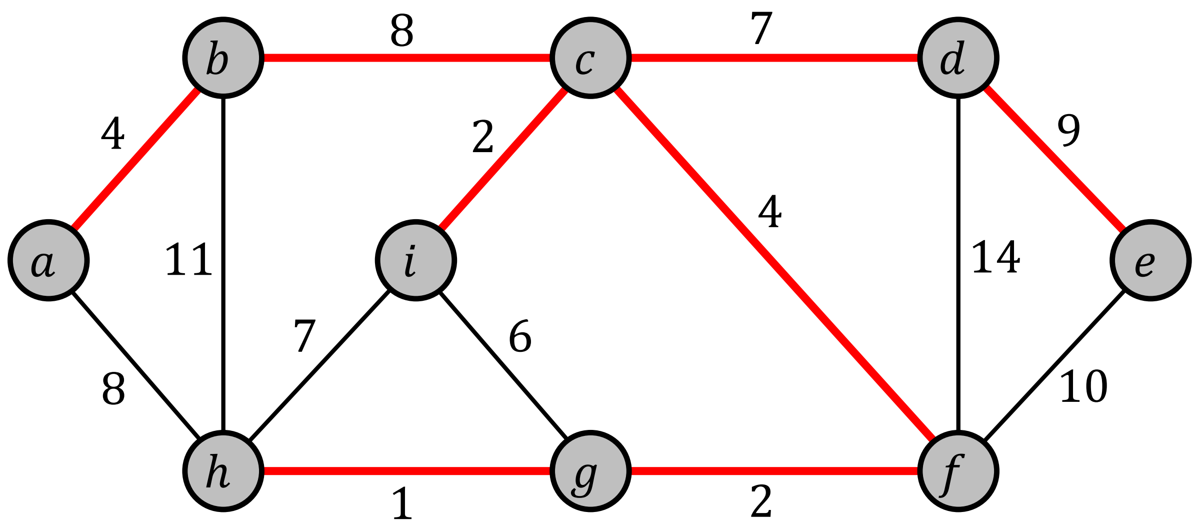
MST: Prim's Algorithm

Step 9: add vertex e through edge (d, e)



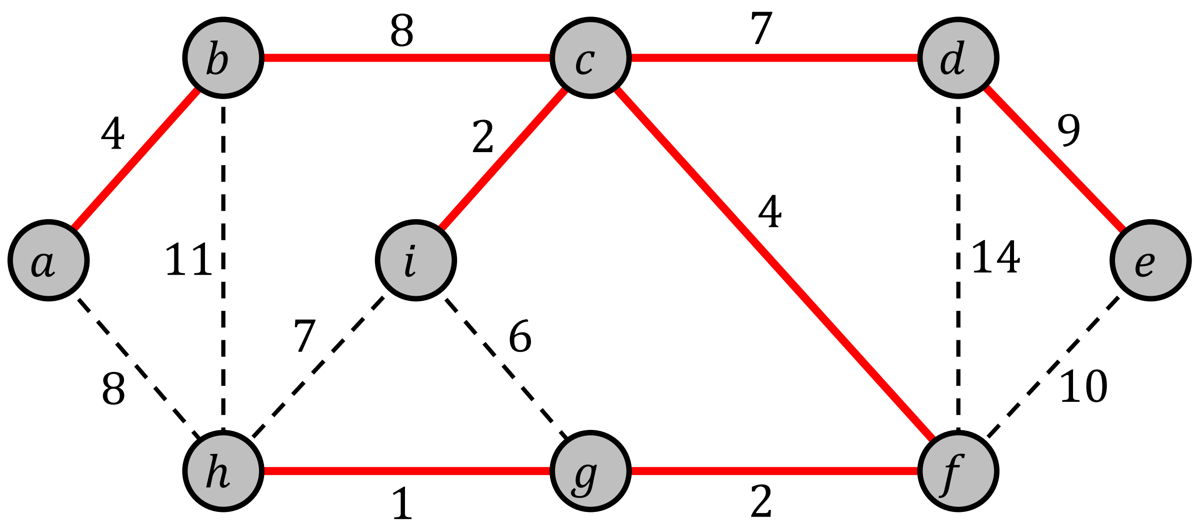
MST: Prim's Algorithm

Step $9'$: update neighbors of e



MST: Prim's Algorithm

Done



Total weight = 37

MST: Prim's Algorithm

MST-Prim ($G = (V, E),\ w,\ r$)

for each vertex $v \in G.V$ *do*
    $v.d \leftarrow \infty$
    $v.\pi \leftarrow NIL$
$r.d \leftarrow 0$
Min-Heap $Q \leftarrow \emptyset$
for each vertex $v \in G.V$ *do*
    INSERT(Q, v)
while $Q \neq \emptyset$ *do*
    $u \leftarrow$ EXTRACT-MIN(Q)
    for each $(u, v) \in G.E$ *do*
        if $v \in Q$ *and* $w(u, v) < v.d$ *then*
            $v.d \leftarrow w(u, v)$
            $v.\pi \leftarrow u$
            DECREASE-KEY($Q,\ v,\ w(u, v)$)

Let $n = |V|$ and $m = |E|$

\# INSERTS $= n$
\# EXTRACT-MINS $= n$
\# DECREASE-KEYS $\leq m$

Total cost

$$\leq\ n(cost_{Insert} + cost_{Extract-Min}) + m\left(cost_{Decrease-Key}\right)$$

MST: Prim's Algorithm

MST-Prim ($G = (V, E)$, w, r)

for each vertex $v \in G.V$ *do*
    $v.d \leftarrow \infty$
    $v.\pi \leftarrow NIL$
$r.d \leftarrow 0$
Min-Heap $Q \leftarrow \emptyset$
for each vertex $v \in G.V$ *do*
    INSERT(Q, v)
while $Q \neq \emptyset$ *do*
    $u \leftarrow$ EXTRACT-MIN(Q)
    for each $(u, v) \in G.E$ *do*
        if $v \in Q$ *and* $w(u, v) < v.d$ *then*
            $v.d \leftarrow w(u, v)$
            $v.\pi \leftarrow u$
            DECREASE-KEY(Q, v, $w(u, v)$)

Let $n = |V|$ and $m = |E|$

For Binary Heap (worst-case costs):

$$cost_{Insert} = \mathrm{O}(\log n)$$
$$cost_{Extract-Min} = \mathrm{O}(\log n)$$
$$cost_{Decrease-Key} = \mathrm{O}(\log n)$$

$\therefore$ Total cost (worst-case)

$$= \mathrm{O}\big((m + n) \log n\big)$$

MST: Prim's Algorithm

```
MST-Prim ( G = (V,E), w, r )
    for each vertex v ∈ G.V do
        v.d ← ∞
        v.π ← NIL
    r.d ← 0
    Min-Heap Q ← ∅
    for each vertex v ∈ G.V do
        INSERT( Q, v )
    while Q ≠ ∅ do
        u ← EXTRACT-MIN( Q )
        for each (u,v) ∈ G.E do
            if v ∈ Q and w(u,v) < v.d then
                v.d ← w(u,v)
                v.π ← u
                DECREASE-KEY( Q, v, w(u,v) )
```

Let $n = |V|$ and $m = |E|$

For Fibonacci Heap (amortized):

$$cost_{Insert} = O(1)$$
$$cost_{Extract-Min} = O(\log n)$$
$$cost_{Decrease-Key} = O(1)$$

$\therefore$ Total cost (amortized)

$$= O(m + n \log n)$$

The Single-Source Shortest Paths (SSSP) Problem

We are given a weighted, directed graph $G = (V, E)$ with vertex set V and edge set E, and a weight function w such that for each edge $(u, v) \in E$, $w(u, v)$ represents its weight.

We are also given a source vertex $s \in V$.

Our goal is to find a shortest path (i.e., a path of the smallest total edge weight) from s to each vertex $v \in V$.

SSSP: Relxation

Initialize-Single-Source ($G = (V, E)$, s)

1. *for* each vertex $v \in G.V$ *do*
2. $\quad v.d \leftarrow \infty$
3. $\quad v.\pi \leftarrow NIL$
4. $s.d \leftarrow 0$

Relax (u, v, w)

1. *if* $u.d + w(u, v) < v.d$ *then*
2. $\quad v.d \leftarrow u.d + w(u, v)$
3. $\quad v.\pi \leftarrow u$

SSSP: Properties of Shortest Paths and Relxation

The ***weight*** $w(p)$ of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

We define the ***shortest-path weight*** $\delta(u, v)$ from u to v by

$$\delta(u, v) = \begin{cases} \min\{w(p) : p \text{ is } u \sim v\}, & \text{if there is a path from } u \text{ to } v, \\ \infty, & \text{otherwise.} \end{cases}$$

A ***shortest path*** from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.

SSSP: Properties of Shortest Paths and Relxation

Triangle inequality (Lemma 24.10 of CLRS)

For any edge $(u, v) \in E$, we have $\delta(s, v) \le \delta(s, u) + w(u, v)$.

Upper-bound inequality (Lemma 24.11 of CLRS)

We always have $v.d \ge \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(u, v)$, it never changes.

No-path property (Corollary 24.12 of CLRS)

If there is no path from s to v, then we always have $v.d = \delta(s, v) = \infty$.

Convergence property (Lemma 24.14 of CLRS)

If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $v.d = \delta(s, v)$ at all times afterward.

SSSP: Properties of Shortest Paths and Relxation

Path-relaxation property (Lemma 24.15 of CLRS)

If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations on the edges of p.

Predecessor-subgraph property (Lemma 24.17 of CLRS)

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

Input: Weighted graph $G = (V, E)$ with vertex set V and edge set E, a weight function w, and a source vertex $s \in G[V]$.

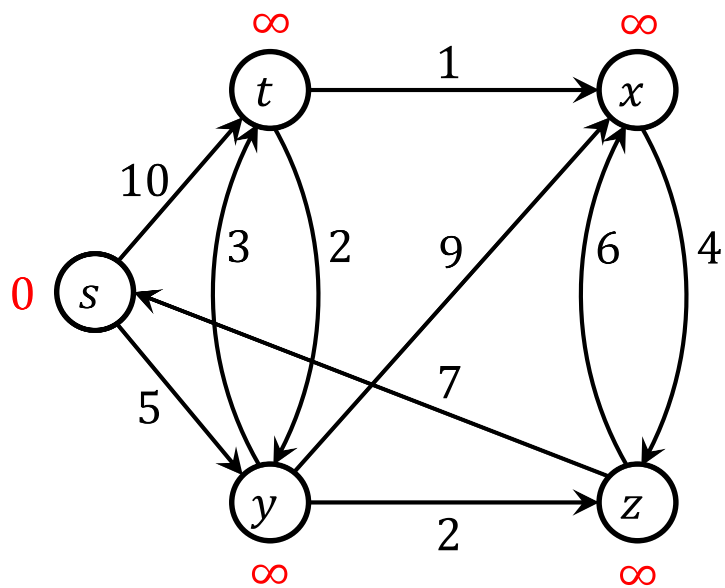
Output: For all $v \in G[V]$, $v.d$ is set to the shortest distance from s to v.

Dijkstra-SSSP ($G = (V, E)$, w, s)

for each vertex $v \in G.V$ *do*
    $v.d \leftarrow \infty$
    $v.\pi \leftarrow NIL$
$s.d \leftarrow 0$
Min-Heap $Q \leftarrow \emptyset$
for each vertex $v \in G.V$ *do*
    Insert(Q, v)
while $Q \neq \emptyset$ *do*
    $u \leftarrow$ *Extract-Min*(Q)
    for each $(u, v) \in G.E$ *do*
        if $u.d + w(u, v) < v.d$ *then*
            $v.d \leftarrow u.d + w(u, v)$
            $v.\pi \leftarrow u$
            Decrease-Key(Q, v, $u.d + w(u, v)$)

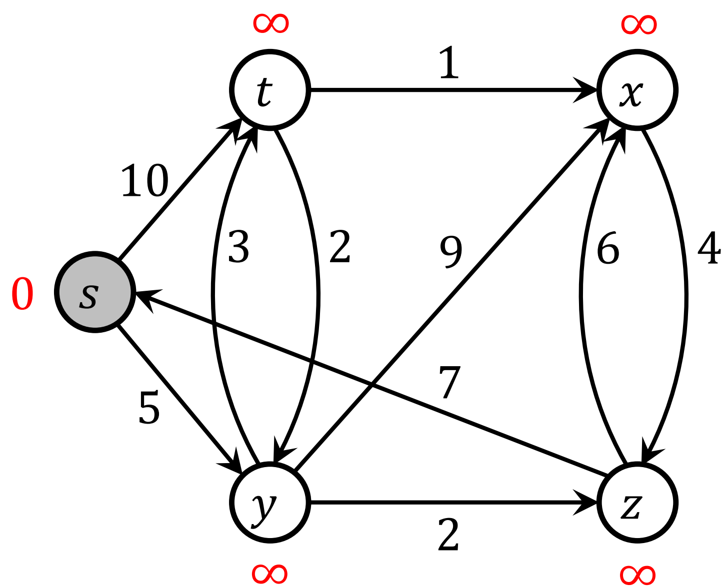
SSSP: Dijkstra's Algorithm

Initial State (with initial tentative distances)



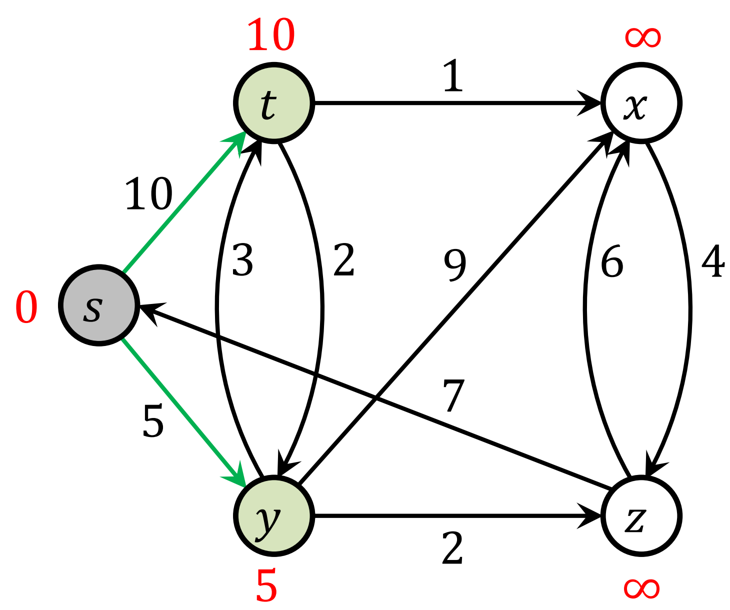
SSSP: Dijkstra's Algorithm

Step 1: add vertex s to SPT



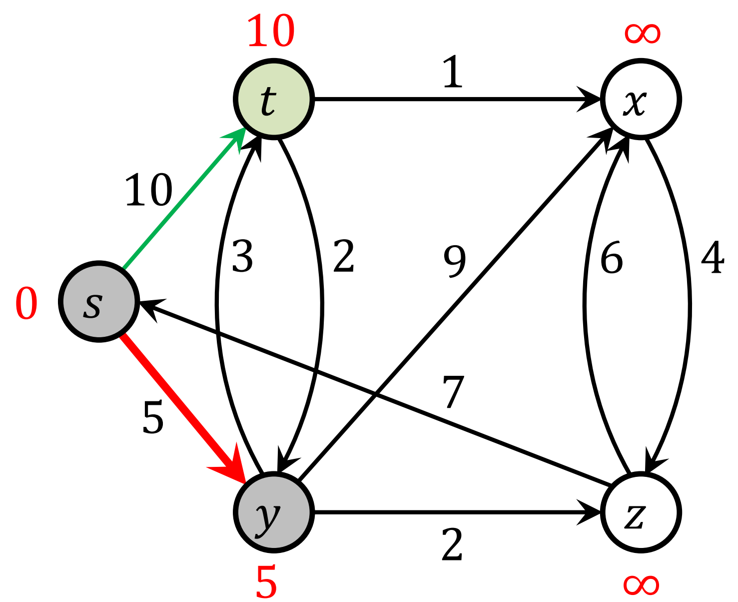
SSSP: Dijkstra's Algorithm

Step $1'$: update neighbors of s



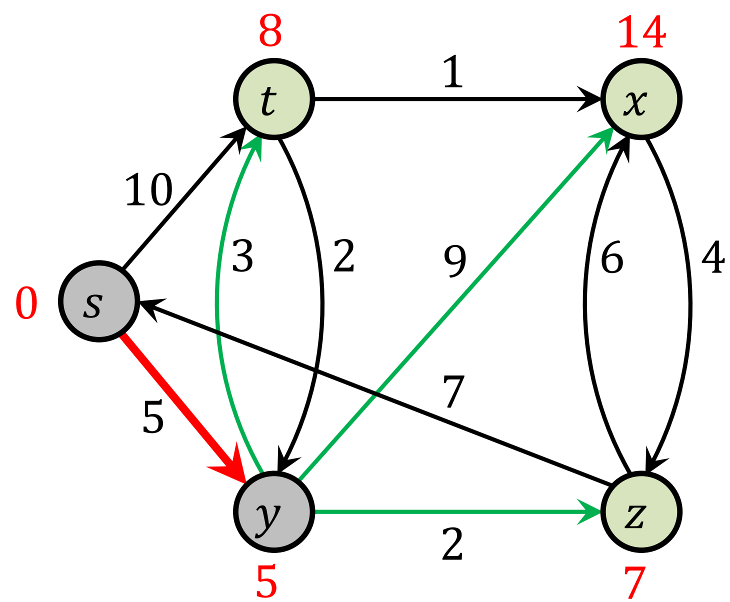
SSSP: Dijkstra's Algorithm

Step 2: add vertex y through edge (s, y)



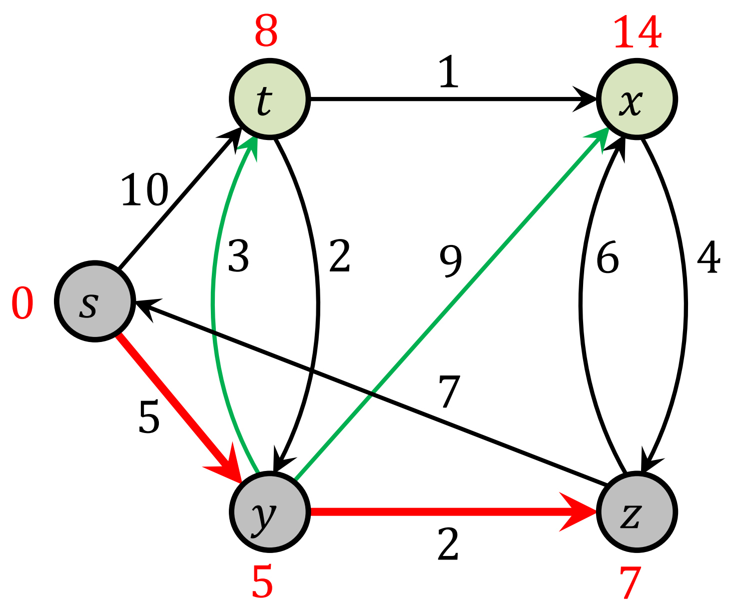
SSSP: Dijkstra's Algorithm

Step $2'$: update neighbors of y



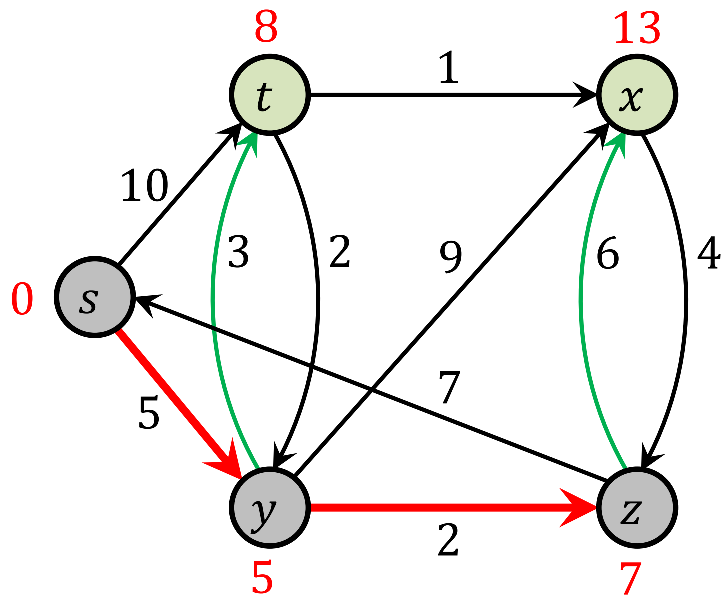
SSSP: Dijkstra's Algorithm

Step 3: add vertex z through edge (y, z)



