CSE 548: Analysis of Algorithms

Prerequisites Review 7
(More Graph Algorithms: Basic and Beyond)

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Breadth-First Search (BFS)

Input: Unweighted directed or undirected graph $G = (V, E)$ with vertex set V and edge set E, and a source vertex $s \in G.V$. For each $v \in V$, the adjacency list of v is $G.Adj[v]$.

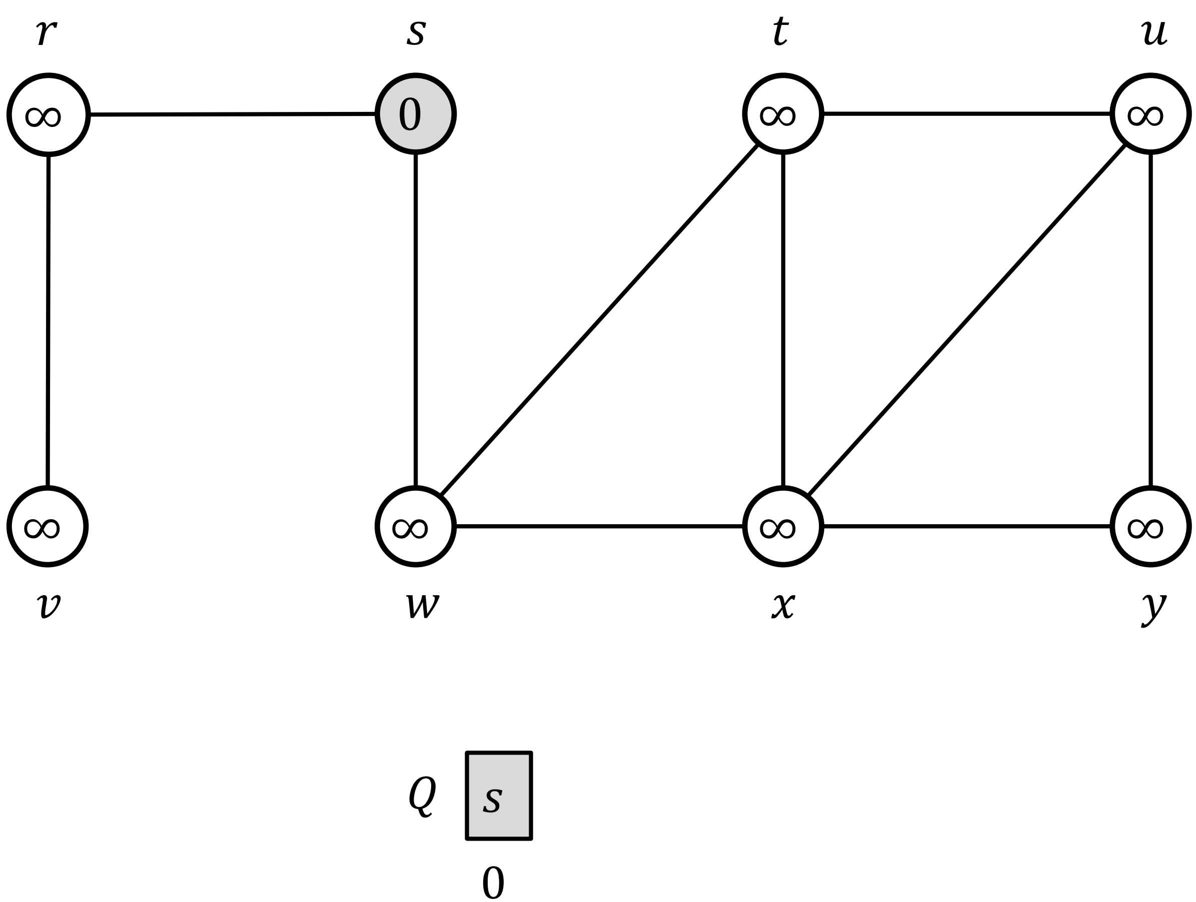
Output: For all $v \in G[V]$, $v.d$ is set to the shortest distance (in terms of the number of edges) from s to v. Also, $v.\pi$ pointers form a breadth-first tree rooted at s that contains all vertices reachable from s.