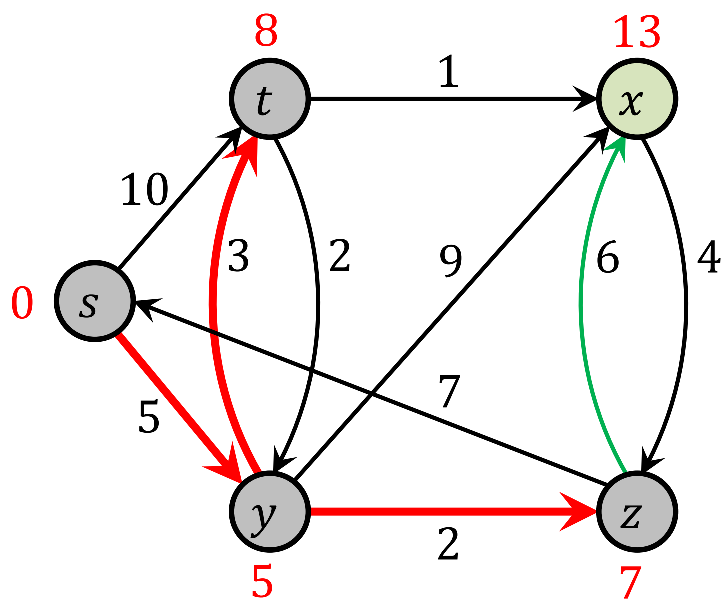
SSSP: Dijkstra's Algorithm

Step $3'$: update neighbors of z



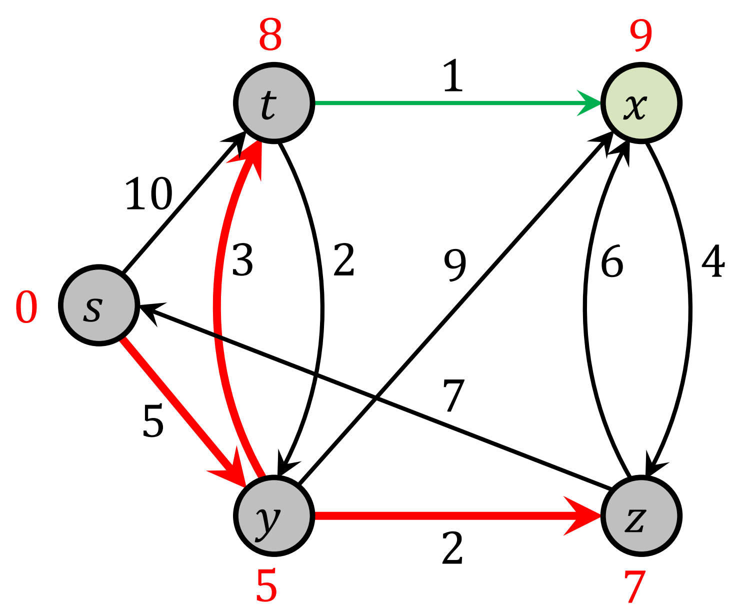
SSSP: Dijkstra's Algorithm

Step 4: add vertex t through edge (y, t)



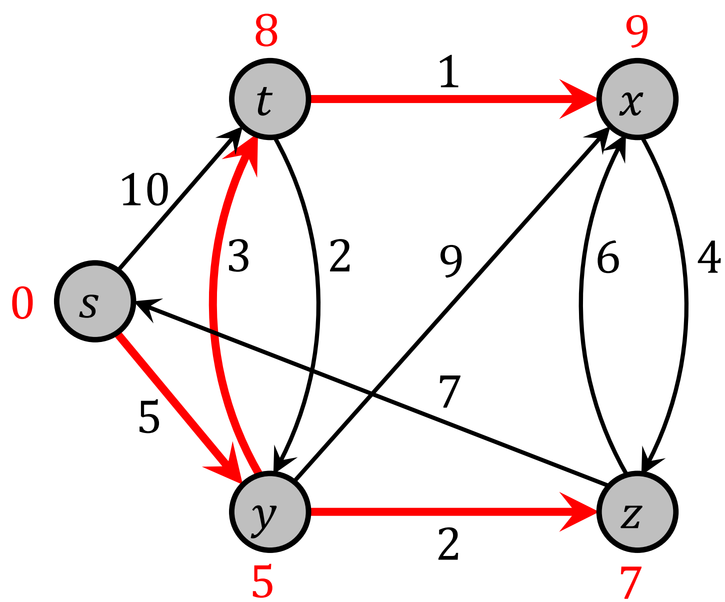
SSSP: Dijkstra's Algorithm

Step $4'$: update neighbors of t



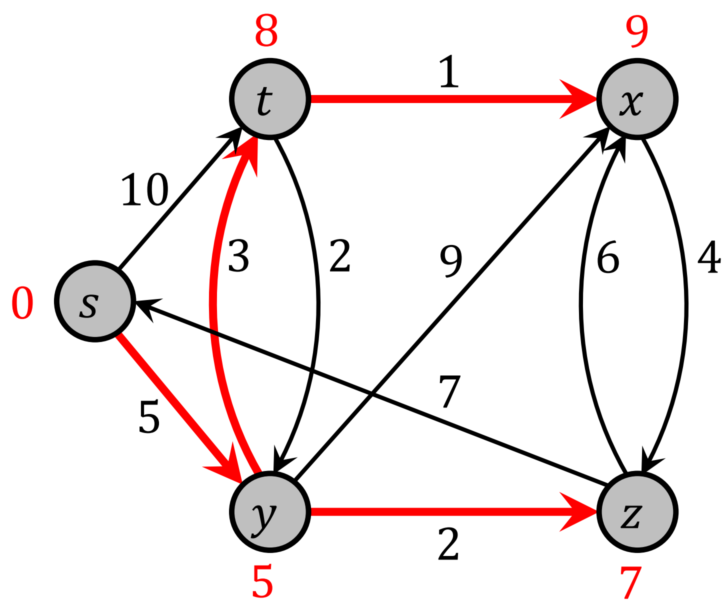
SSSP: Dijkstra's Algorithm

Step 5: add vertex x through edge (t, x)



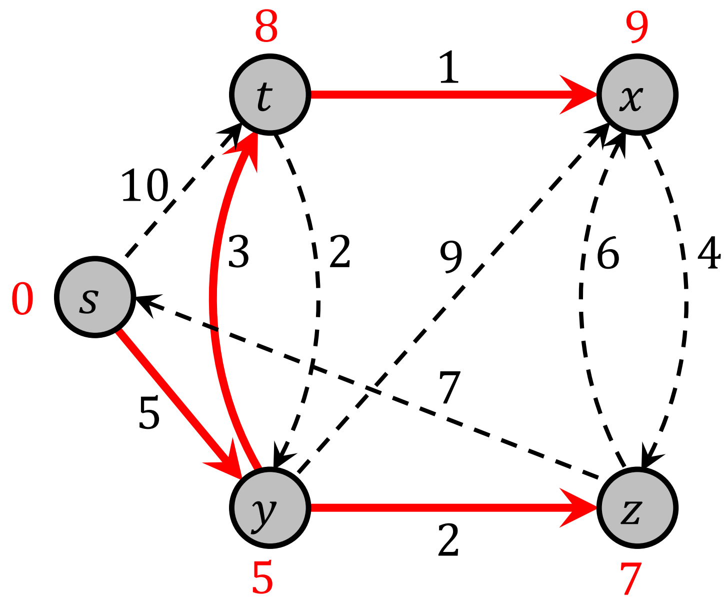
SSSP: Dijkstra's Algorithm

Step $5'$: update neighbors of x



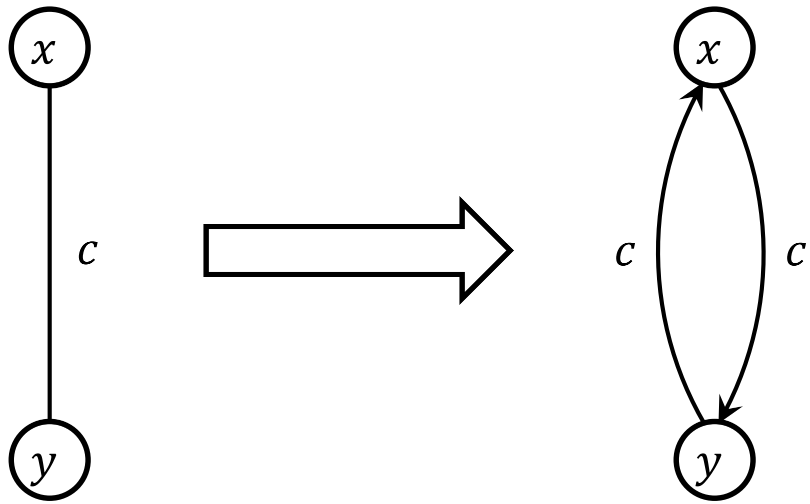
SSSP: Dijkstra's Algorithm

Done



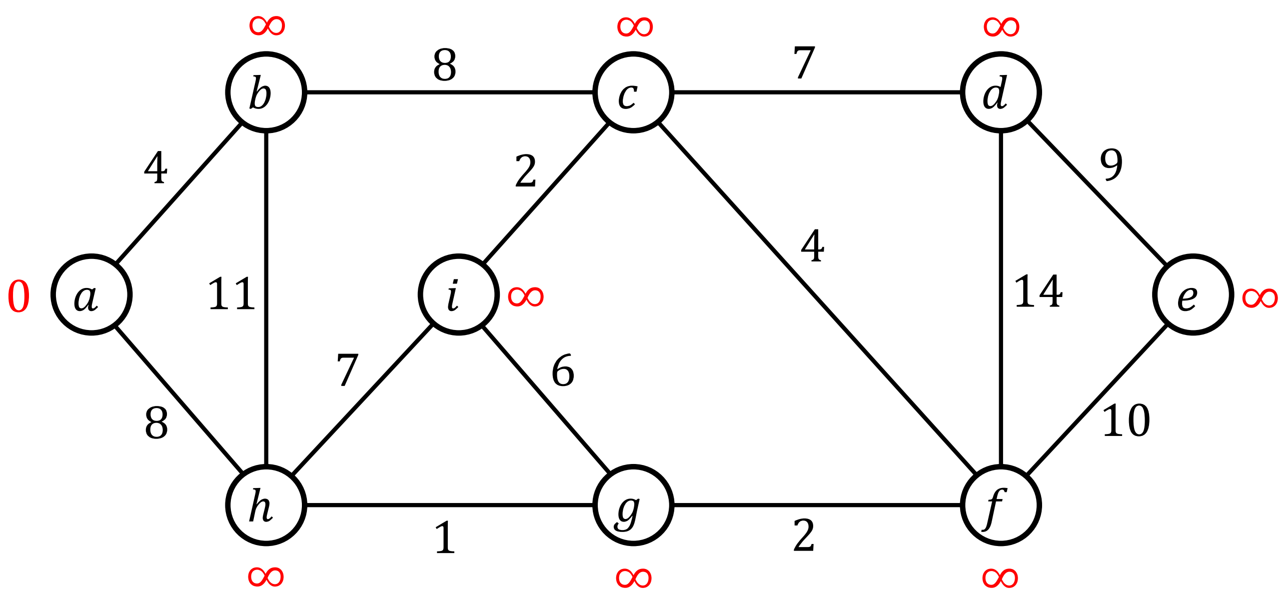
SSSP: Dijkstra’s Algorithm

One undirected edge ⇒ Two directed edges



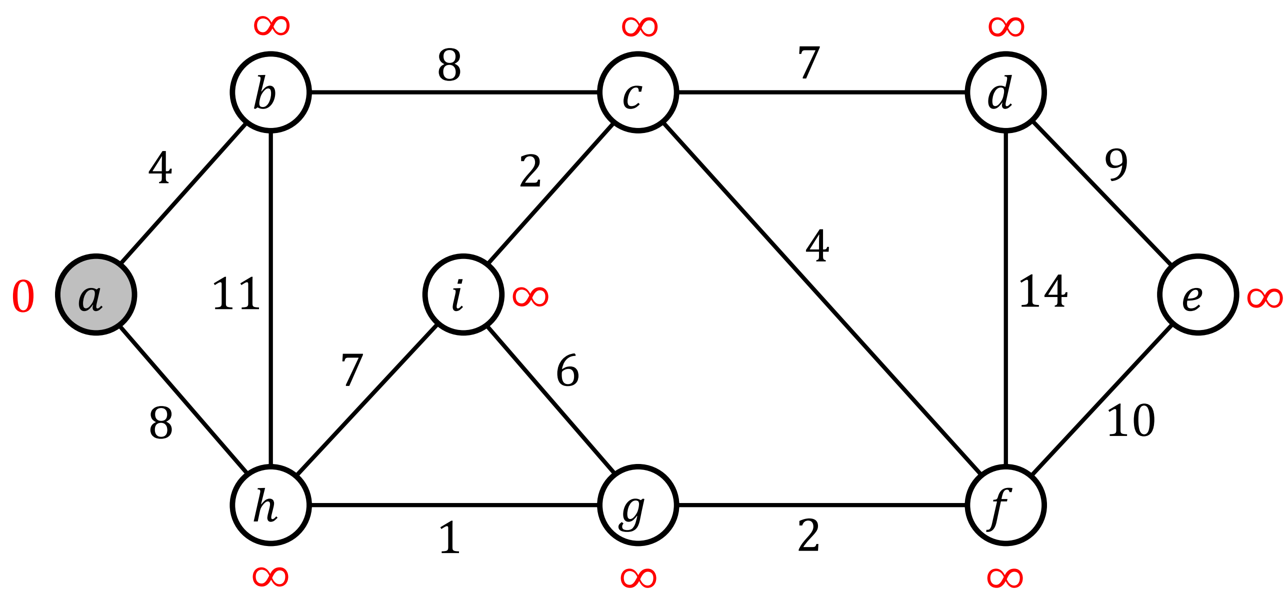
SSSP: Dijkstra's Algorithm

Initial State (with initial tentative distances)



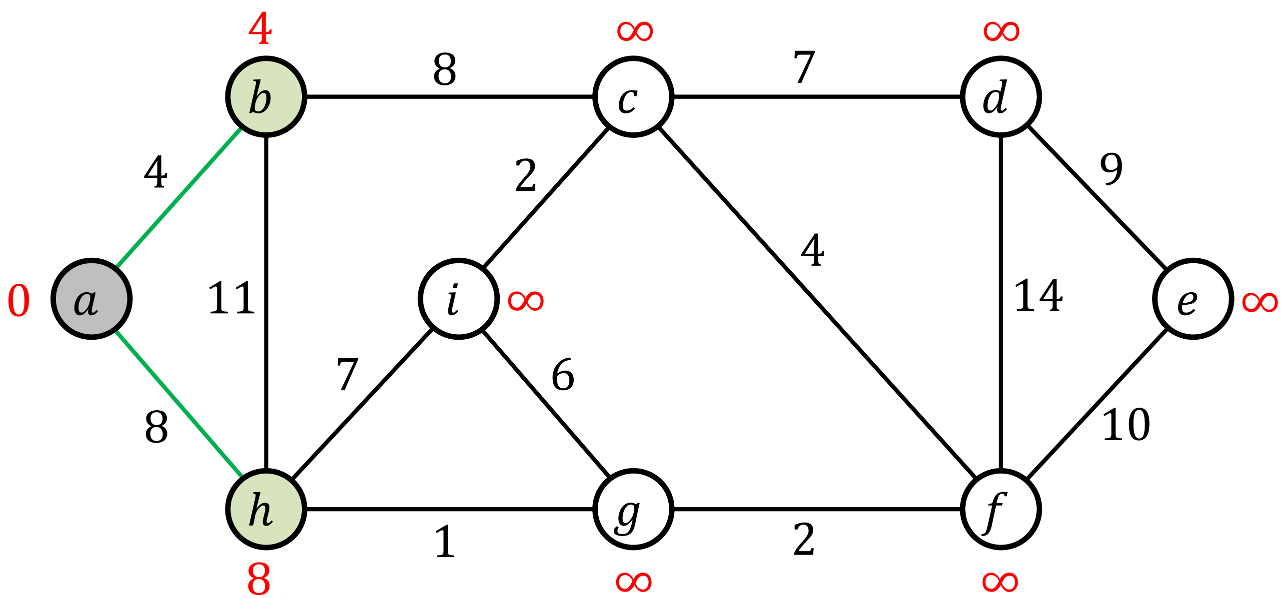
SSSP: Dijkstra's Algorithm

Step 1: add vertex a to SPT



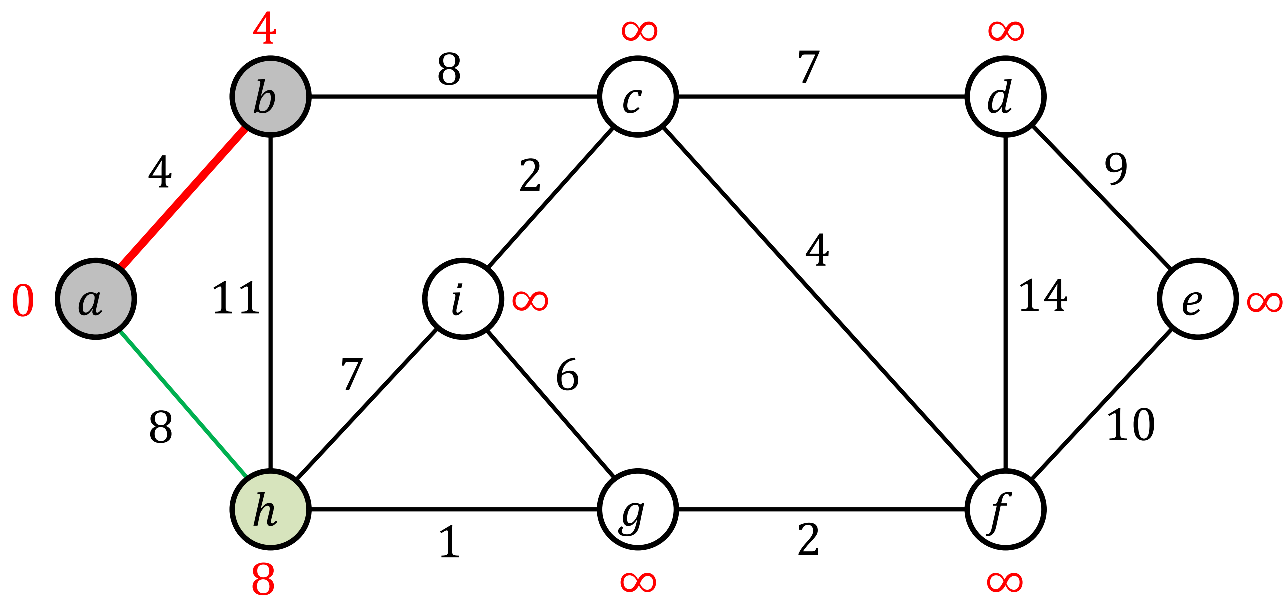
SSSP: Dijkstra's Algorithm

Step $1'$: update neighbors of a



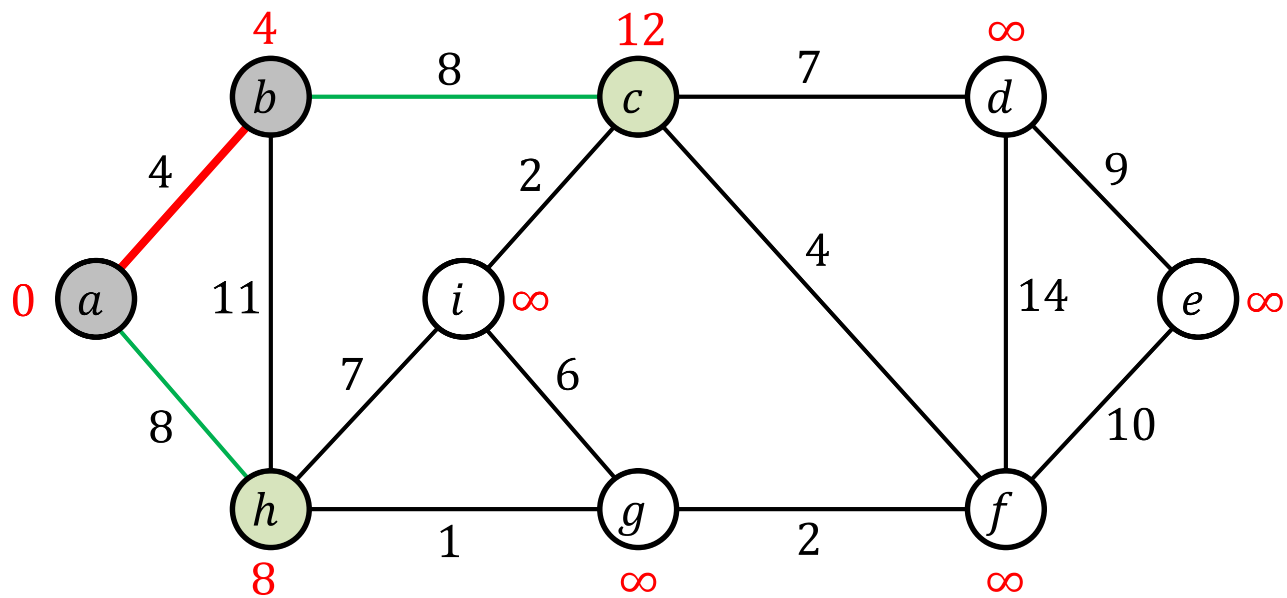
SSSP: Dijkstra's Algorithm

Step 2: add vertex b through edge (a, b)