BFS (G, s)

for each vertex $u \in G.V \setminus \{s\}$ *do*
    $u.color \leftarrow$ WHITE, $u.d \leftarrow \infty$, $u.\pi \leftarrow NIL$
$s.color \leftarrow$ GRAY, $s.d \leftarrow 0$, $s.\pi \leftarrow NIL$
Queue $Q \leftarrow \emptyset$
Enqueue(Q, s)
while $Q \neq \emptyset$ *do*
    $u \leftarrow$ *Dequeue*(Q)
    for each $v \in G.Adj[u]$ *do*
        if $v.color =$ WHITE *then*
            $v.color \leftarrow$ GRAY, $v.d \leftarrow u.d + 1$, $v.\pi \leftarrow u$
            Enqueue(Q, v)
    $u.color \leftarrow$ BLACK

Breadth-First Search (BFS)

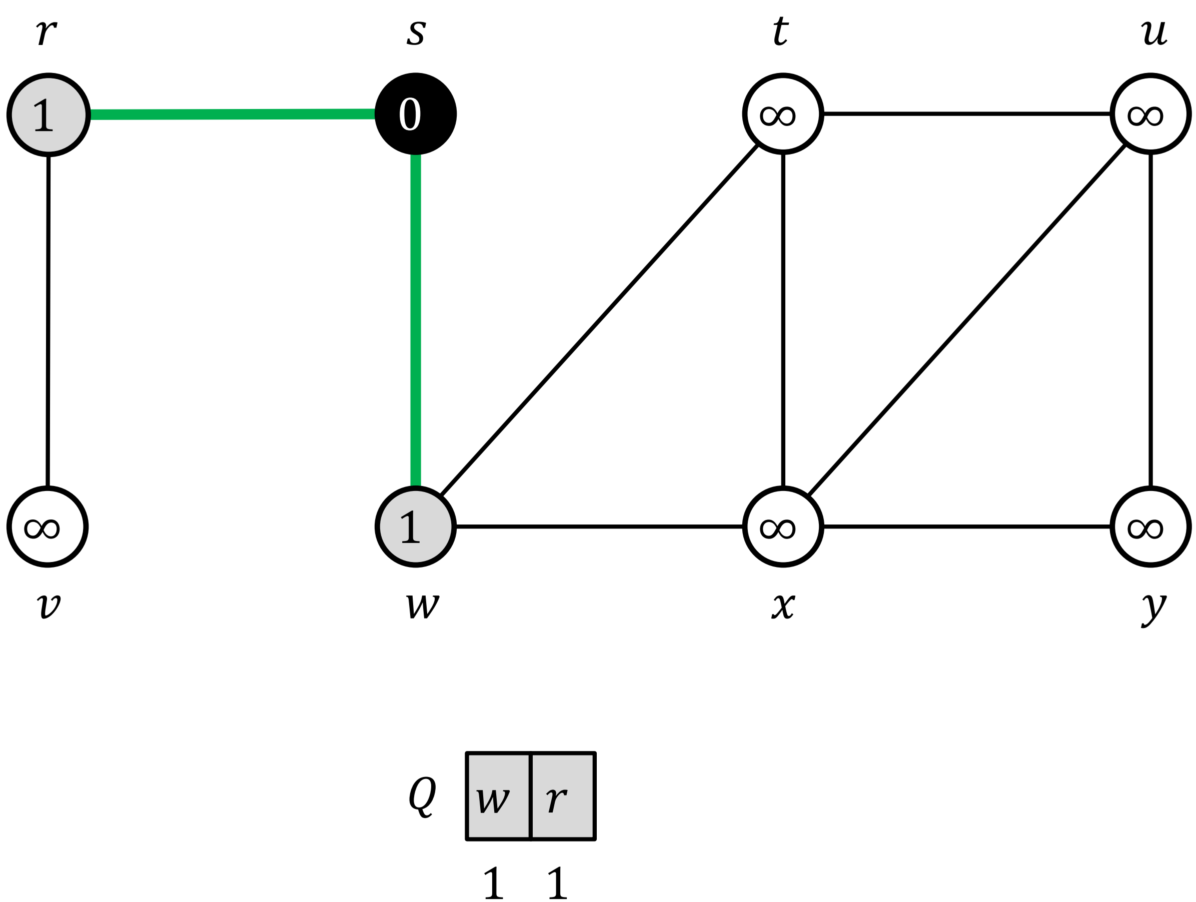
ENQUEUE (Q, s)