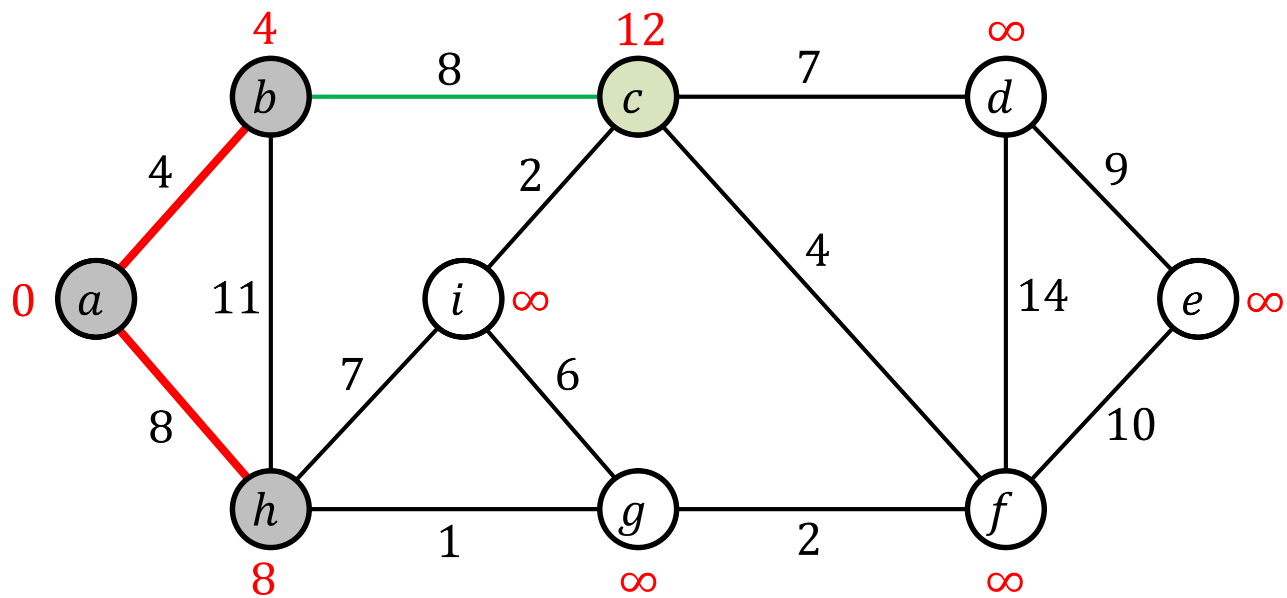
SSSP: Dijkstra's Algorithm

Step $2'$: update neighbors of b



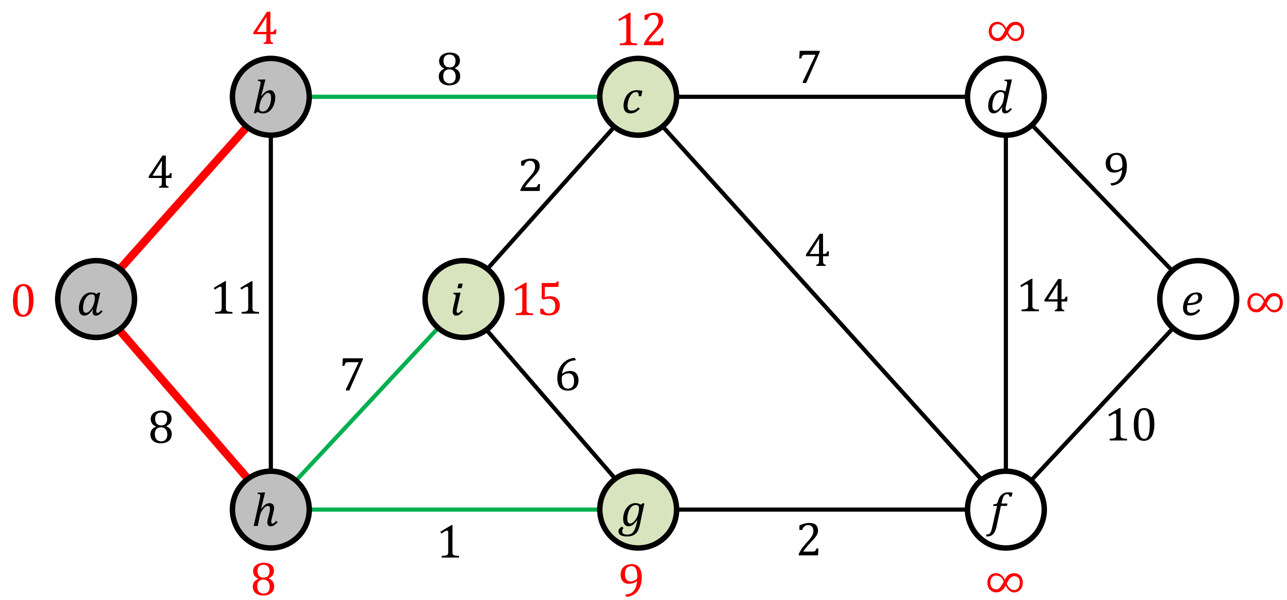
SSSP: Dijkstra's Algorithm

Step 3: add vertex h through edge (a, h)



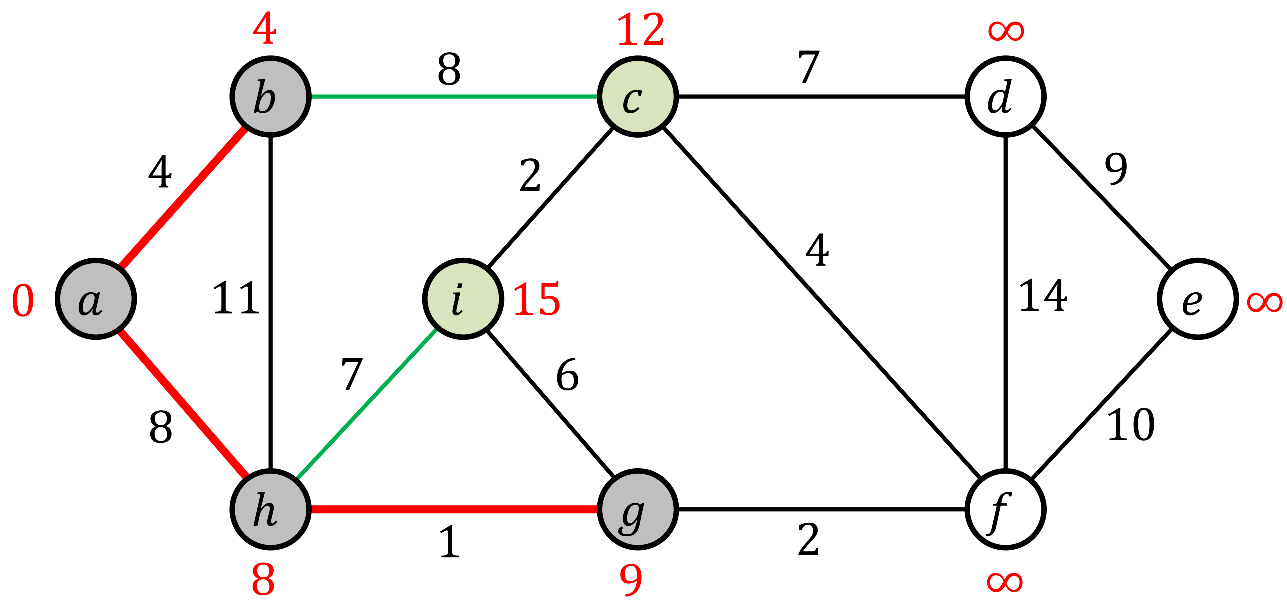
SSSP: Dijkstra's Algorithm

Step $3'$: update neighbors of h



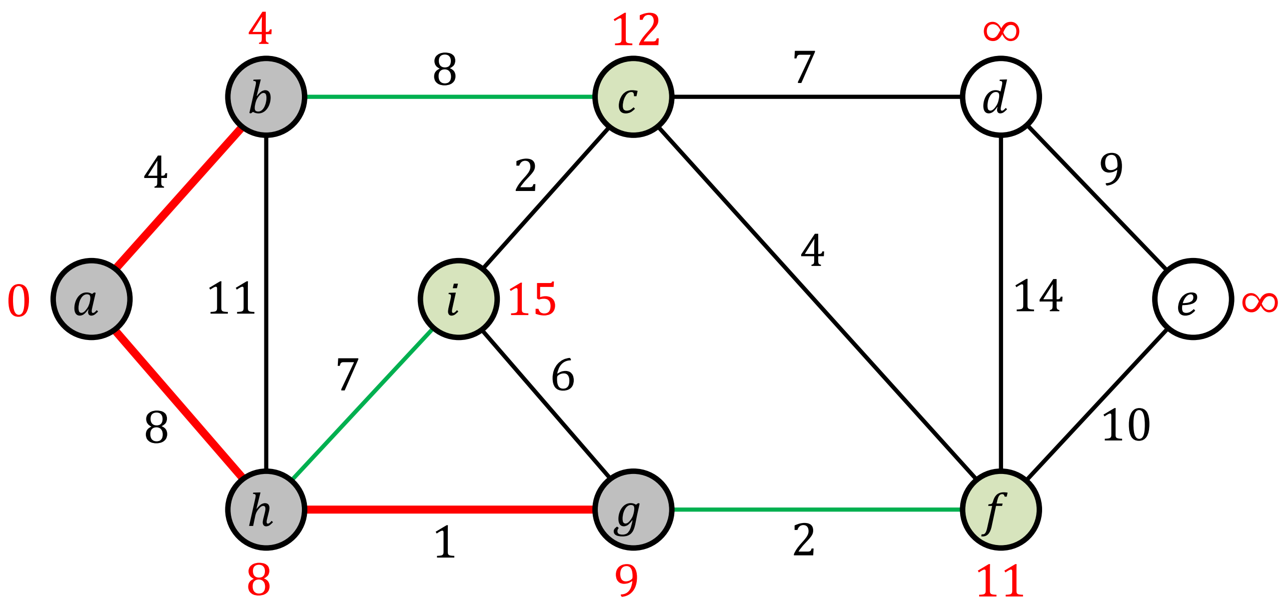
SSSP: Dijkstra's Algorithm

Step 4: add vertex g through edge (h, g)



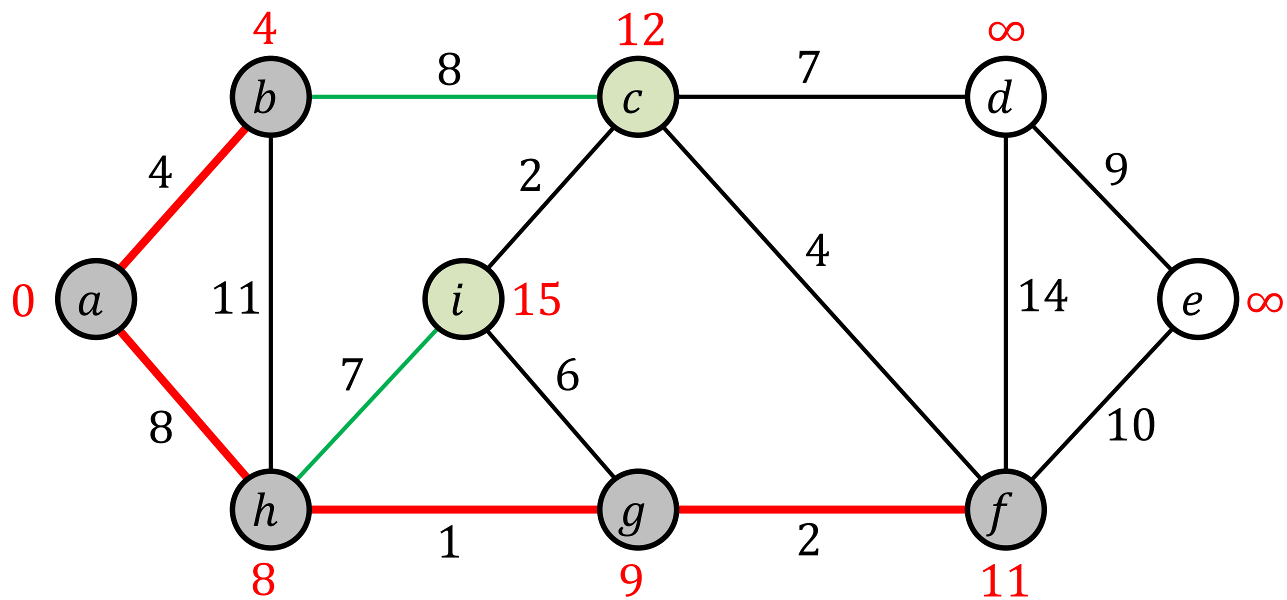
SSSP: Dijkstra's Algorithm

Step $4'$: update neighbors of g



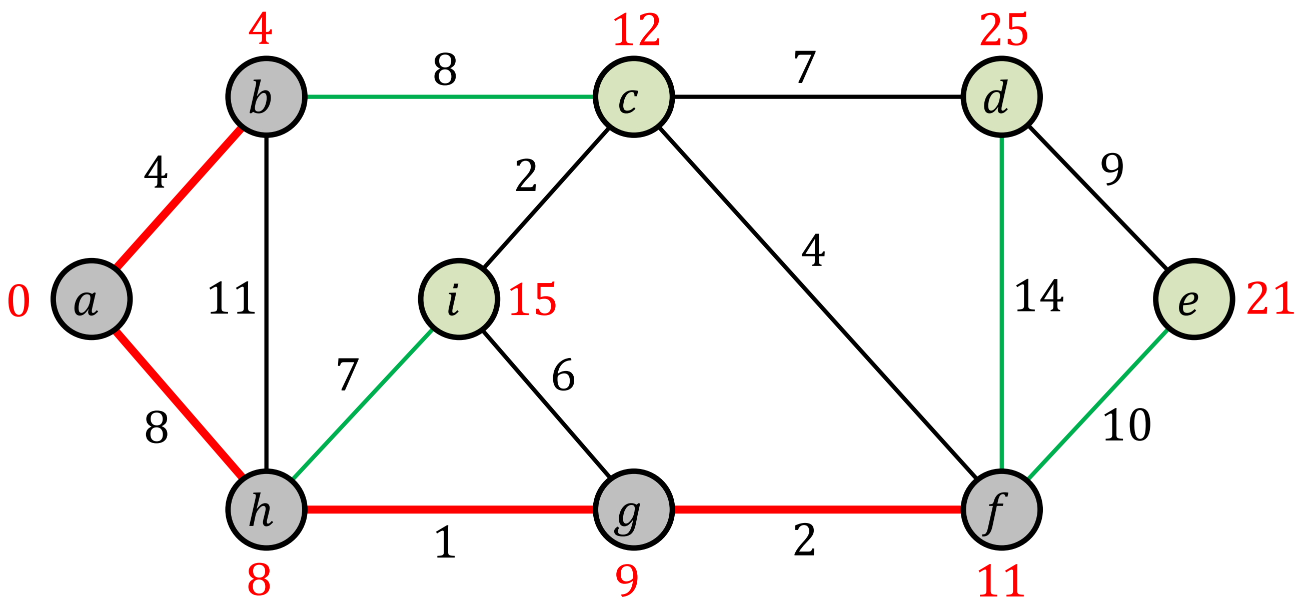
SSSP: Dijkstra's Algorithm

Step 5: add vertex f through edge (g, f)



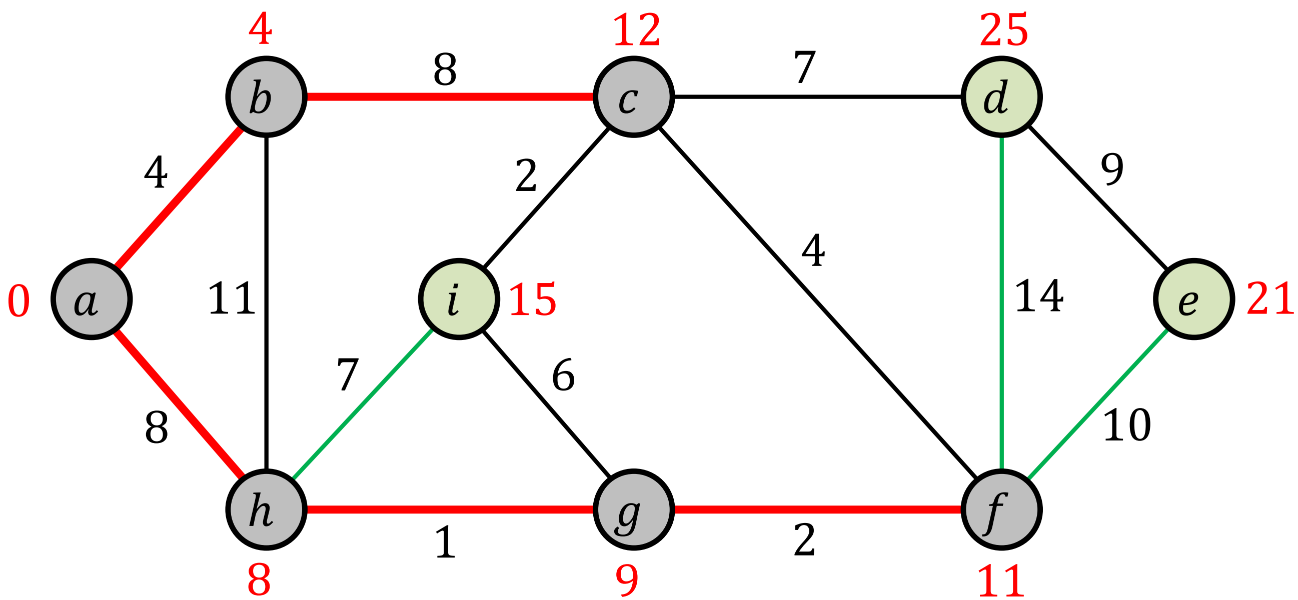
SSSP: Dijkstra's Algorithm

Step $5'$: update neighbors of f



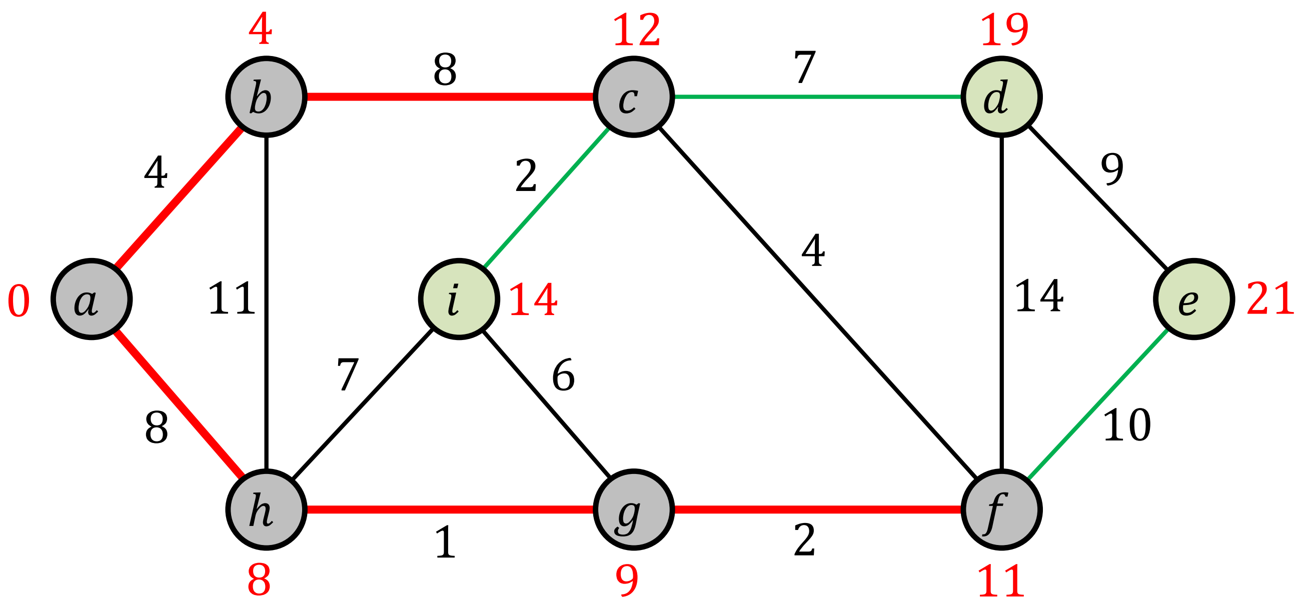
SSSP: Dijkstra's Algorithm

Step 6: add vertex c through edge (b, c)



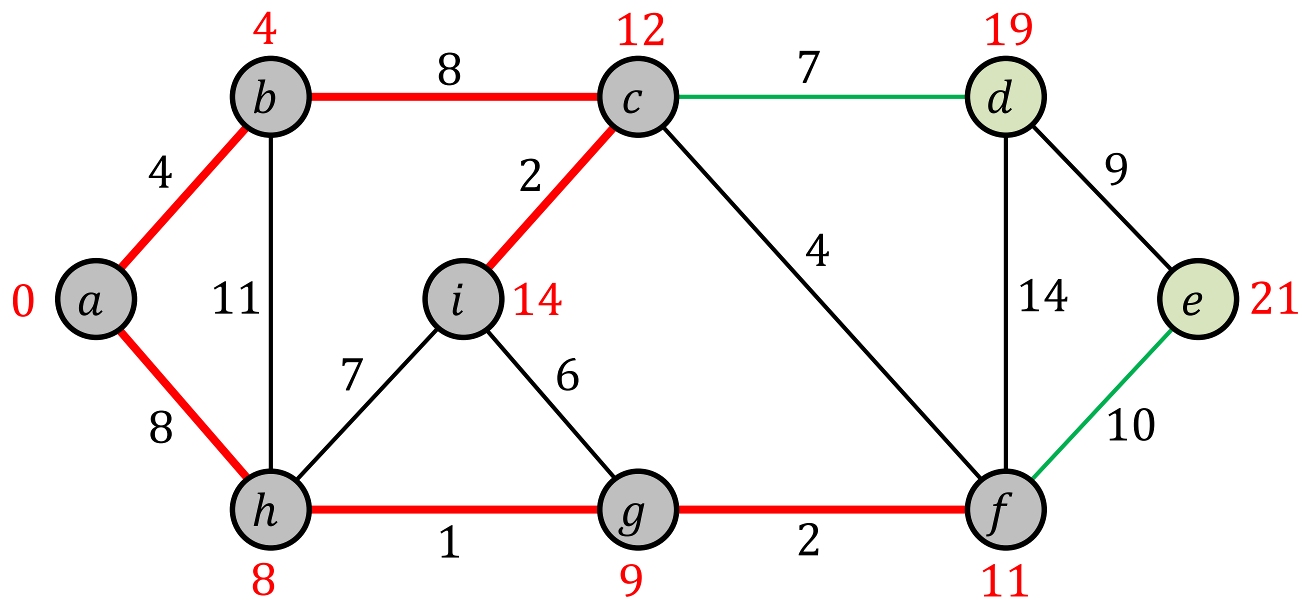
SSSP: Dijkstra's Algorithm

Step $6'$: update neighbors of c



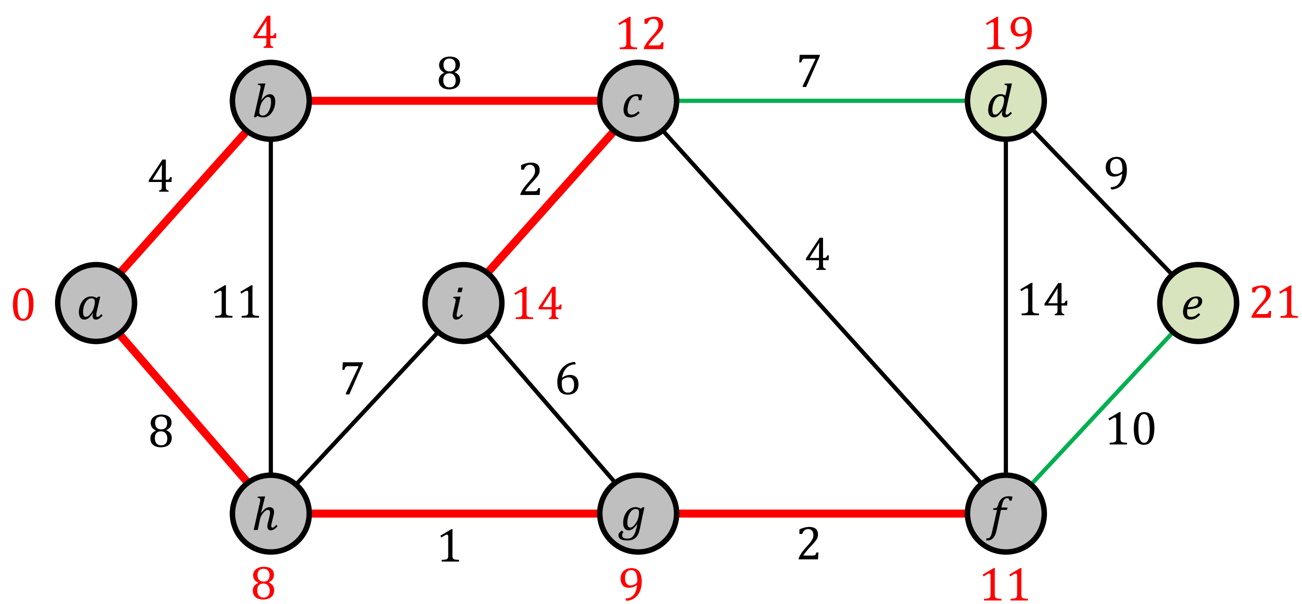
SSSP: Dijkstra's Algorithm

Step 7: add vertex i through edge (c, i)



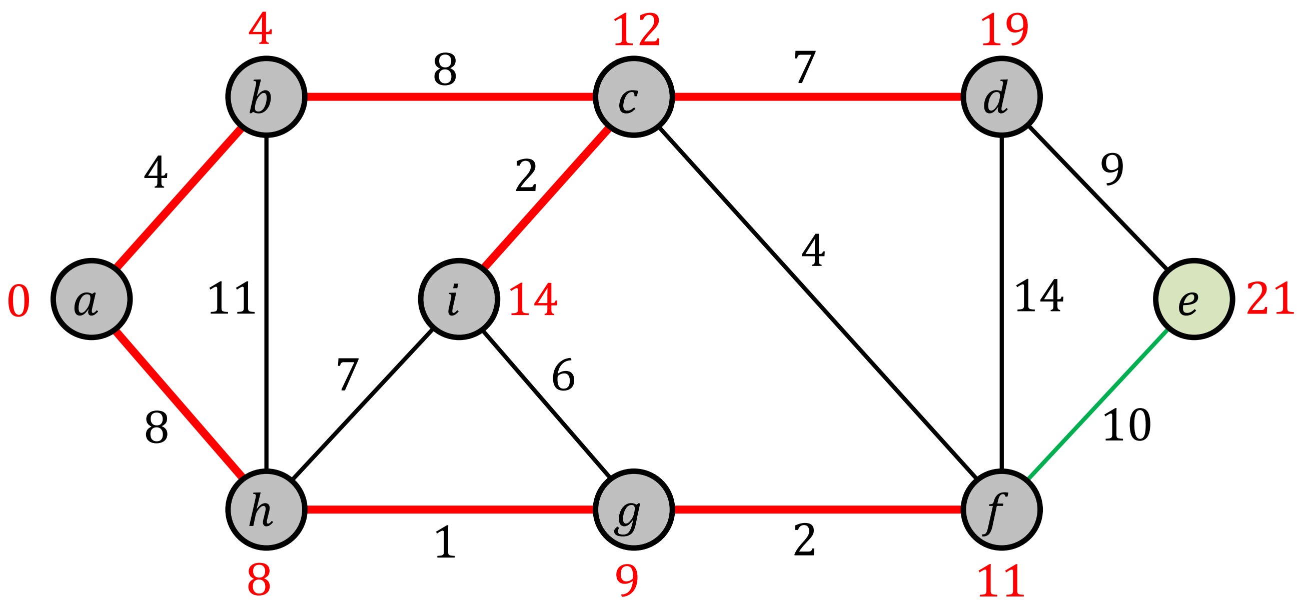
SSSP: Dijkstra's Algorithm

Step $7'$: update neighbors of i



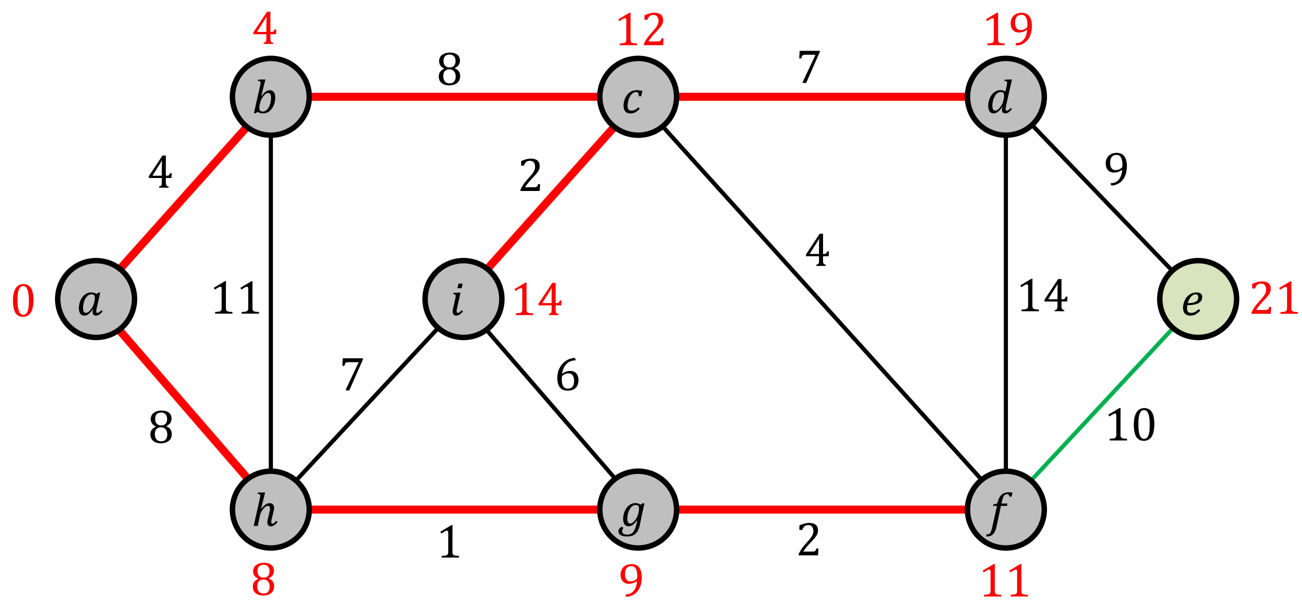
SSSP: Dijkstra's Algorithm

Step 8: add vertex d through edge (c, d)



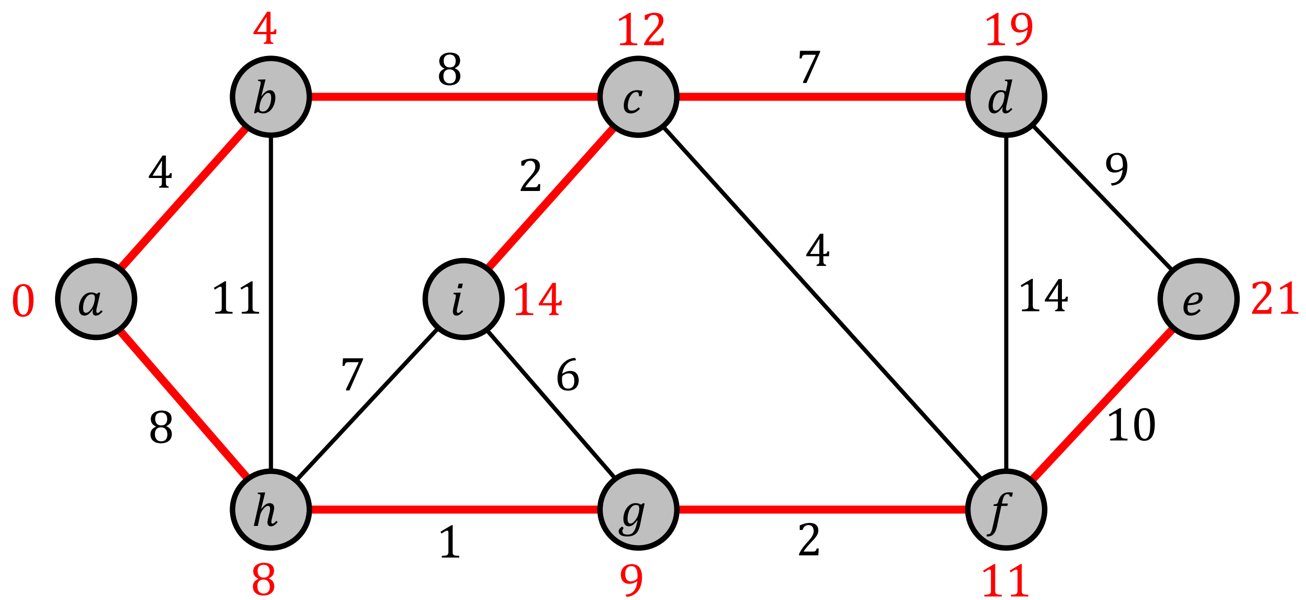
SSSP: Dijkstra's Algorithm

Step $8'$: update neighbors of d



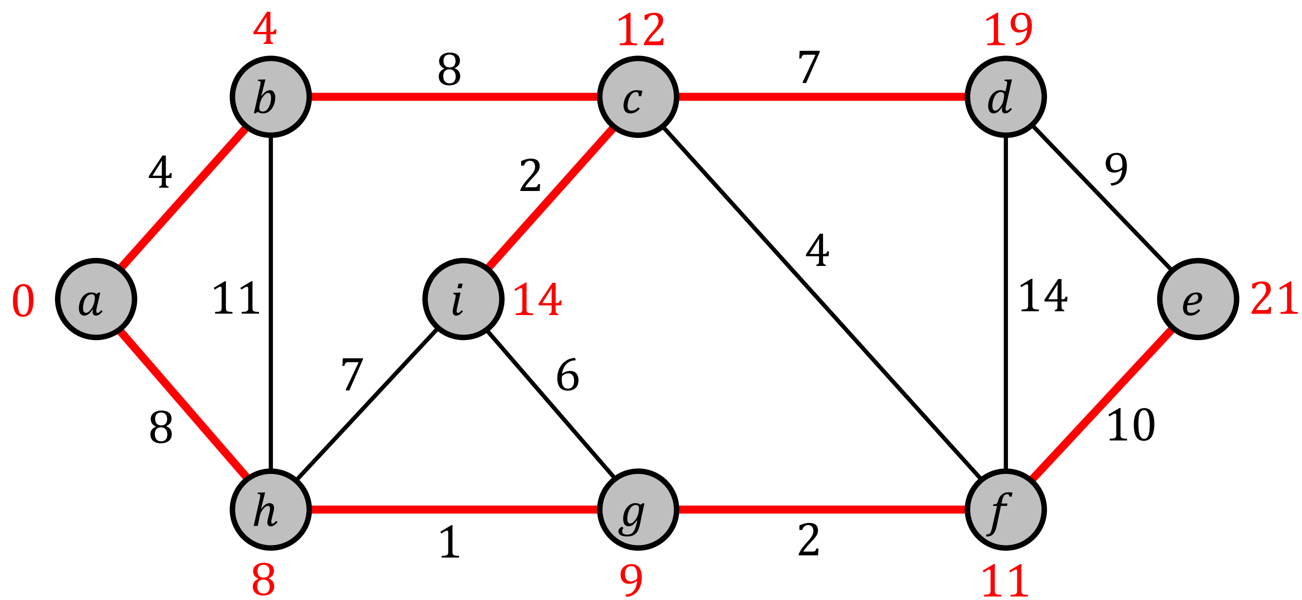
SSSP: Dijkstra's Algorithm

Step 9: add vertex e through edge (f, e)



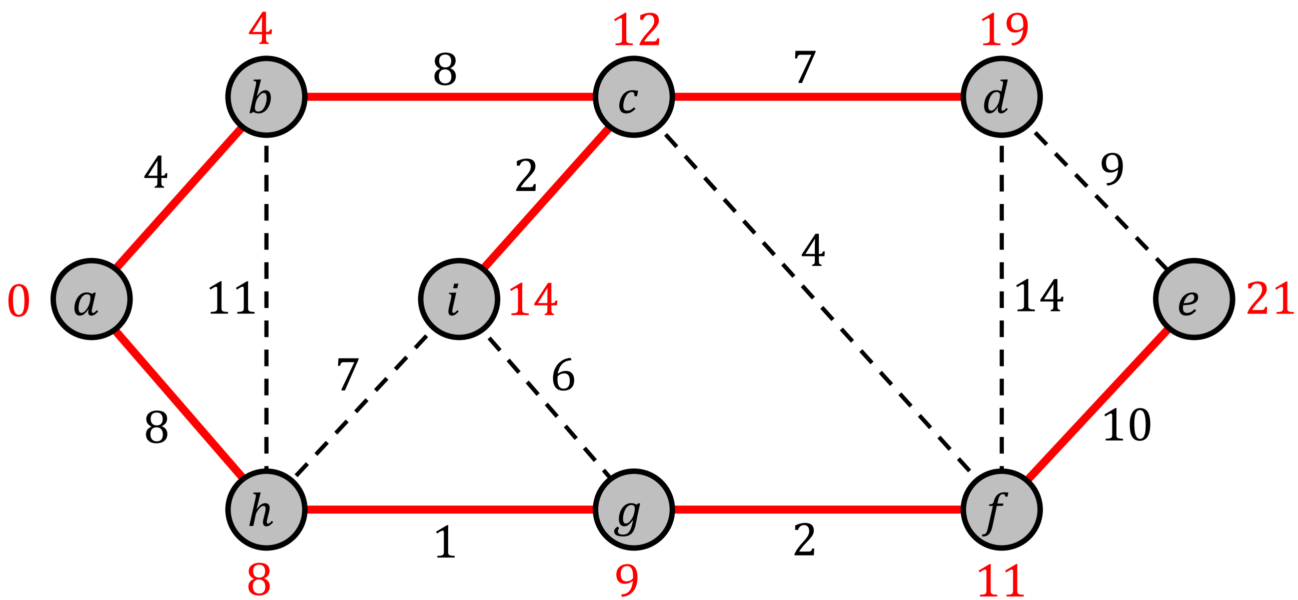
SSSP: Dijkstra's Algorithm

Step $9'$: update neighbors of e



SSSP: Dijkstra's Algorithm

Done



Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

Input: Weighted graph $G = (V, E)$ with vertex set V and edge set E, a weight function w, and a source vertex $s \in G[V]$.

Output: For all $v \in G[V]$, $v.d$ is set to the shortest distance from s to v.

```
Dijkstra-SSSP ( G = (V,E), w, s )

for each vertex v ∈ G.V do
    v.d ← ∞
    v.π ← NIL
s.d ← 0
Min-Heap Q ← ∅
for each vertex v ∈ G.V do
    INSERT( Q, v )
while Q ≠ ∅ do
    u ← EXTRACT-MIN( Q )
    for each (u,v) ∈ G.E do
        if u.d + w(u,v) < v.d then
            v.d ← u.d + w(u,v)
            v.π ← u
            DECREASE-KEY( Q, v, u.d + w(u,v) )
```

Let $n = |G[V]|$ and $m = |G[E]|$

INSERTS $= n$
EXTRACT-MINS $= n$
DECREASE-KEYS $\leq m$

Total cost

$$\leq n(cost_{Insert} + cost_{Extract-Min}) + m(cost_{Decrease-Key})$$

Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

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Output: For all $v \in G[V]$, $v.d$ is set to the shortest distance from s to v.

Dijkstra-SSSP ($G = (V, E),\ w,\ s$)

for each vertex $v \in G.V$ *do*
    $v.d \leftarrow \infty$
    $v.\pi \leftarrow NIL$
$s.d \leftarrow 0$
Min-Heap $Q \leftarrow \emptyset$
for each vertex $v \in G.V$ *do*
    INSERT(Q, v)
while $Q \neq \emptyset$ *do*
    $u \leftarrow$ EXTRACT-MIN(Q)
    for each $(u, v) \in G.E$ *do*
        if $u.d + w(u, v) < v.d$ *then*
            $v.d \leftarrow u.d + w(u, v)$
            $v.\pi \leftarrow u$
            DECREASE-KEY($Q,\ v,\ u.d + w(u, v)$)

Let $n = |G[V]|$ and $m = |G[E]|$

For Binary Heap (worst-case costs):

$$cost_{Insert} = O(\log n)$$
$$cost_{Extract-Min} = O(\log n)$$
$$cost_{Decrease-Key} = O(\log n)$$

$\therefore$ Total cost (worst-case)

$$= O\big((m + n) \log n\big)$$

Dijkstra's SSSP Algorithm with a Min-Heap
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                v.π ← u
                DECREASE-KEY( Q, v, u.d + w(u,v) )
```

Let $n = |G[V]|$ and $m = |G[E]|$

For Fibonacci Heap (amortized):

$$cost_{Insert} = O(1)$$
$$cost_{Extract-Min} = O(\log n)$$
$$cost_{Decrease-Key} = O(1)$$

$\therefore$ Total cost (amortized)

$$= O(m + n \log n)$$

Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

Input: Weighted graph $G = (V, E)$ with vertex set V and edge set E, a weight function w, and a source vertex $s \in G[V]$.

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                v.π ← u
                DECREASE-KEY( Q, v, u.d + w(u, v) )
```

Let $n = |G[V]|$ and $m = |G[E]|$

\# *Inserts* $= n$
\# *Extract-Mins* $= n$
\# *Decrease-Keys* $\leq m$

Total cost

$$\leq n(cost_{Insert} + cost_{Extract-Min}) + m(cost_{Decrease-Key})$$

Dijkstra's SSSP Algorithm with a Min-Heap
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$\therefore$ Total cost (worst-case)

$$= O\big((m+n)\log n\big)$$

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    for each vertex v ∈ G.V do
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```

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$$cost_{Insert} = O(1)$$
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$$cost_{Decrease-Key} = O(1)$$

$\therefore$ Total cost (worst-case)

$$= O(m + n \log n)$$

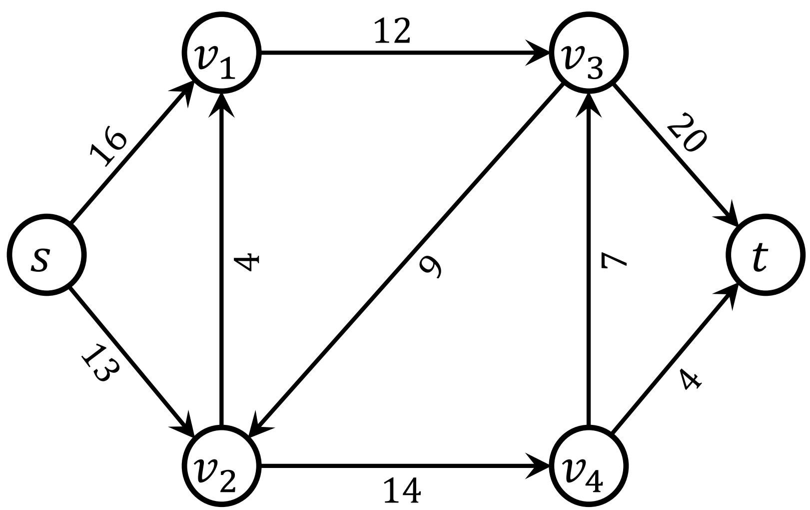
Flow Networks

A ***flow network*** $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v)$. Also, if $(u, v) \in E$ then $(v, u) \notin E$. If $(u, v) \notin E$, then we define $c(u, v) = 0$ for convenience, and we disallow self-loops.

We distinguish two vertices in a flow network: a ***source*** s and a ***sink*** t .

For convenience, we assume that each vertex lies on some path from s to t.

The graph is therefore connected and, since each vertex other than s has at least one entering edge, $|E| \geq |V| - 1$.



Flows and the Maximum Flow Problem

Let $G = (V, E)$ be a flow network with a capacity function c. Let s be the source of the network and let t be the sink.

A ***flow*** in G is a real-valued function $f: V \times V \to \mathbb{R}$ that satisfies the following two properties:

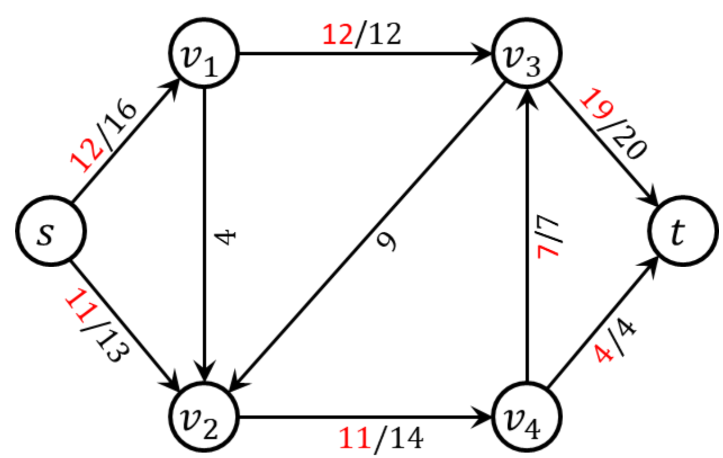
Capacity constraint: For all $u, v \in V$, we require $0 \le f(u, v) \le c(u, v)$.

Flow conservation: For all $u \in V \setminus \{s, t\}$, we require $\sum_{v \in \mathrm{V}} f(v, u) = \sum_{v \in \mathrm{V}} f(u, v)$.

When $(u, v) \notin E$, there can be no flow from u to v, and $f(u, v) = 0$.