Breadth-First Search (BFS)

DEQUEUE (Q) → s

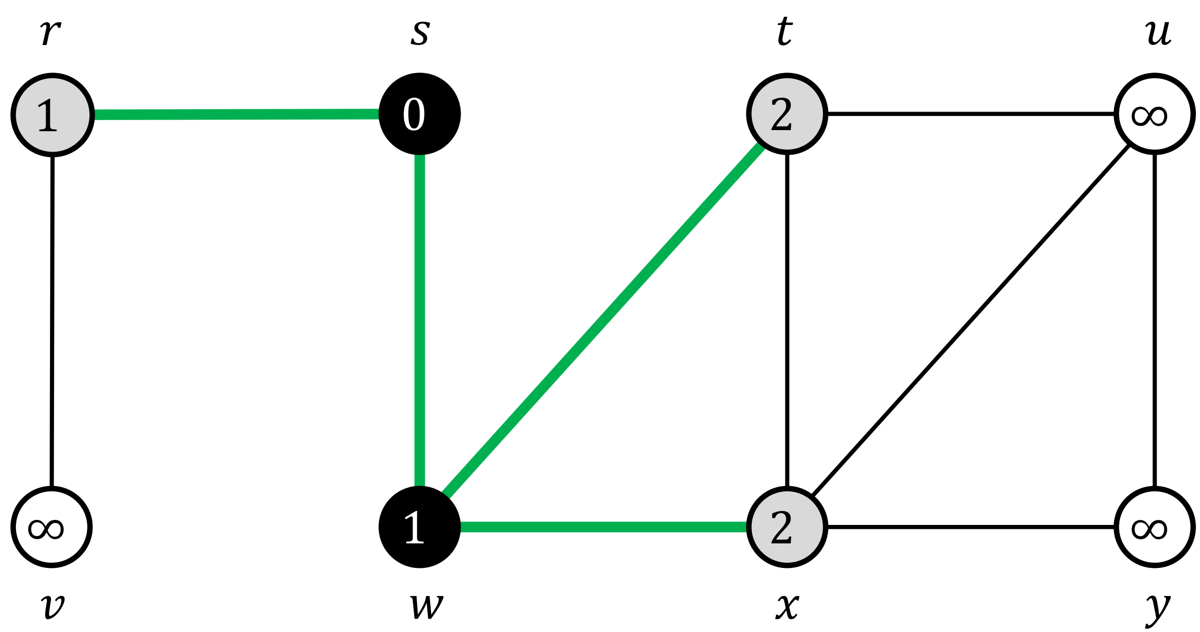
ENQUEUE (Q, w), ENQUEUE (Q, r)

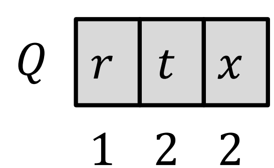


Breadth-First Search (BFS)

DEQUEUE (Q) → w

ENQUEUE (Q, t), ENQUEUE (Q, x)