The ***value*** $|f|$ of a flow f is defined as: $|f| = \sum_{v \in \mathrm{V}} f(s, v) - \sum_{v \in \mathrm{V}} f(v, s)$.

In the ***maximum-flow problem***, we are given a flow network G with source s and sink t, and we wish to find a flow of maximum value.



Maximum Flow: The Ford-Fulkerson Method

Input: A flow network $G = (V, E)$ with a capacity function c, a source vertex s and a sink vertex t.

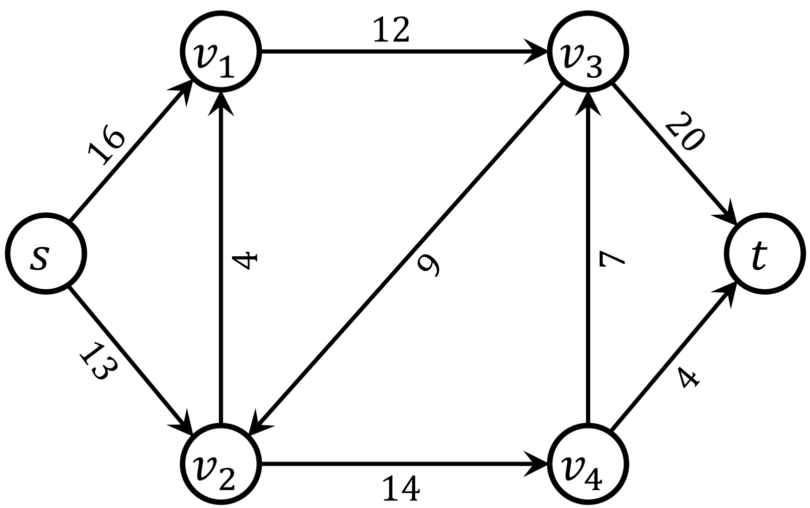
Output: A maximum s *to* t flow.

```
FORD-FULKERSON ( G = (V, E), s, t )

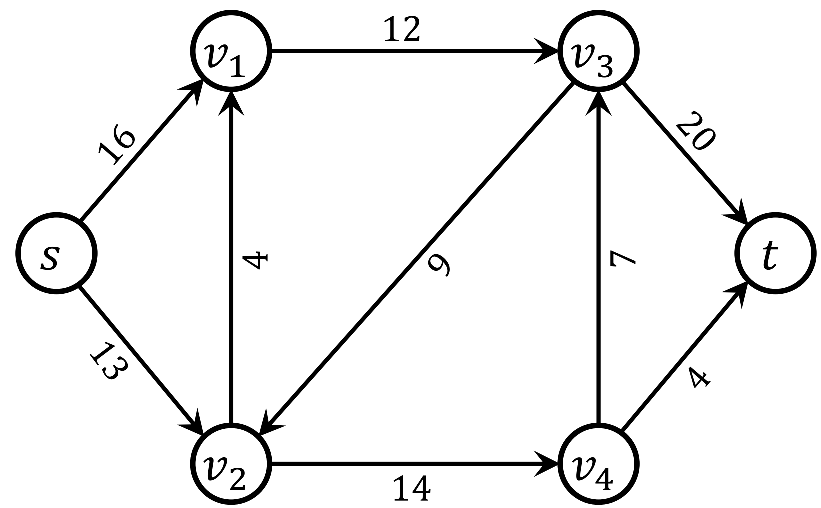
    for each edge (u, v) ∈ G.E do
        (u, v).f ← 0
    while there exists a path p from s to t in the residual network G_f do
        c_f(p) ← min{c_f(u, v) | (u, v) is in p}
        for each edge (u, v) in p do
            if (u, v) ∈ G.E then
                (u, v).f ← (u, v).f + c_f(p)
            else (v, u).f ← (v, u).f − c_f(p)
```

Maximum Flow: The Ford-Fulkerson Method

Original Network

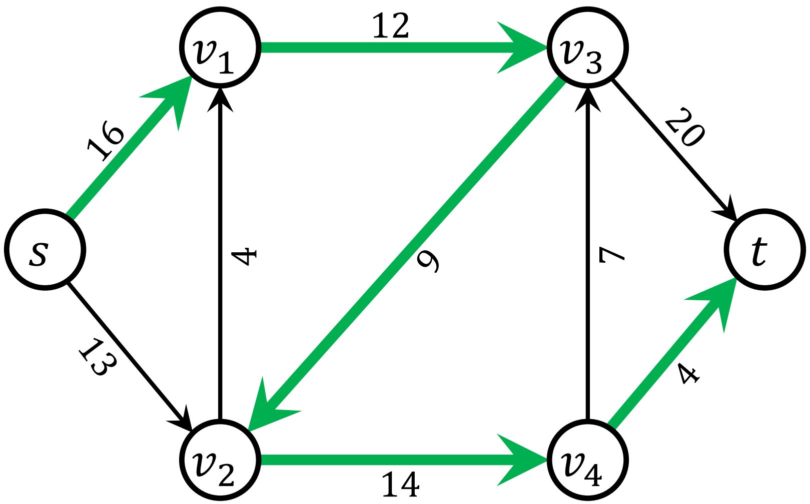


Original Network



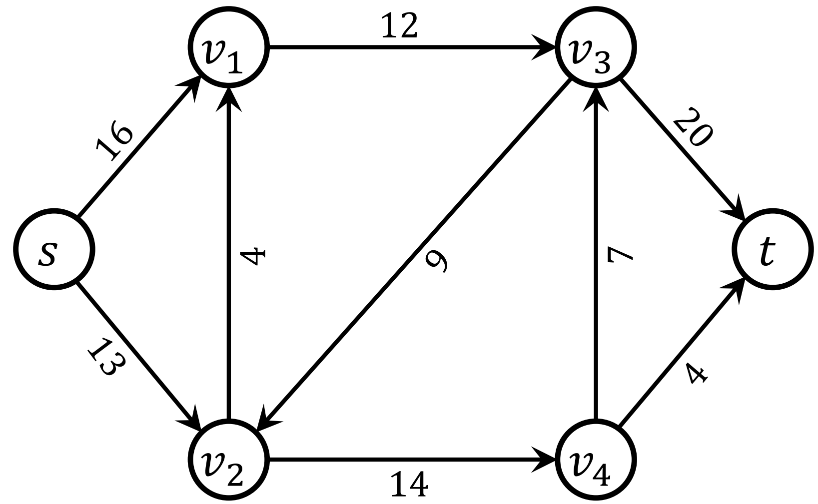
Maximum Flow: The Ford-Fulkerson Method

Step 1: Augmenting Path



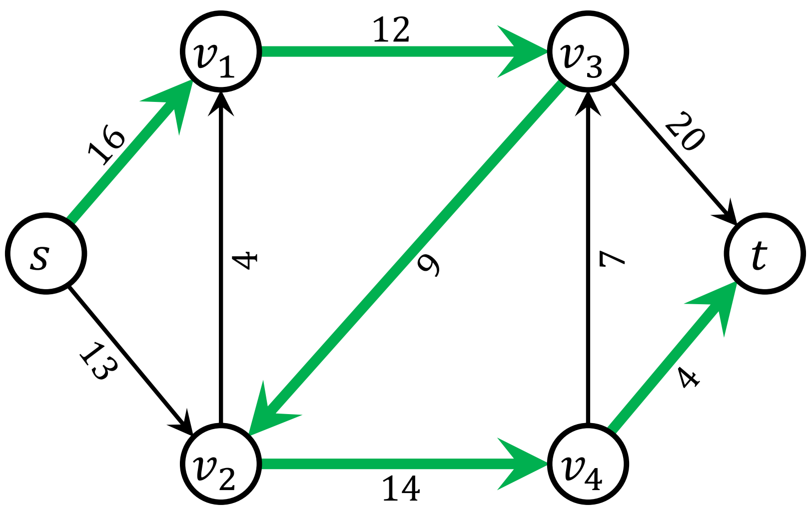
Residual capacity $= 4$

Original Network



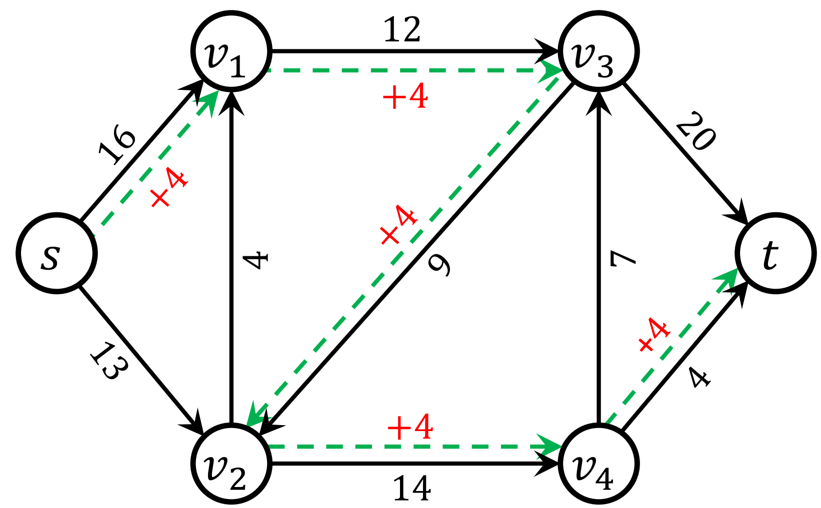
Maximum Flow: The Ford-Fulkerson Method

Step 1: Augmenting Path



Residual capacity $= 4$

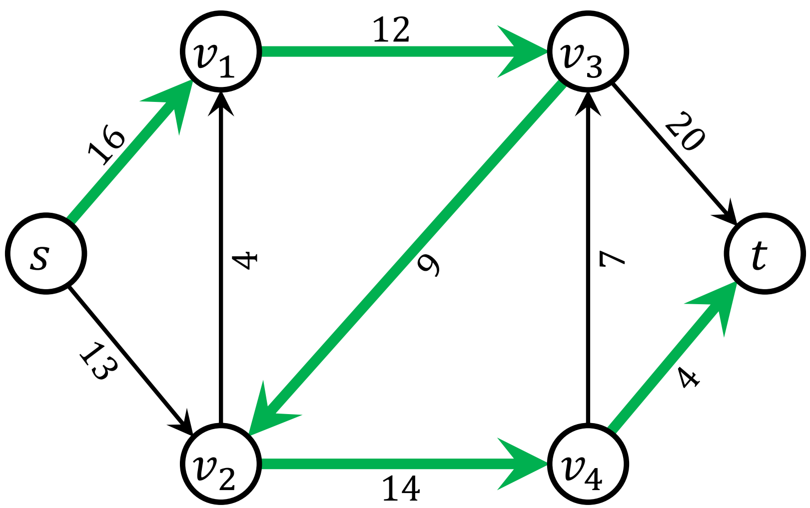
Step 1: Updating Flow



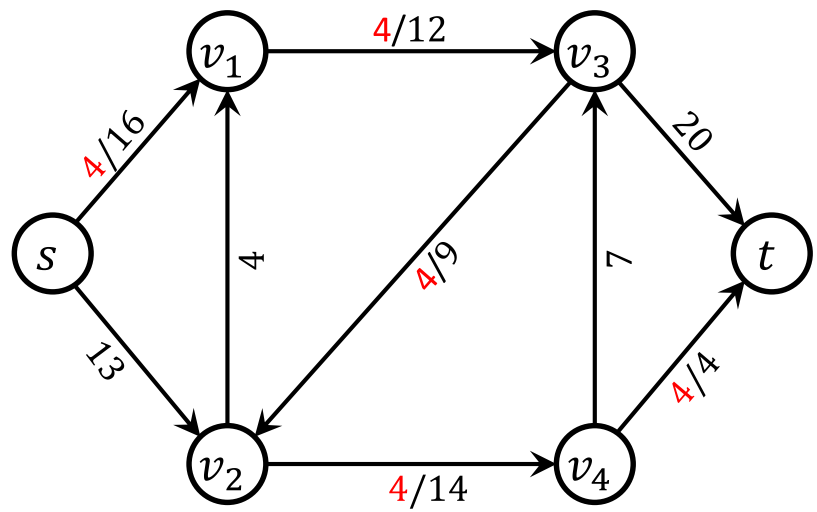
Increase flow by 4 along path
$s \rightarrow v_1 \rightarrow v_3 \rightarrow v_2 \rightarrow v_4 \rightarrow t$

Maximum Flow: The Ford-Fulkerson Method

Step 1: Augmenting Path



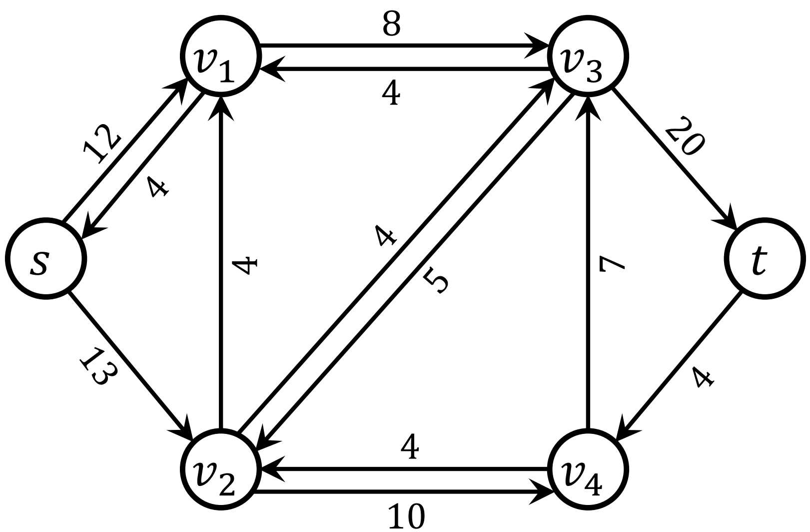
Step 1: Updated Flow



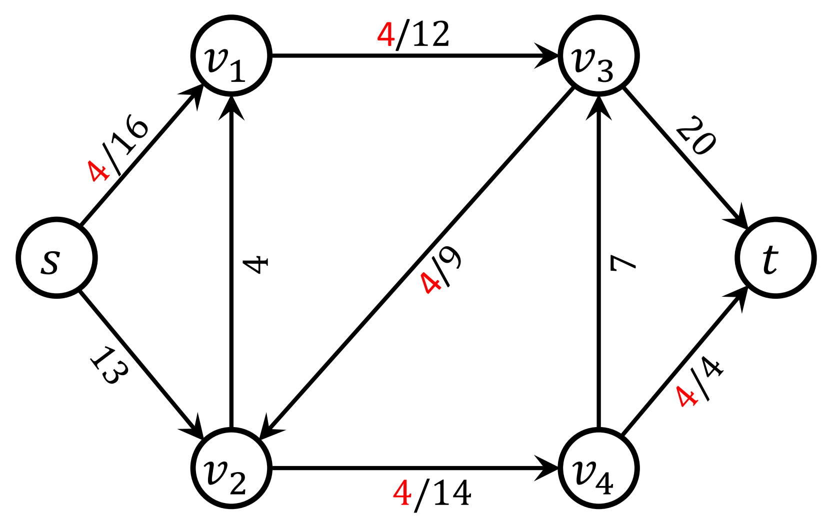
Current s to t flow $= 4$

Maximum Flow: The Ford-Fulkerson Method

Step 2: Residual Network

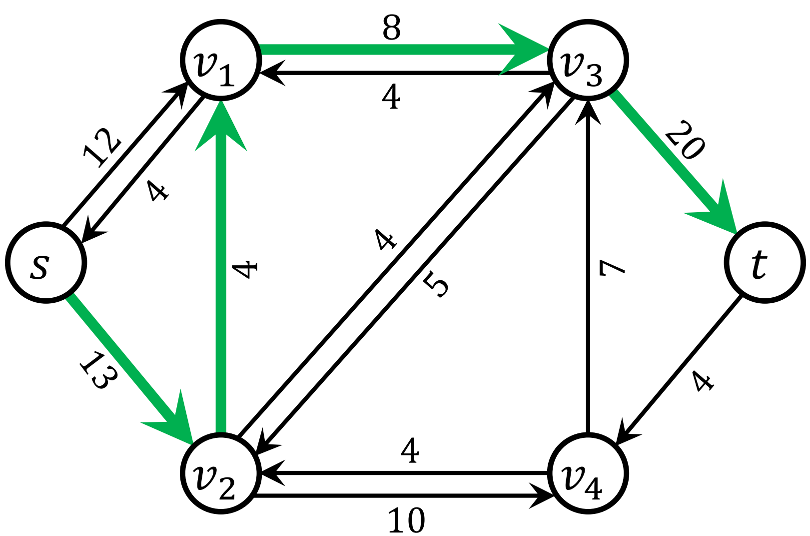


Step 1: Updated Flow



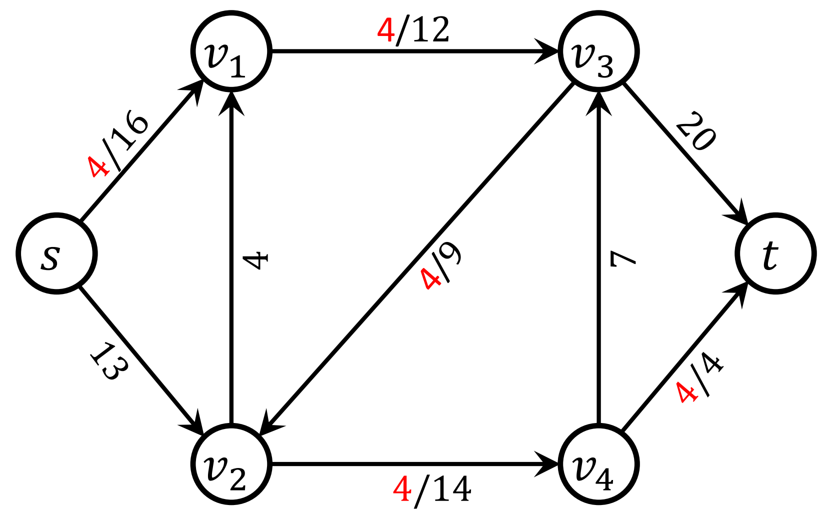
Maximum Flow: The Ford-Fulkerson Method

Step 2: Augmenting Path



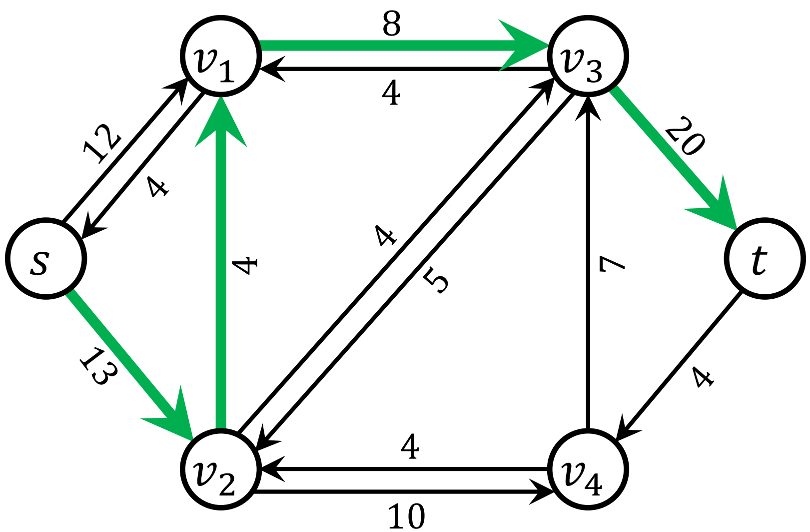
Residual capacity $= 4$

Step 1: Updated Flow



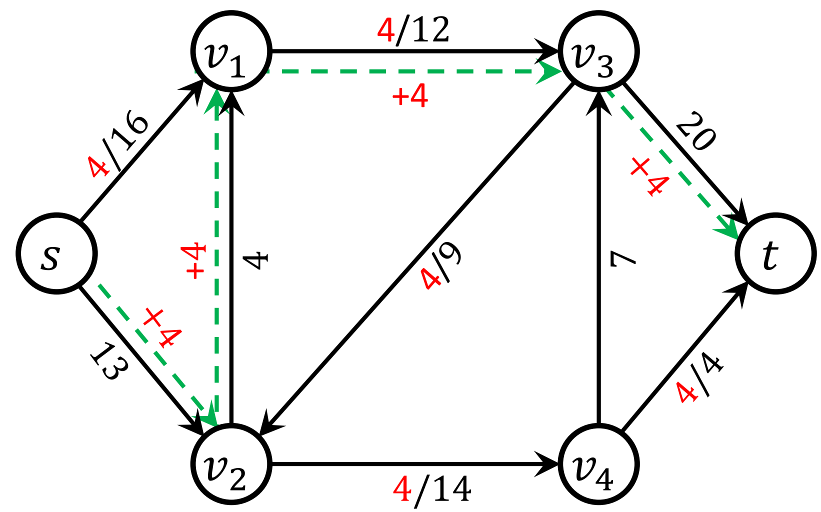
Maximum Flow: The Ford-Fulkerson Method

Step 2: Augmenting Path



Residual capacity $= 4$

Step 2: Updating Flow

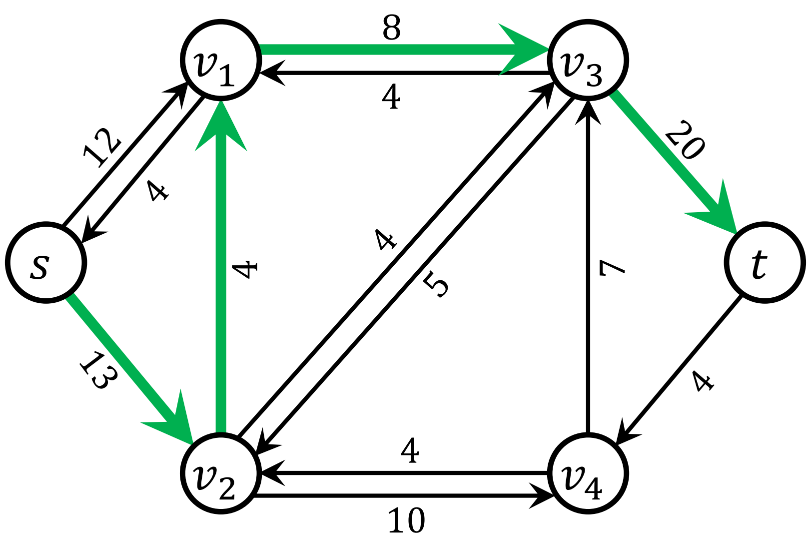


Increase flow by 4 along path

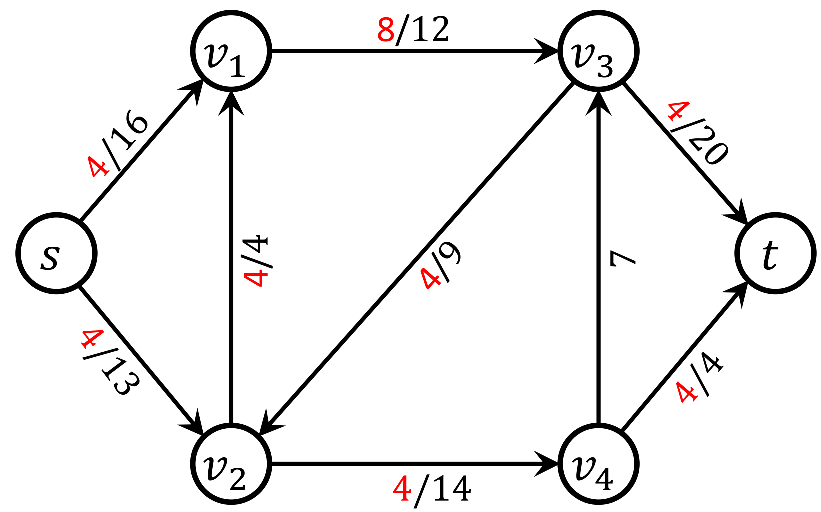
$\boldsymbol{s \to v_4 \to v_1 \to v_3 \to t}$