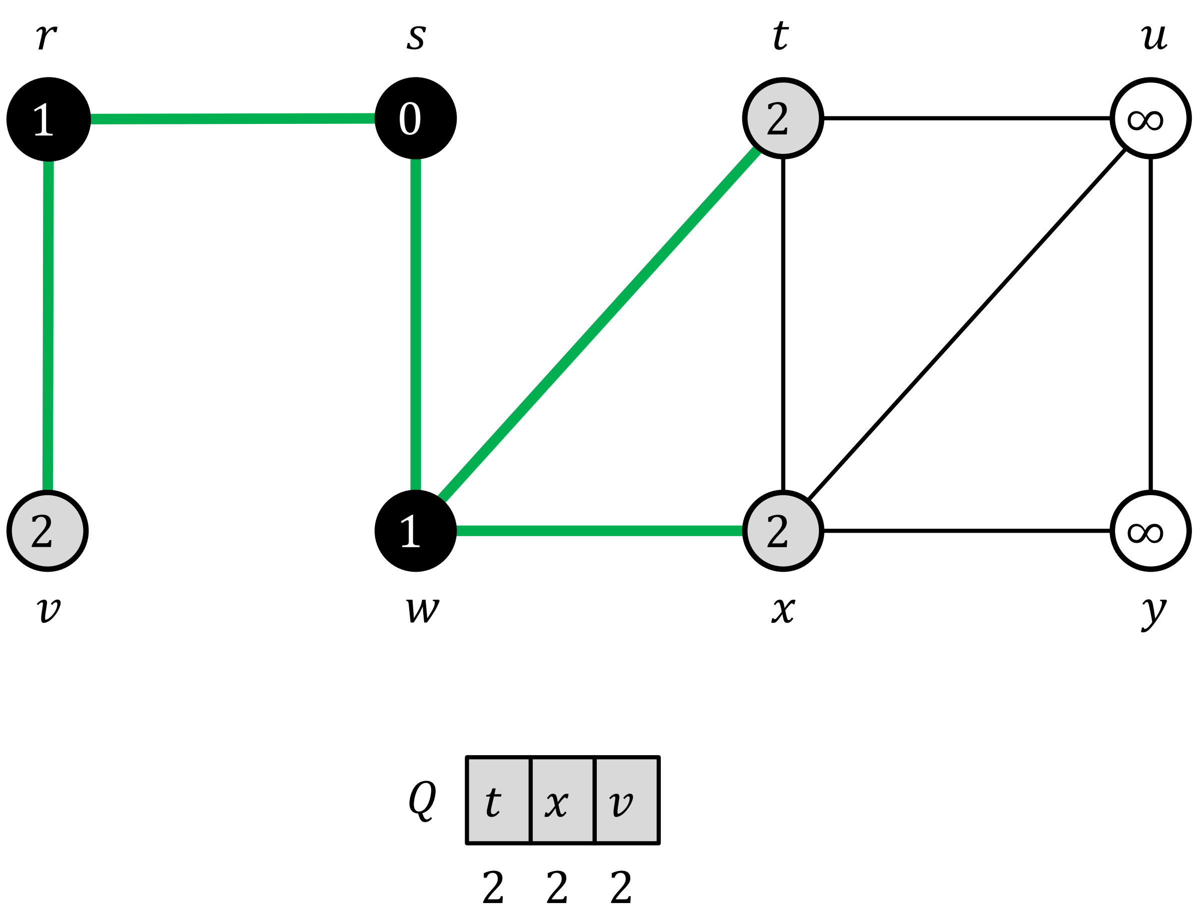




Breadth-First Search (BFS)

DEQUEUE (Q) $\rightarrow r$

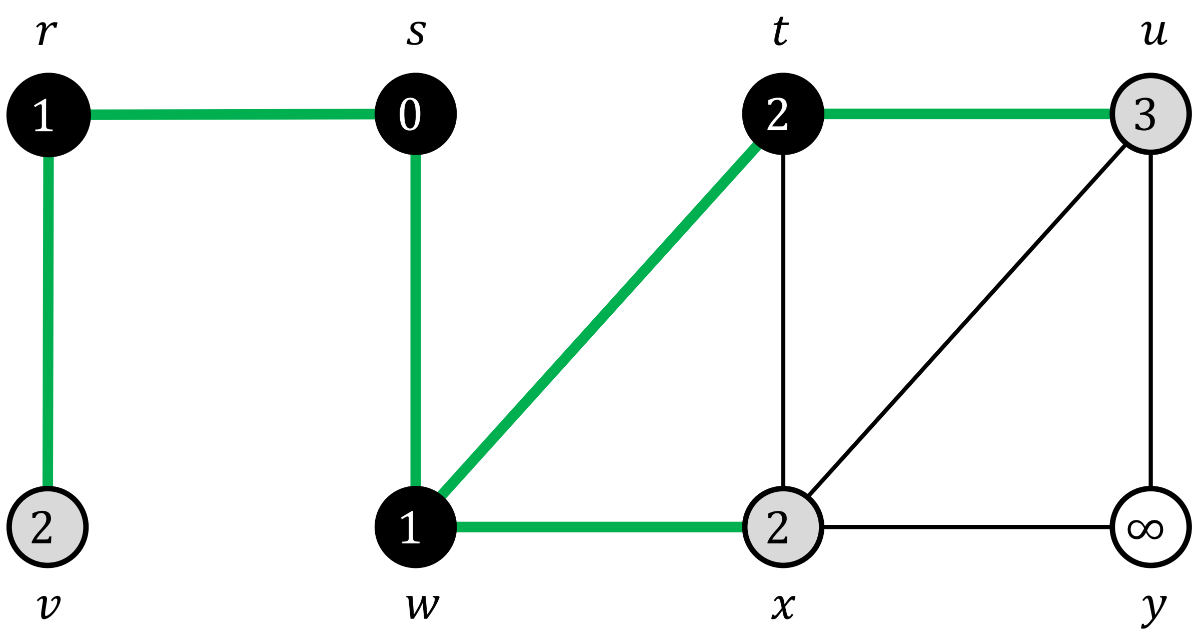
ENQUEUE (Q, x), ENQUEUE (Q, v)

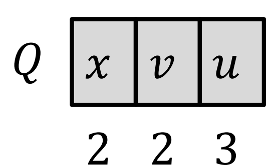


Breadth-First Search (BFS)

DEQUEUE (Q) → t

ENQUEUE (Q, u)

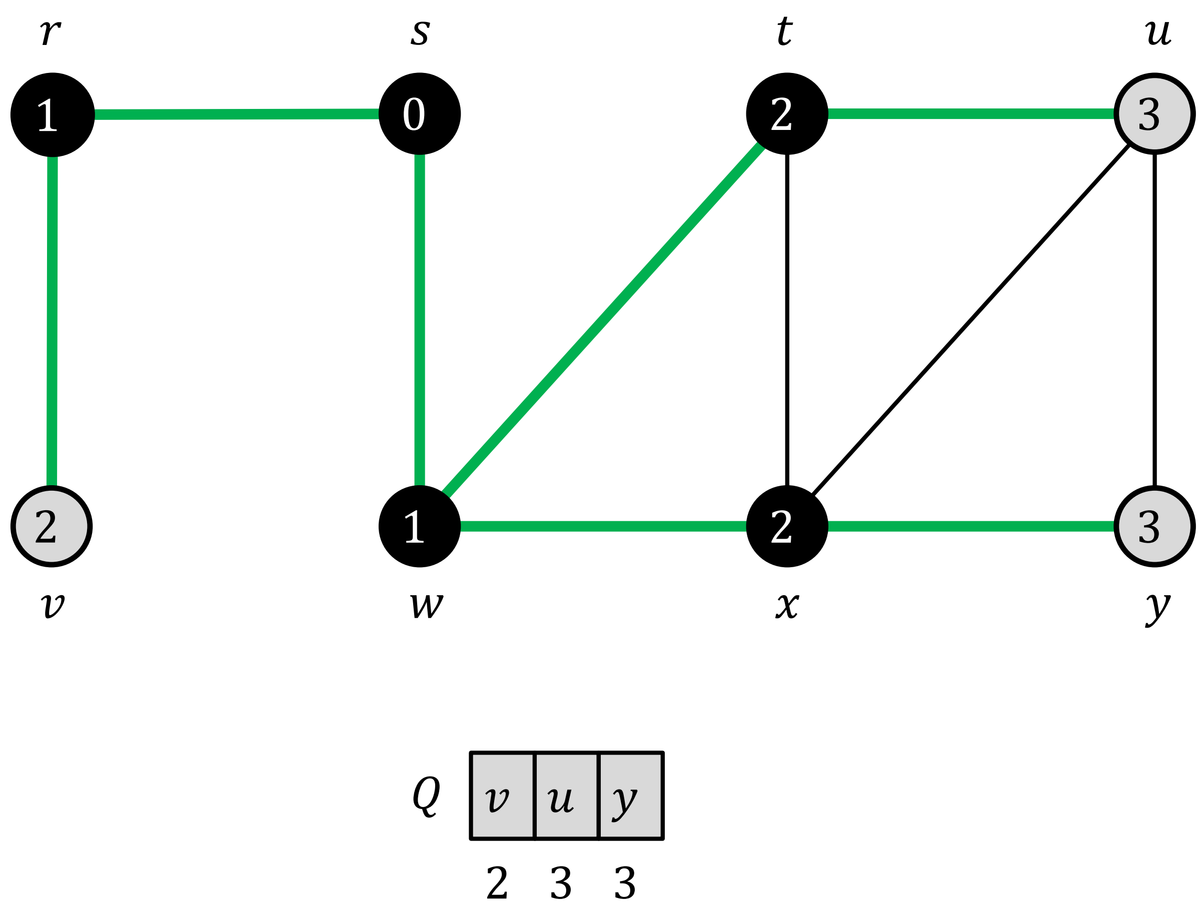