Maximum Flow: The Ford-Fulkerson Method

Step 2: Augmenting Path



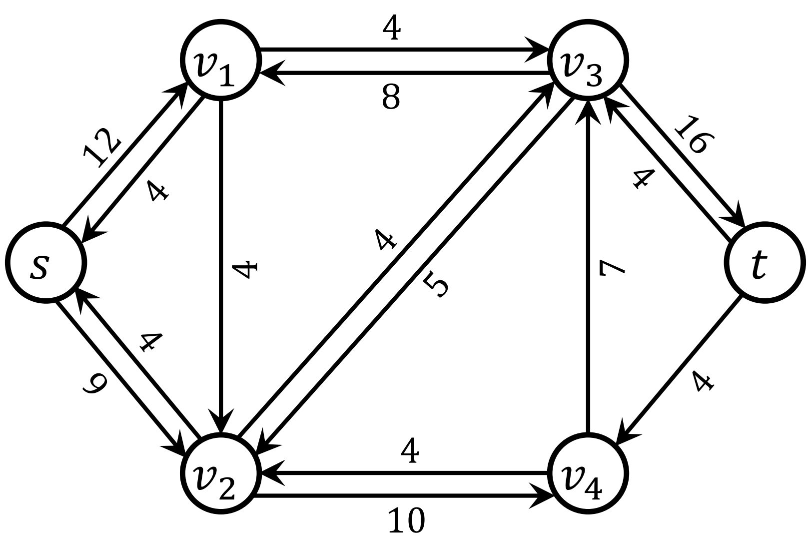
Step 2: Updated Flow



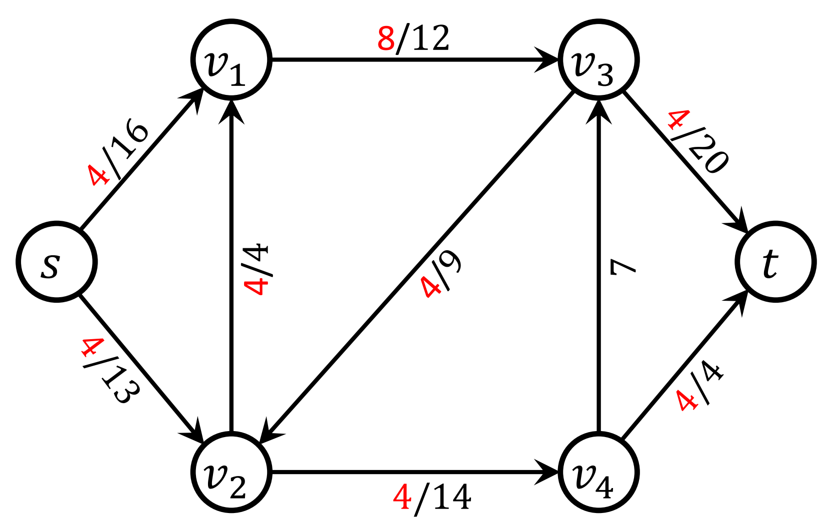
Current s to t flow $= 8$

Maximum Flow: The Ford-Fulkerson Method

Step 3: Residual Network

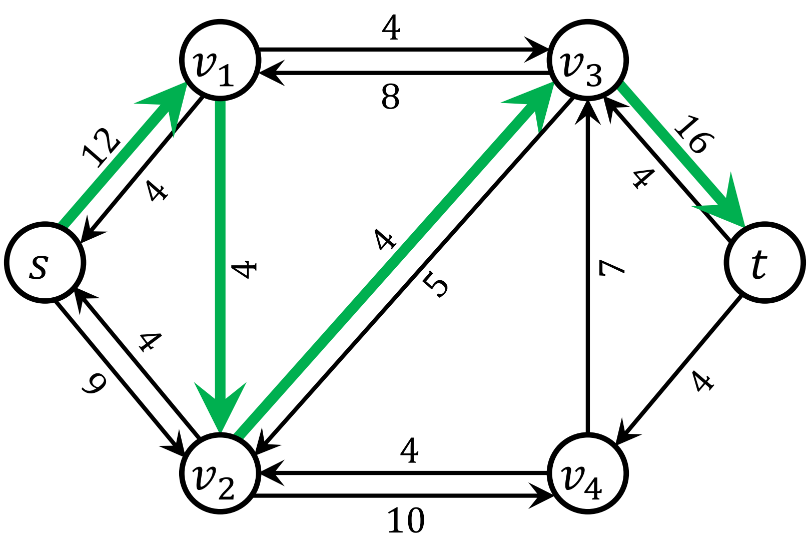


Step 2: Updated Flow



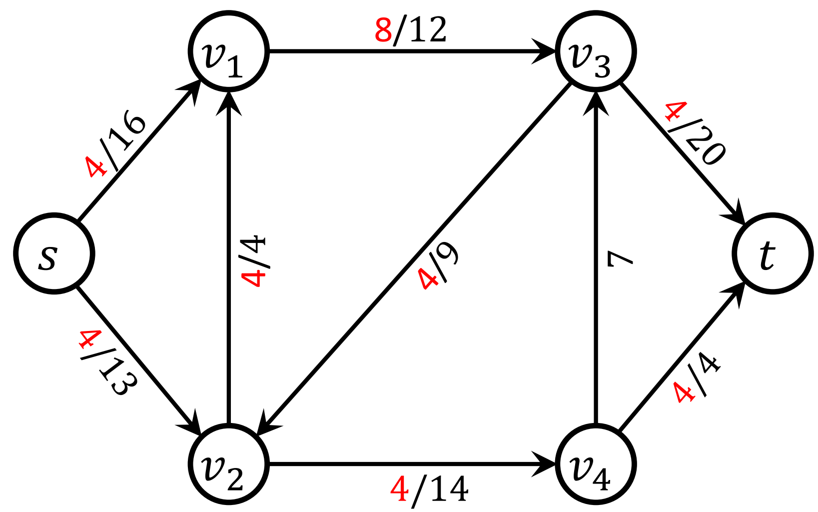
Maximum Flow: The Ford-Fulkerson Method

Step 3: Augmenting Path



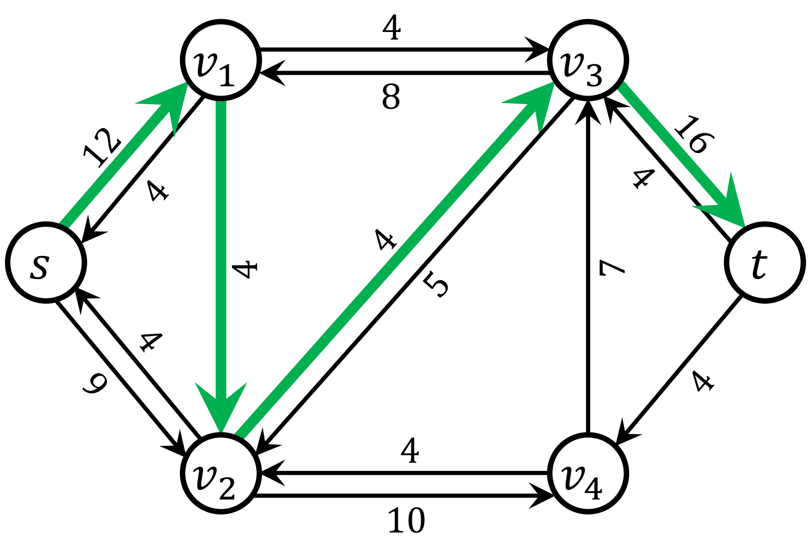
Residual capacity $= 4$

Step 2: Updated Flow



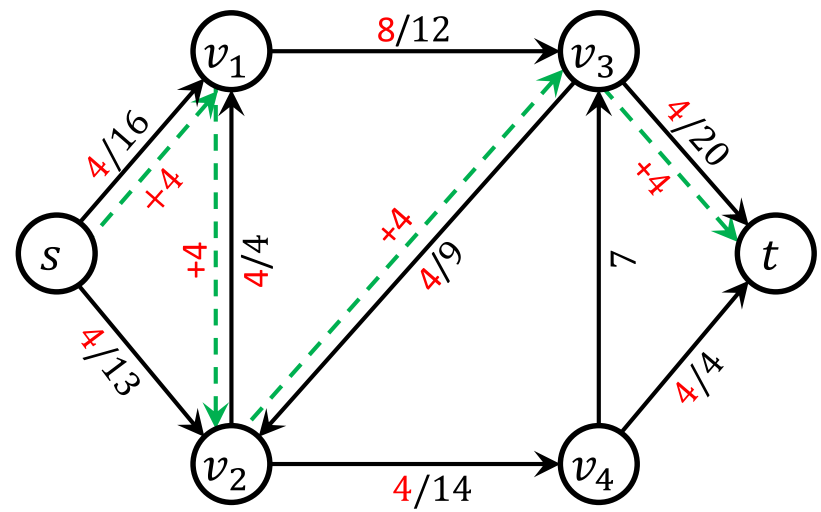
Maximum Flow: The Ford-Fulkerson Method

Step 3: Augmenting Path



Residual capacity $= 4$

Step 3: Updating Flow

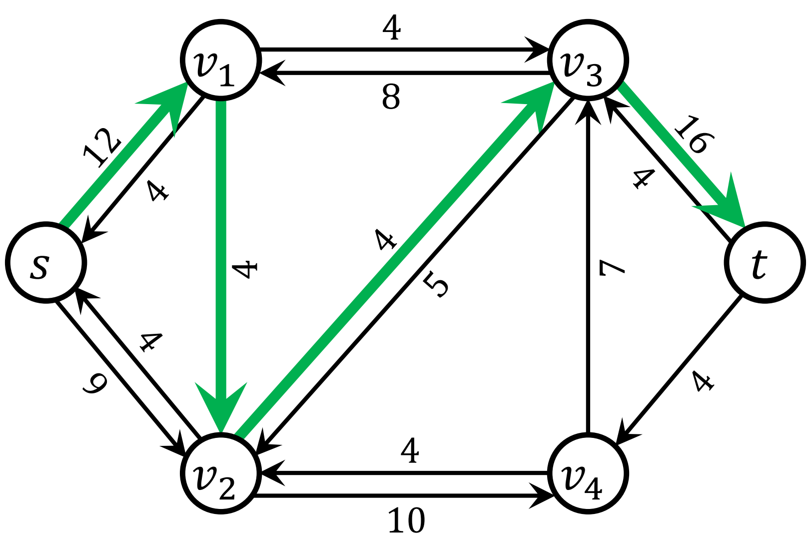


Increase flow by 4 along path

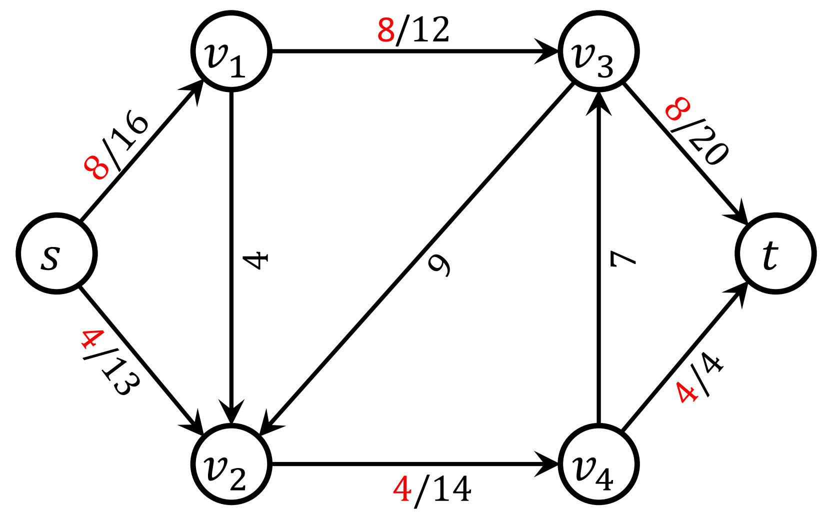
$s \to v_1 \to v_4 \to v_3 \to t$

Maximum Flow: The Ford-Fulkerson Method

Step 3: Augmenting Path



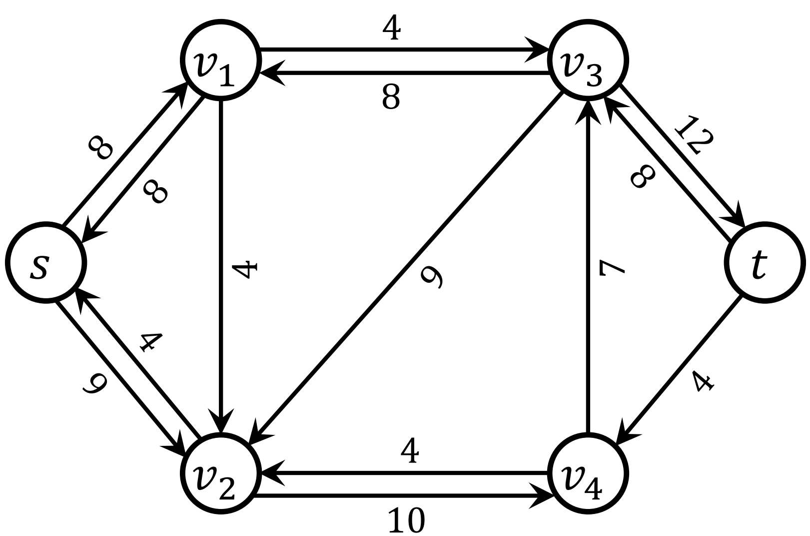
Step 3: Updated Flow



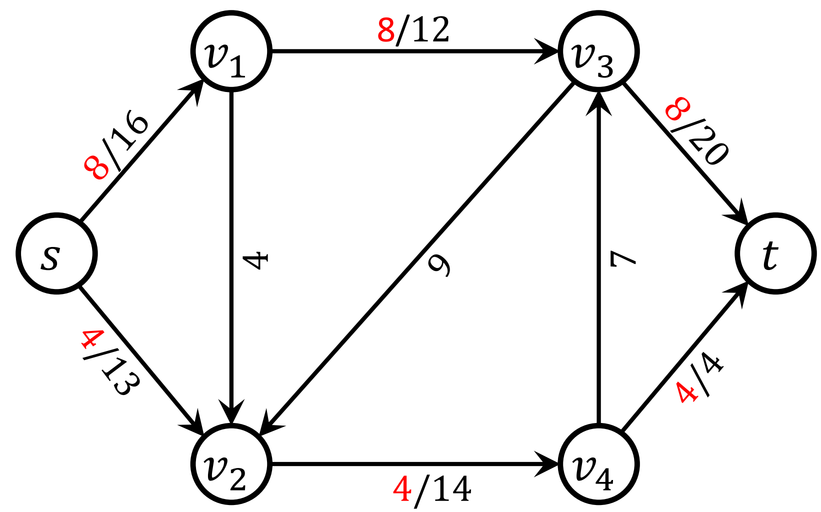
Current s to t flow $= 12$

Maximum Flow: The Ford-Fulkerson Method

Step 4: Residual Network

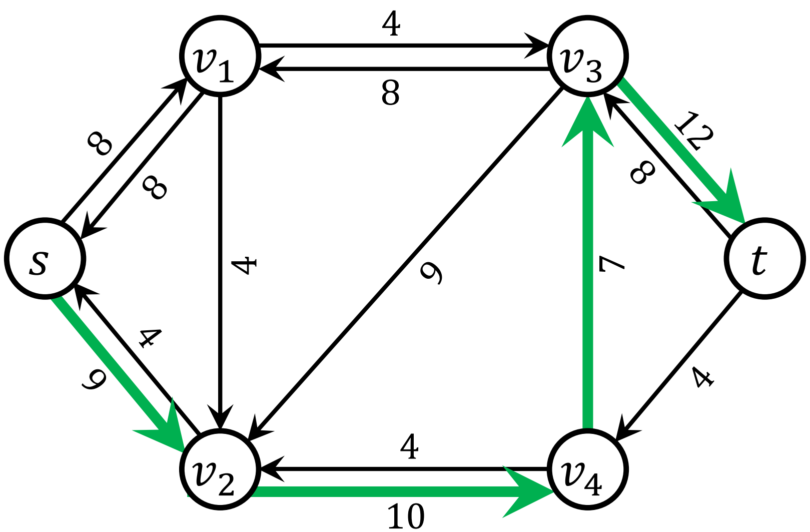


Step 3: Updated Flow



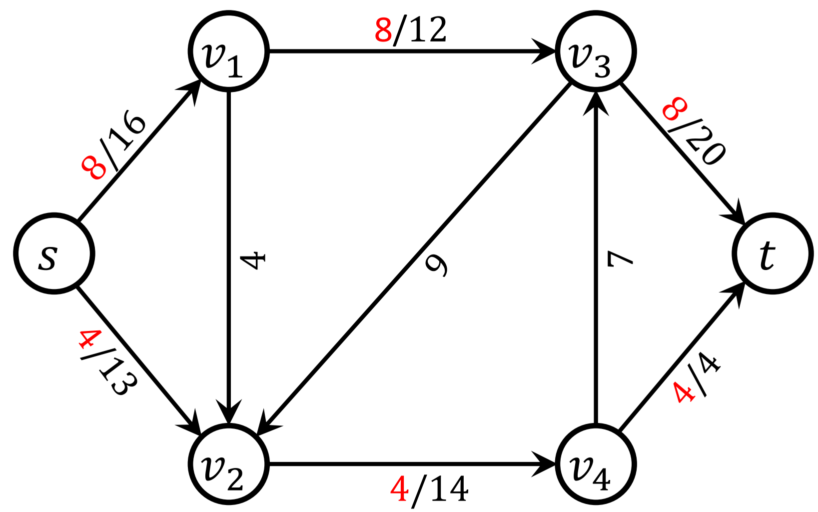
Maximum Flow: The Ford-Fulkerson Method

Step 4: Augmenting Path



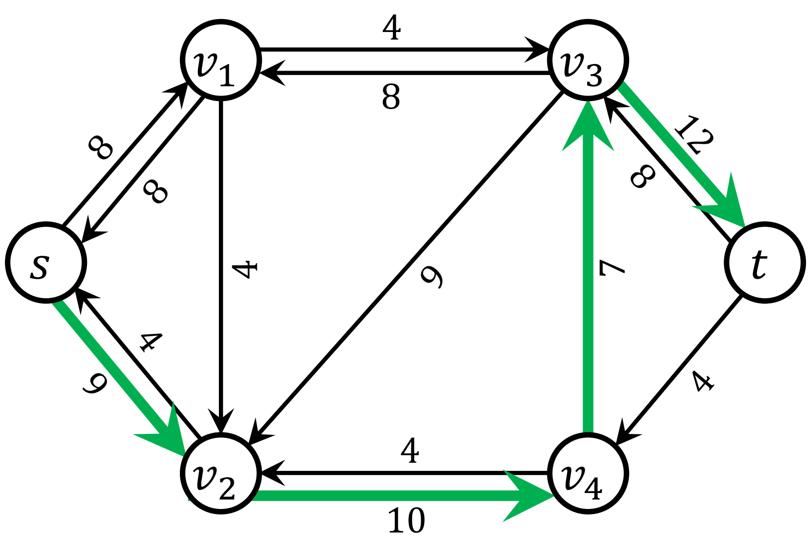
Residual capacity $= 7$

Step 3: Updated Flow

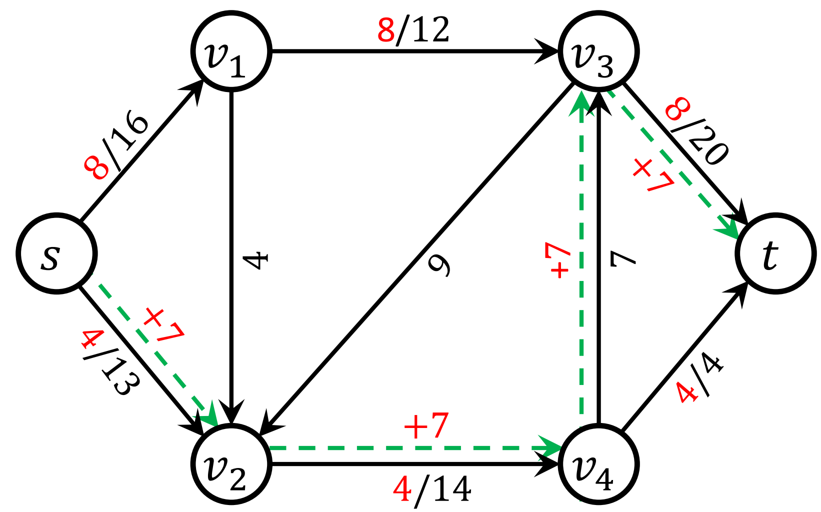


Maximum Flow: The Ford-Fulkerson Method

Step 4: Augmenting Path



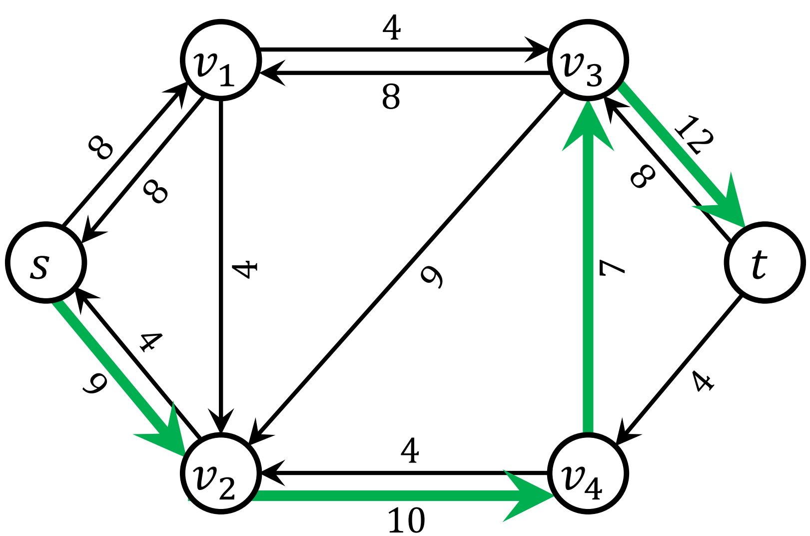
Step 4: Updating Flow



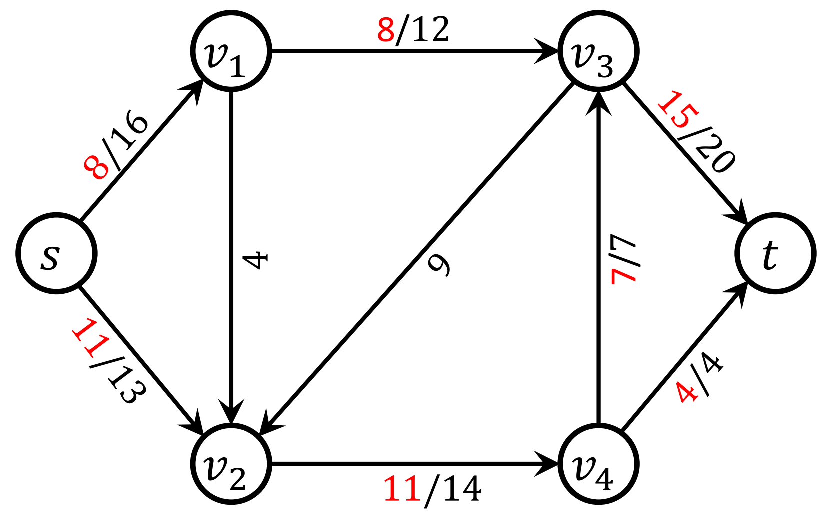
Increase flow by 7 along path
$\boldsymbol{s \to v_2 \to v_4 \to v_3 \to t}$

Maximum Flow: The Ford-Fulkerson Method

Step 4: Augmenting Path



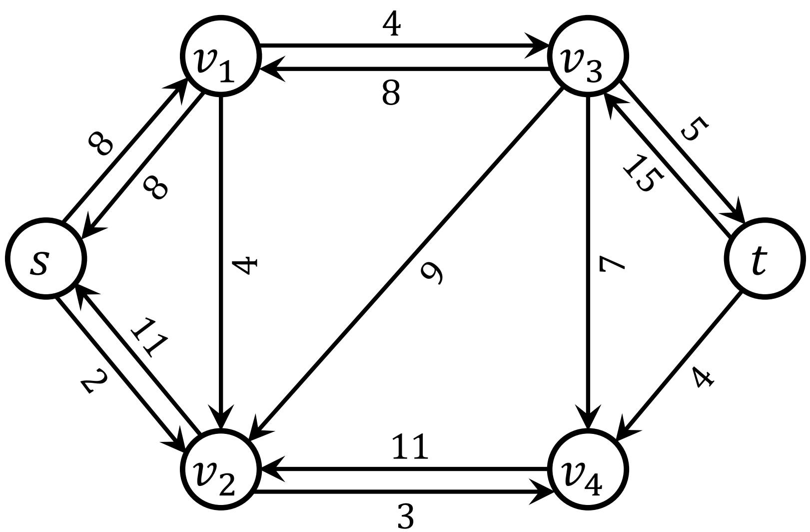
Step 4: Updated Flow



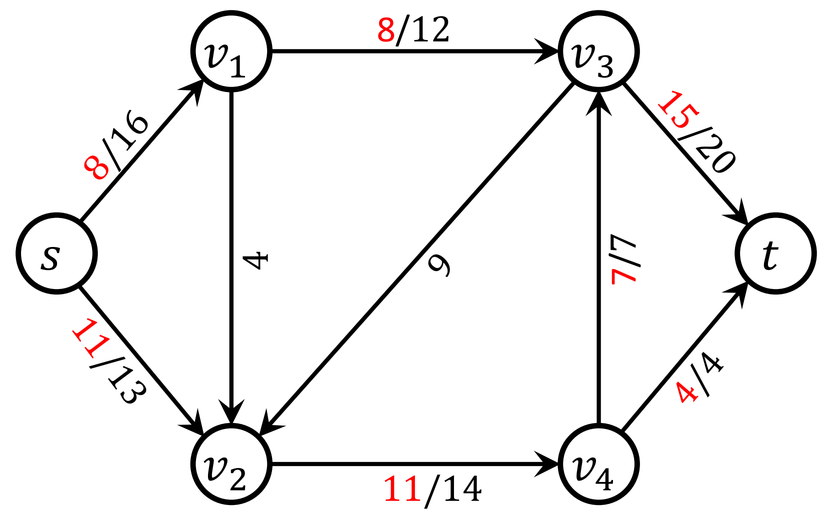
Current s to t flow $= 19$

Maximum Flow: The Ford-Fulkerson Method

Step 5: Residual Network

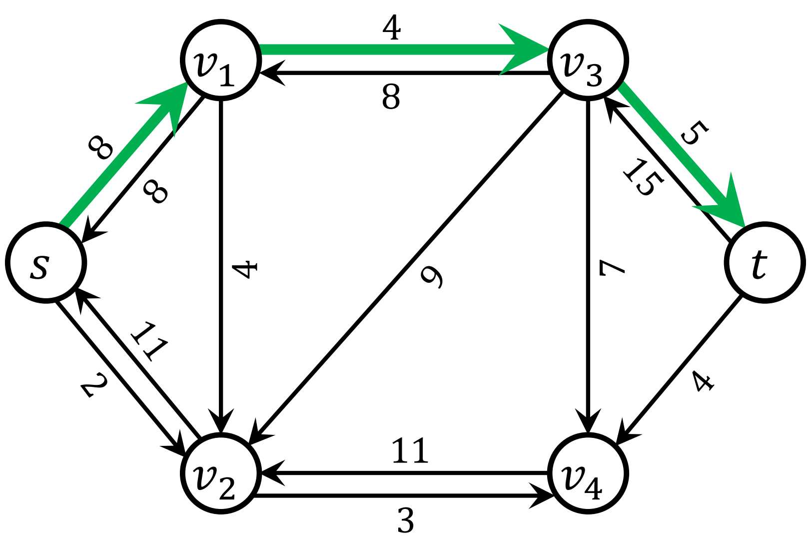


Step 4: Updated Flow



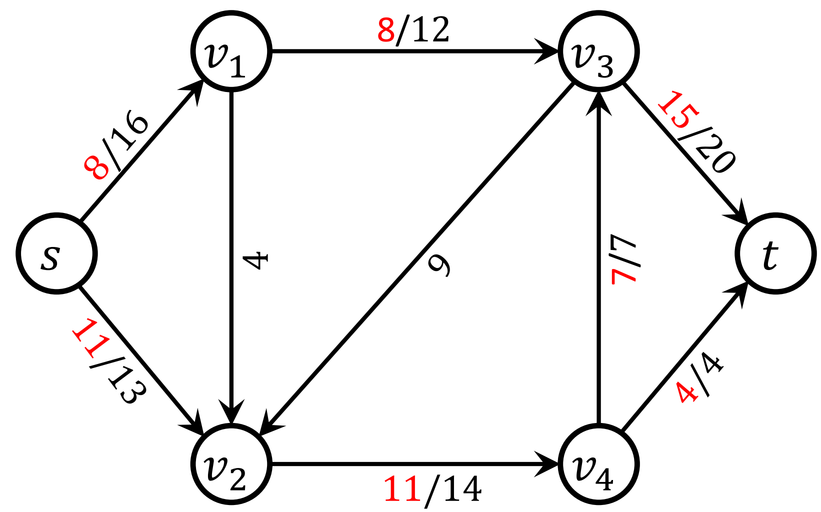
Maximum Flow: The Ford-Fulkerson Method

Step 5: Augmenting Path



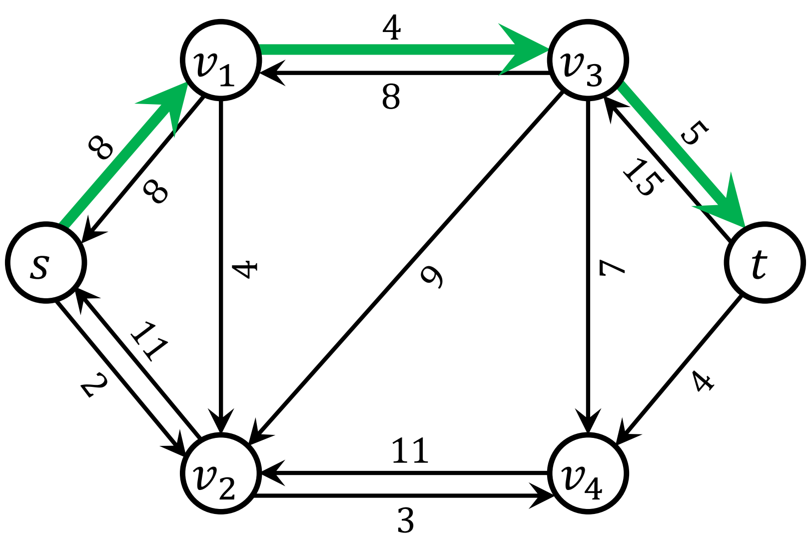
Residual capacity $= 4$

Step 4: Updated Flow



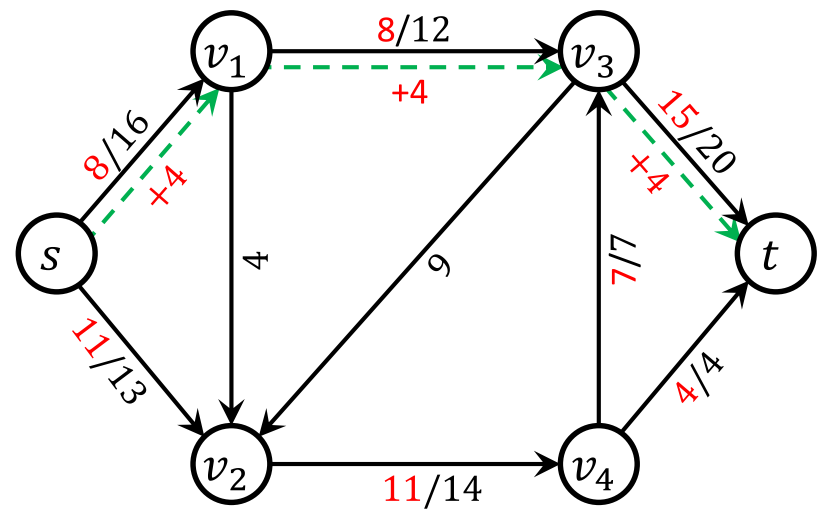
Maximum Flow: The Ford-Fulkerson Method

Step 5: Augmenting Path



Residual capacity $= 4$

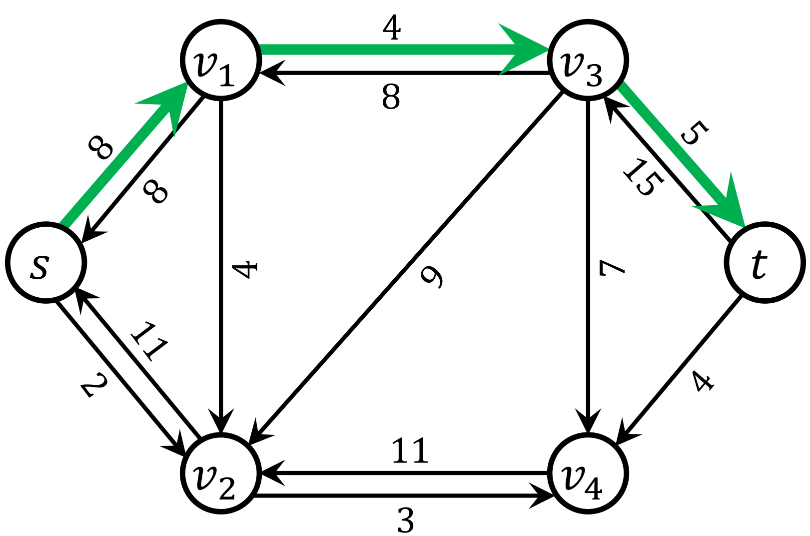
Step 5: Updating Flow



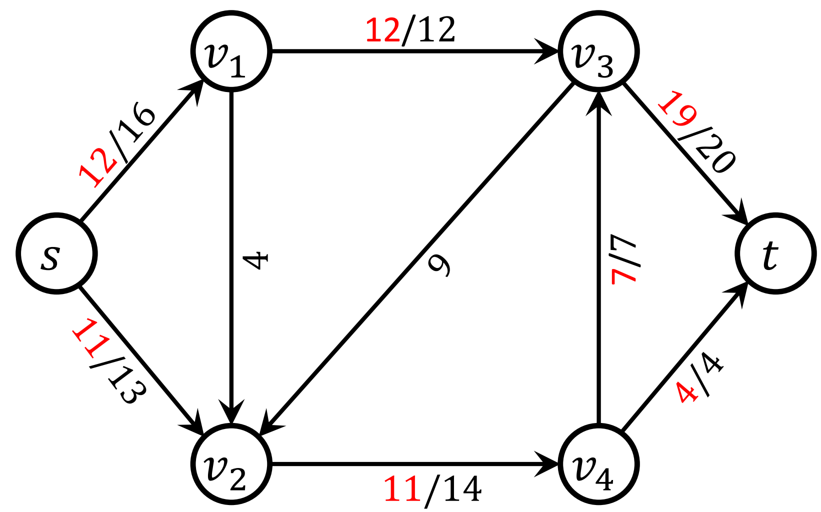
Increase flow by 4 along path
$$s \to v_1 \to v_3 \to t$$

Maximum Flow: The Ford-Fulkerson Method

Step 5: Augmenting Path



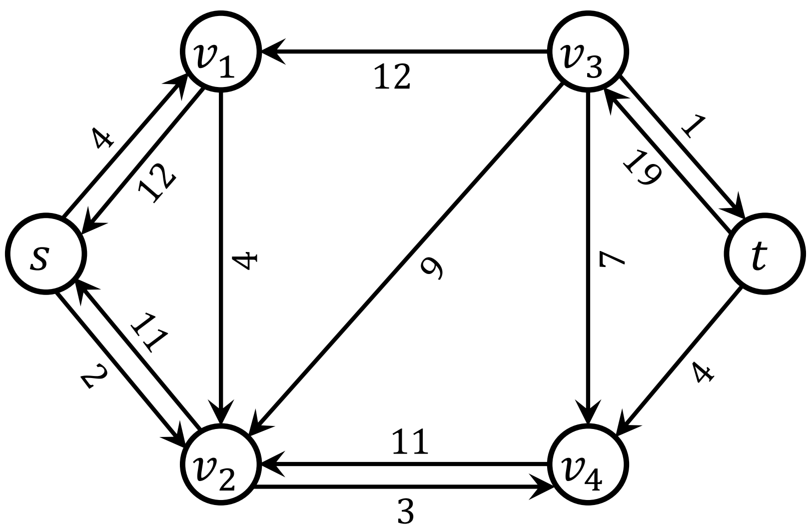
Step 5: Updated Flow



Current s to t flow $= 23$

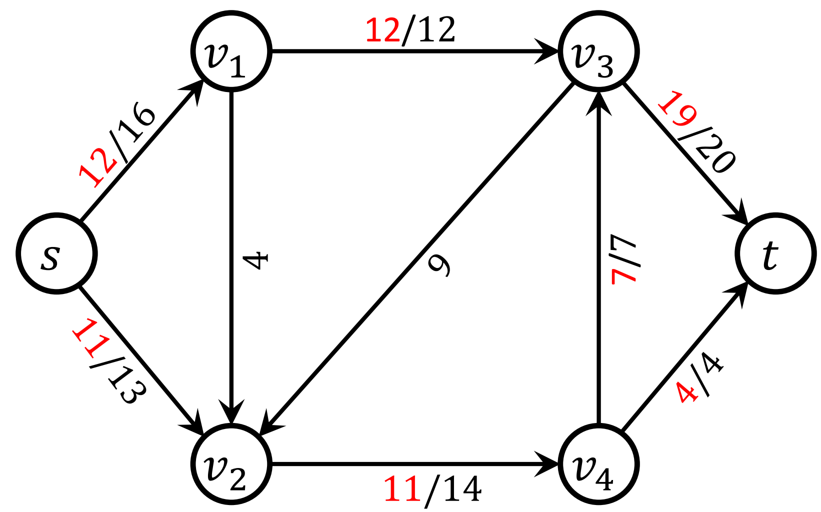
Maximum Flow: The Ford-Fulkerson Method

Step 6: Residual Network



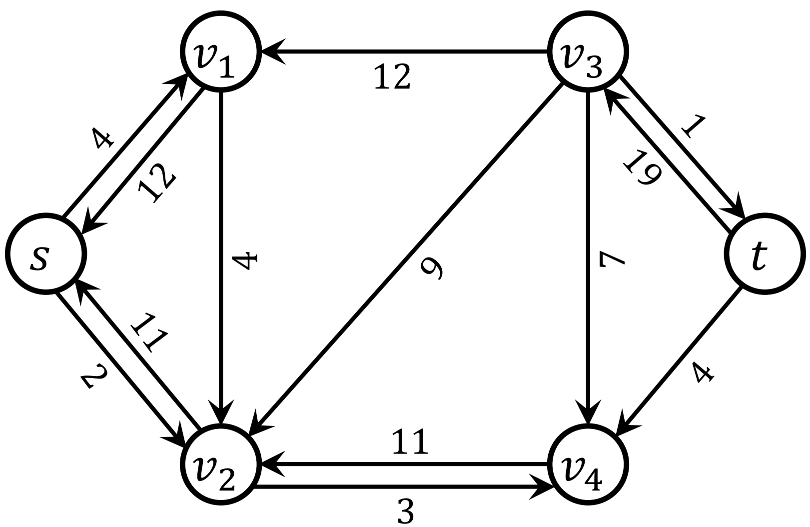
No augmenting path!

Step 5: Updated Flow



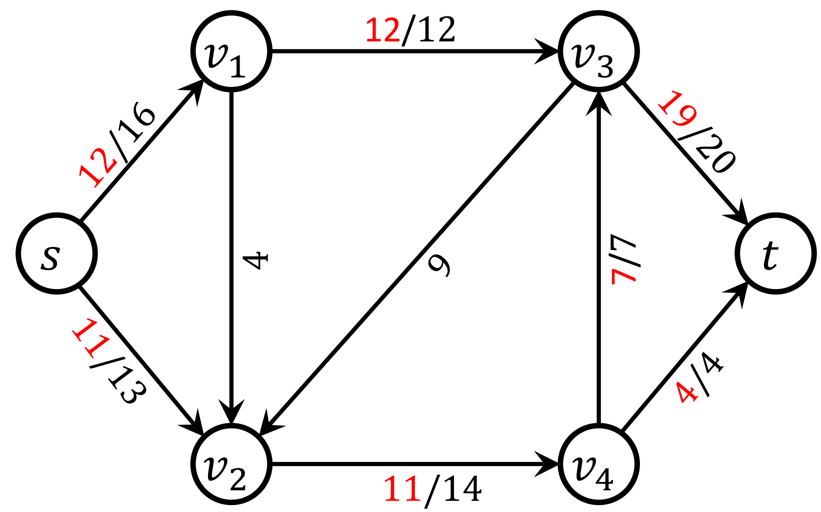
Maximum Flow: The Ford-Fulkerson Method

Step 6: Residual Network



No augmenting path!

Step 6: No Update



Done!

Maximum s to t flow $= 23$

The Ford-Fulkerson Method: Running Time

The running time of *FORD-FULKERSON* depends on how we find the augmenting paths. If we choose them poorly, the algorithm might not even terminate: the value of the flow will increase with successive augmentations, but it need not even converge to the maximum flow value (e.g., might happen when the capacities are irrational numbers).

In practice, the capacities are often integral. If the capacities are rational numbers, we can apply an appropriate scaling transformation to make them all integral. If f^* denotes a maximum flow in the transformed network, then a straightforward implementation of *FORD-FULKERSON* requires to find an augmenting path at most $|f^*|$ times, since each augmentation increases the flow value by at least one unit.