Breadth-First Search (BFS)

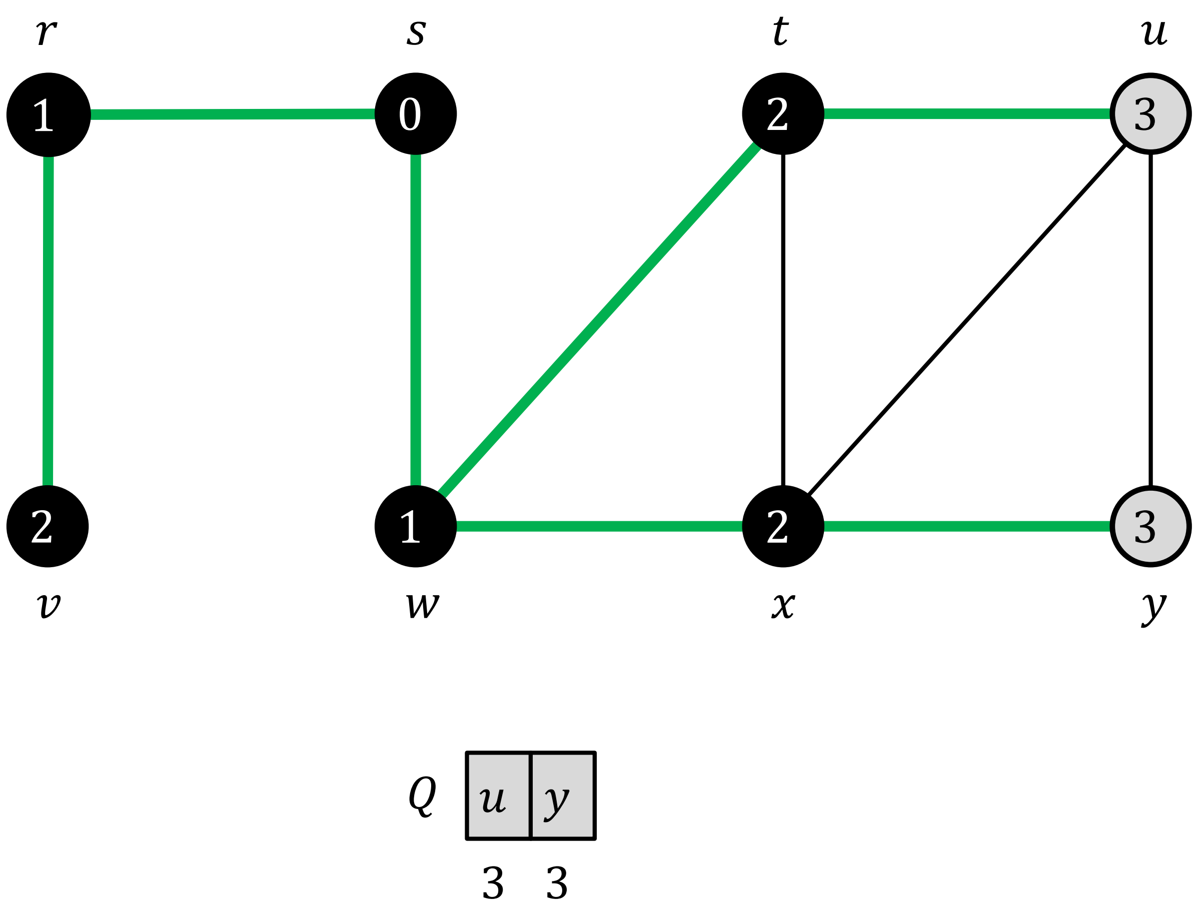
DEQUEUE (Q) → x

ENQUEUE (Q, y)