The Ford-Fulkerson Method: Running Time

```
FORD-FULKERSON ( G = (V,E), s, t )
    for each edge (u,v) ∈ G.E do
        (u,v).f ← 0
    while there exists a path p from s to t in the residual network G_f do
        c_f(p) ← min{c_f(u,v) | (u,v) is in p}
        for each edge (u,v) in p do
            if (u,v) ∈ G.E then
                (u,v).f ← (u,v).f + c_f(p)
            else (v,u).f ← (v,u).f − c_f(p)
```

Once the residual network G_f is known, an augmenting path can be found in $O(m+n)$ time using either a depth-first or a breadth-first search, where $n = |V|$ and $m = |E|$. It is also easy to maintain the network, capacities and flows in a way that allows one to find G_f and update the flows in $O(m+n)$ time during each augmentation.

The running time of FORD-FULKERSON is thus $O\big((m+n)|f^*|\big)$ which is simply $O(m|f^*|)$ as $m = \Omega(n)$.

The Edmonds-Karp Algorithm

Ford-Fulkerson ($G = (V, E),\ s,\ t$)

1. *for* each edge $(u, v) \in G.E$ *do*
2. $\quad (u, v).f \leftarrow 0$
3. *while* there exists a path p from s to t in the residual network G_f *do*
4. $\quad c_f(p) \leftarrow \min\{c_f(u, v) \mid (u, v)$ is in $p\}$
5. $\quad$ *for* each edge (u, v) in p *do*
6. $\quad\quad$ *if* $(u, v) \in G.E$ *then*
7. $\quad\quad\quad (u, v).f \leftarrow (u, v).f + c_f(p)$
8. $\quad\quad$ *else* $(v, u).f \leftarrow (v, u).f - c_f(p)$

The Edmonds-Karp algorithm is an implementation of the *Ford-Fulkerson* method in which the augmenting path p in line 3 is found using a breadth-first search. That is, p is chosen as a shortest path from s to t in the residual network, where each edge has unit distance (weight).

One can show that the Edmonds-Karp algorithm runs in $O(m^2 n)$ time, where $n = |G.V|$ and $m = |G.E|$.

The Edmonds-Karp Algorithm

LEMMA 26.7 (CLRS): If the Edmonds-Karp algorithm is run on a flow network $G = (V, E)$ with source s and and sink t, then for all vertices $v \in G.V \setminus \{s, t\}$, the shortest path distance $\delta_f(s, v)$ in the residual network G_f increases monotonically with each flow augmentation.

PROOF: Let's assume for contradiction that for some vertex $v \in G.V \setminus \{s, t\}$, there is a flow augmentation that causes the shortest-path distance from s to v to decrease.

Let f be the flow just before the first augmentation that decreases some shortest-path distance, and let f' be the flow just afterward.

Let v be the vertex with the minimum $\delta_{f'}(s, v)$ whose distance was decreased by the augmentation, so that $\delta_{f'}(s, v) < \delta_f(s, v)$.

The Edmonds-Karp Algorithm

PROOF (CONTINUED): Let $p = s \rightsquigarrow u \to v$ be a shortest path from s to v in G_f, so that $(u, v) \in E_f$, and $\delta_{f'}(s, u) = \delta_{f'}(s, v) - 1$.

Because of the way we chose v, we know that $\delta_{f'}(s, u) \geq \delta_f(s, u)$.

Then we must have $(u, v) \notin E_f$, as otherwise by triangle inequality:

$$\delta_f(s, v) \leq \delta_f(s, u) + 1 \leq \delta_{f'}(s, u) + 1 = \delta_{f'}(s, v),$$

which contradicts our assumption that $\delta_{f'}(s, v) < \delta_f(s, v)$.

How can we have $(u, v) \notin E_f$ and $(u, v) \in E_{f'}$? The augmentation must have increased the flow from v to u. The Edmonds-Karp algorithm always augments flow along shortest paths, and therefore the shortest path from s to u in G_f has (v, u) as its last edge.

Therefore, $\delta_f(s, v) = \delta_f(s, u) - 1 \leq \delta_{f'}(s, u) - 1 = \delta_{f'}(s, v) - 2$, which contradicts our assumption that $\delta_{f'}(s, v) < \delta_f(s, v)$.

The Edmonds-Karp Algorithm

THEOREM 26.8 (CLRS): If the Edmonds-Karp algorithm is run on a flow network $G = (V, E)$ with source s and sink t, then the total number of flow augmentations performed by the algorithm is $O(mn)$, where $n = |G.V|$ and $m = |G.E|$.

PROOF: We say that an edge (u, v) in a residual network G_f is ***critical*** on an augmenting path p if $c_f(p) = c_f(u, v)$.

After we have augmented flow along an augmenting path, any critical edge on the path disappears from the residual network. Moreover, at least one edge on any augmenting path must be critical.

We will show that each of the m edges can become critical at most $n/2$ times.

The Edmonds-Karp Algorithm

PROOF (CONTINUED): Let $u, v \in G.V$ be connected by $(u, v) \in G.E$.

Since augmenting paths are shortest paths, when (u, v) is critical for the first time, we have $\delta_f(s, v) = \delta_f(s, u) + 1$.

Once the flow is augmented, the edge (u, v) disappears from the residual network. It cannot reappear later on another augmenting path until after the flow from u to v is decreased, which occurs only if (v, u) appears on an augmenting path. If f' is the flow in G when this event occurs, then we have $\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1$.

Since $\delta_f(s, v) \leq \delta_{f'}(s, v)$ by Lemma 26.7, we have:

$$\delta_{f'}(s, u) = \delta_{f'}(s, v) + 1 \geq \delta_f(s, v) + 1 = \delta_f(s, u) + 2.$$

The Edmonds-Karp Algorithm

PROOF (CONTINUED): Consequently, from the time (u, v) becomes critical to the time when it next becomes critical, the distance of u from s increases by at least 2. The distance of u from s is initially at least 0. The intermediate vertices on a shortest path from s to u cannot contain s, u, or t (since (u, v) on an augmenting path implies that $u \neq t$). Therefore, until u becomes unreachable from s, if ever, its distance is at most $n - 2$. Thus, after the first time that (u, v) becomes critical, it can become critical at most $(n - 2)/2 = n/2 - 1$ times more, for a total of at most $n/2$ times.

Since there are $O(m)$ pairs of vertices that can have an edge between them in a residual network, the total number of critical edges during the entire execution of the algorithm is $O(mn)$. Each augmenting path has at least one critical edge, and hence the theorem follows.

Cuts of Flow Networks (Max-flow Min-cut Theorem)

A ***cut*** (S, T) of flow network $G = (V, E)$ is a partition of V into S and $T = V \setminus S$ such that $s \in S$ and $t \in T$.

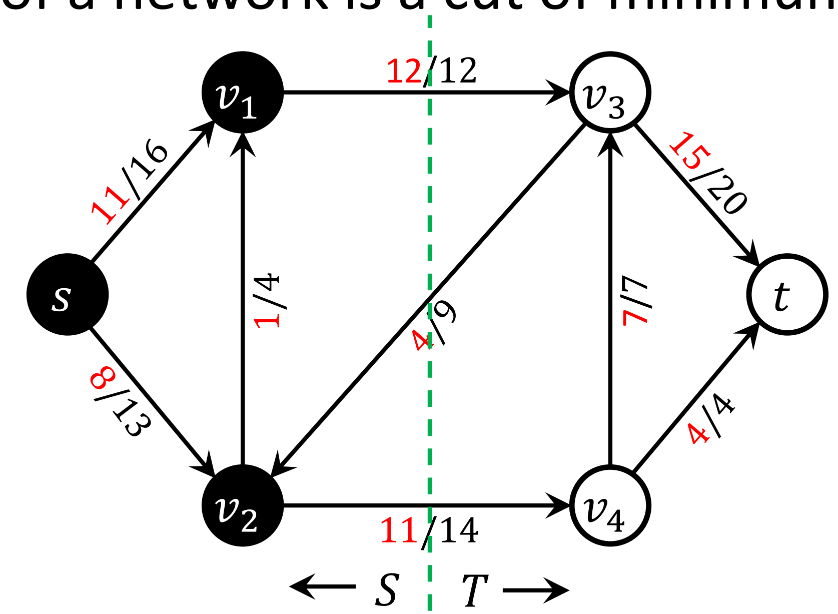
(Unlike the "cut" used for MST's, here the graph is directed, and we insist that $s \in S$ and $t \in T$.)

If f is a flow, then the ***net flow*** $f(S, T)$ across (S, T) is defined to be

$$f(S,T) = \sum_{u\in S} \sum_{v\in T} f(u,v) - \sum_{u\in S} \sum_{v\in T} f(v,u).$$

The capacity of the cut (S, T) is: $c(S,T) = \sum_{u\in S} \sum_{v\in T} c(u,v)$.

A ***minimum cut*** of a network is a cut of minimum capacity.



Cuts of Flow Networks (Max-flow Min-cut Theorem)

LEMMA 26.4 (CLRS): Let f be a flow in a flow network G with source s and sink t, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$.

COROLLARY 26.5 (CLRS): The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G.

Corollary 26.5 implies that in a flow network:

$$value\ of\ a\ maximum\ flow \leq\ capacity\ of\ a\ minimum\ cut$$

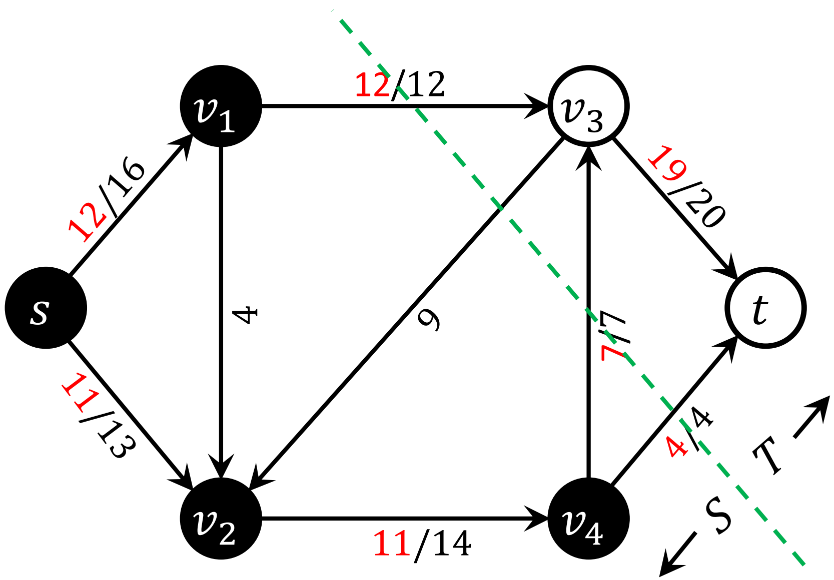
Cuts of Flow Networks (Max-flow Min-cut Theorem)

Theorem 26.6 below says that, in fact, in a flow network:

$$value\ of\ a\ maximum\ flow = \ capacity\ of\ a\ minimum\ cut$$

THEOREM 26.6 (CLRS): If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.



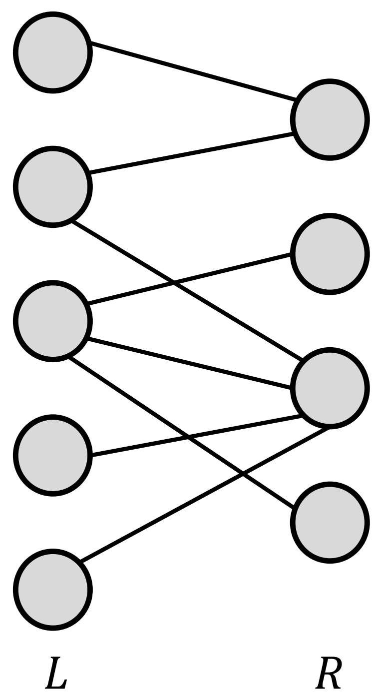
The Maximum Matching Problem

Given an undirected graph $G = (V, E)$, a ***matching*** is a subset of edges $M \subseteq E$ such that for all vertices $v \in V$, at most one edge of M is incident on v. We say that a vertex $v \in V$ is matched by the matching M if some edge in M is incident on v; otherwise, v is unmatched.

A ***maximum matching*** is a matching of maximum cardinality, that is, a matching M such that for any matching M', we have $|M| \geq |M'|$.

The Maximum Bipartite Matching Problem

We shall restrict our attention to finding ***maximum matchings in bipartite graphs***: graphs in which the vertex set can be partitioned into $V = L \cup R$, where L and R are disjoint and all edges in E go between L and R. We further assume that every vertex in V has at least one incident edge.



A bipartite graph

The Maximum Bipartite Matching Problem

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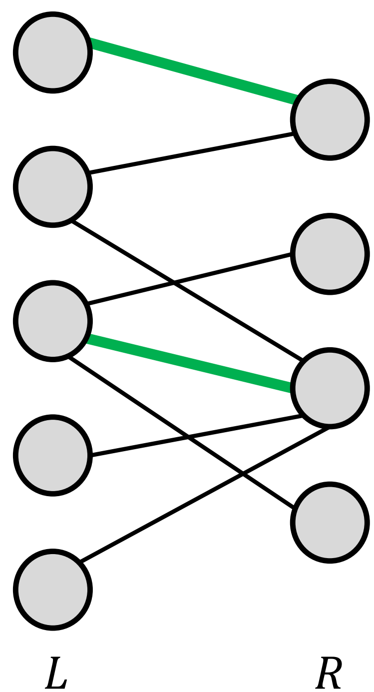

A bipartite matching

The Maximum Bipartite Matching Problem

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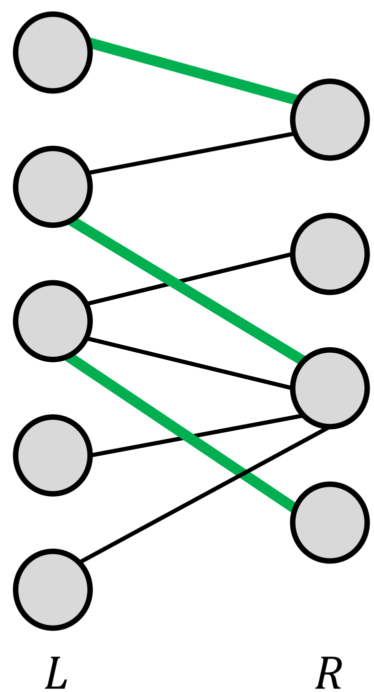
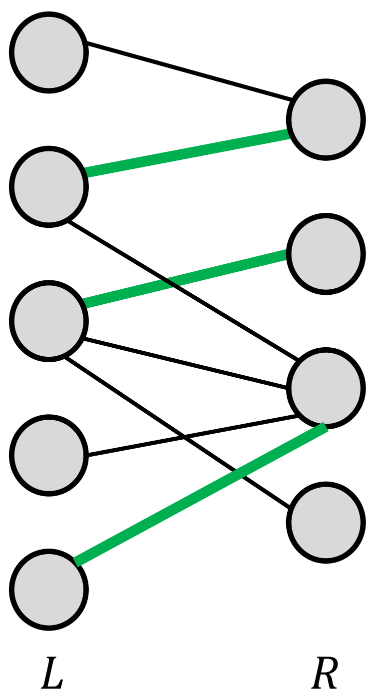


A maximum bipartite matching

The Maximum Bipartite Matching Problem

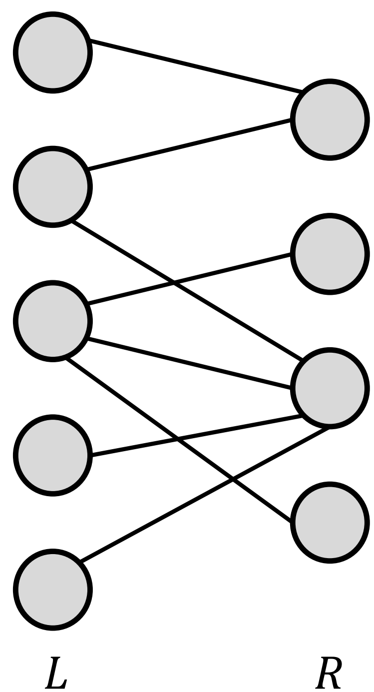
We shall restrict our attention to finding ***maximum matchings in bipartite graphs***: graphs in which the vertex set can be partitioned into $V = L \cup R$, where L and R are disjoint and all edges in E go between L and R. We further assume that every vertex in V has at least one incident edge.



Another maximum bipartite matching

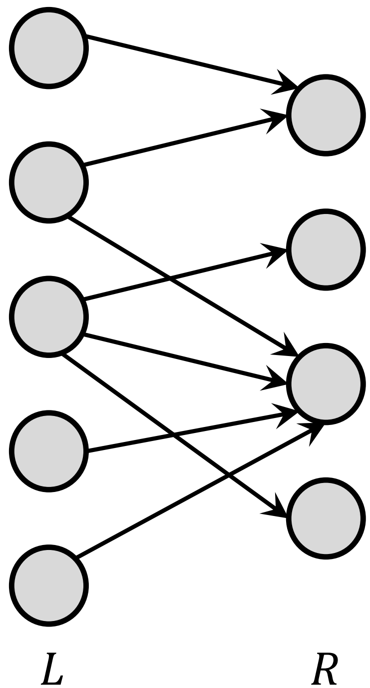
Maximum Bipartite Matching using Network Flow

Given a bipartite graph $G = (V, E)$ with $V = L \cup R$, where L and R are disjoint and all edges in E go between L and R.



Maximum Bipartite Matching using Network Flow

First, direct all edges from L to R.

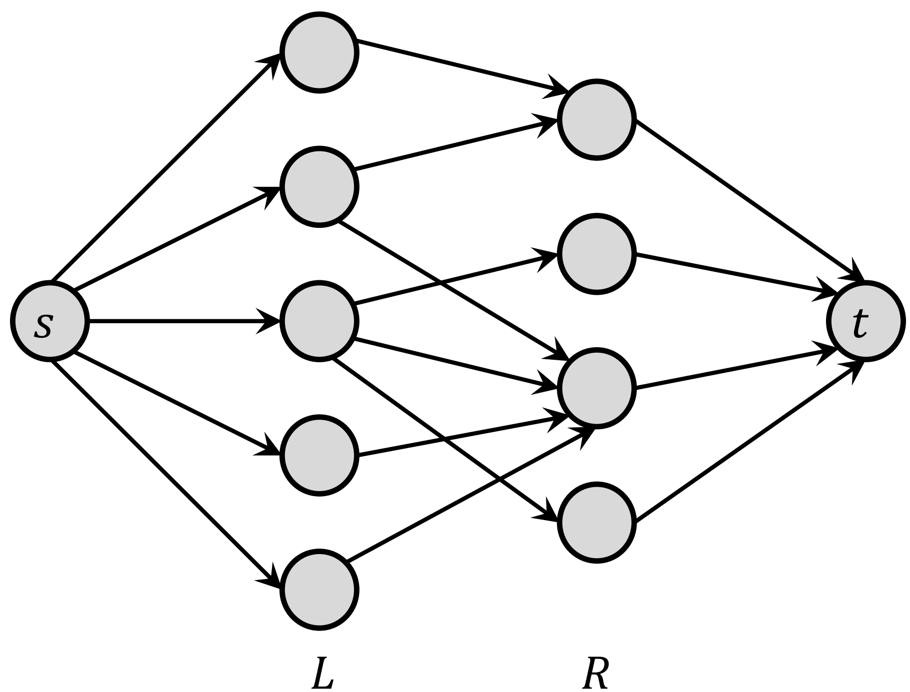


Maximum Bipartite Matching using Network Flow

Then add a source s and a sink t.

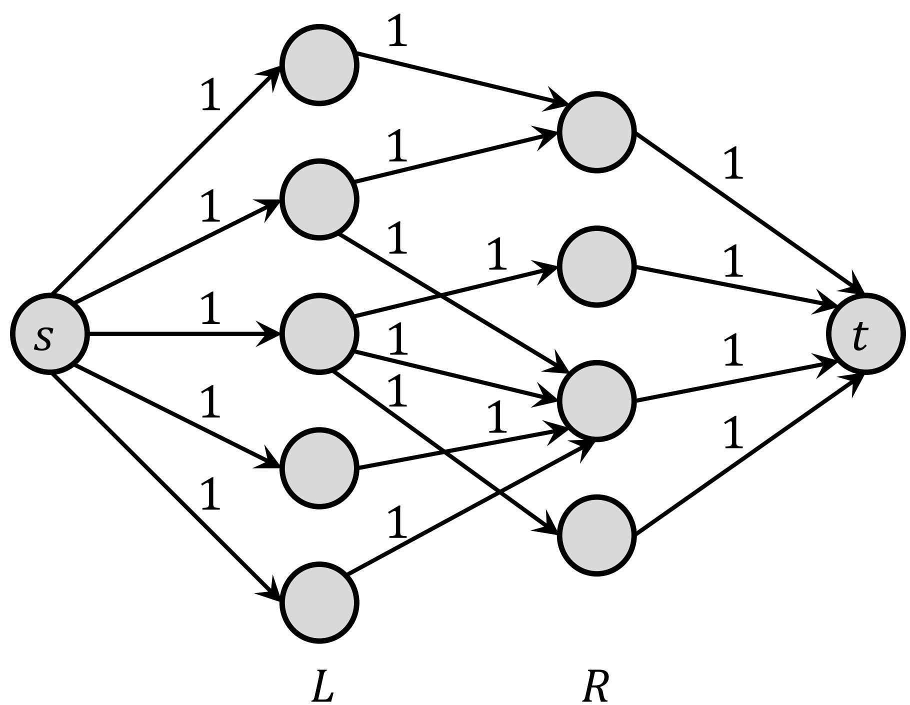
For every vertex $v \in L$, add an edge (s, v) directed from s to v.

For every vertex $v \in R$, add an edge (v, t) directed from v to t.



Maximum Bipartite Matching using Network Flow

For every edge (u, v) in this new directed graph, set capacity $c(u, v) = 1$.



Maximum Bipartite Matching using Network Flow

Now, find a maximum s to t flow f^* in this new graph using *FORD-FULKERSON*.

One can show that $|f^*|$ will always be an integer and will be equal to the maximum matching in the original bipartite graph.

Since $|f^*| < n = |L \cup R|$, running time will be $O(mn)$, where $m = |E|$.