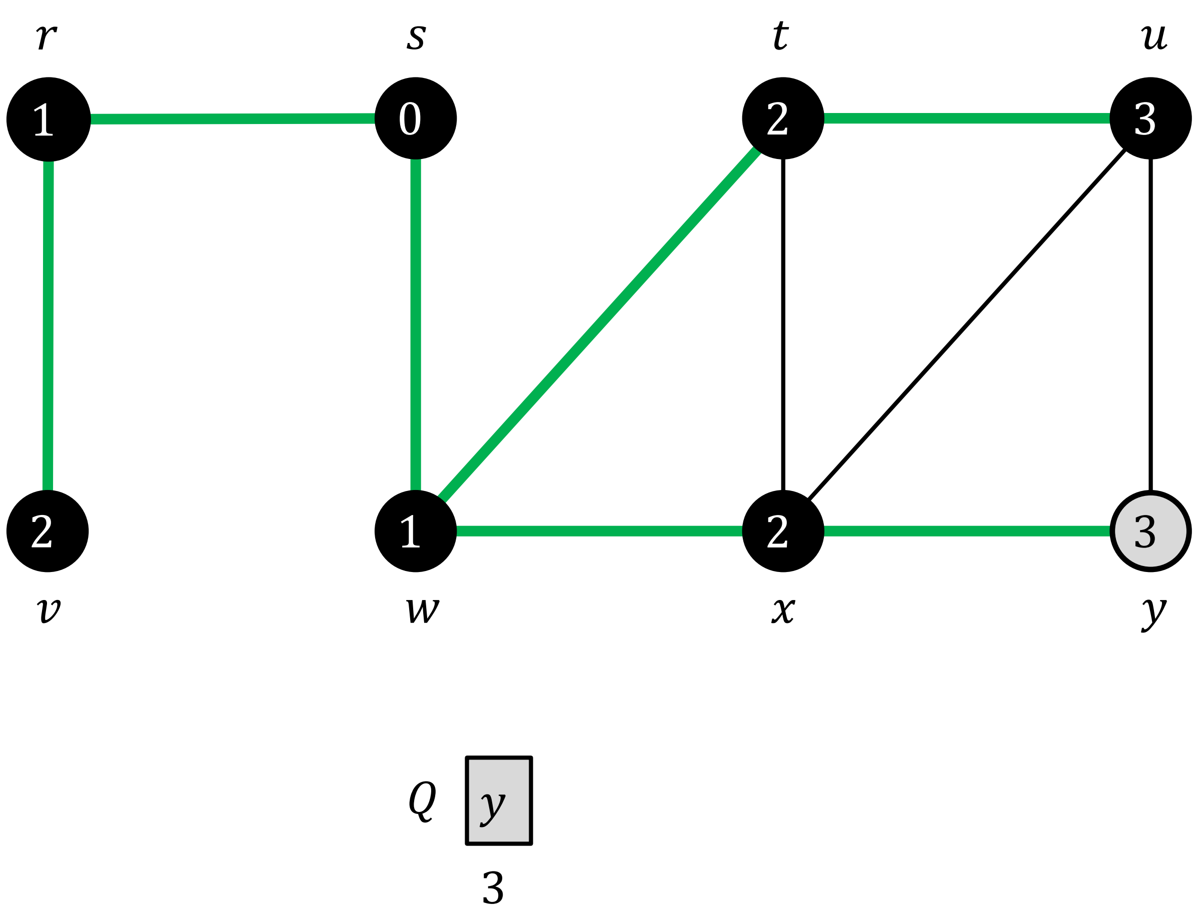
Breadth-First Search (BFS)

DEQUEUE (Q) → v



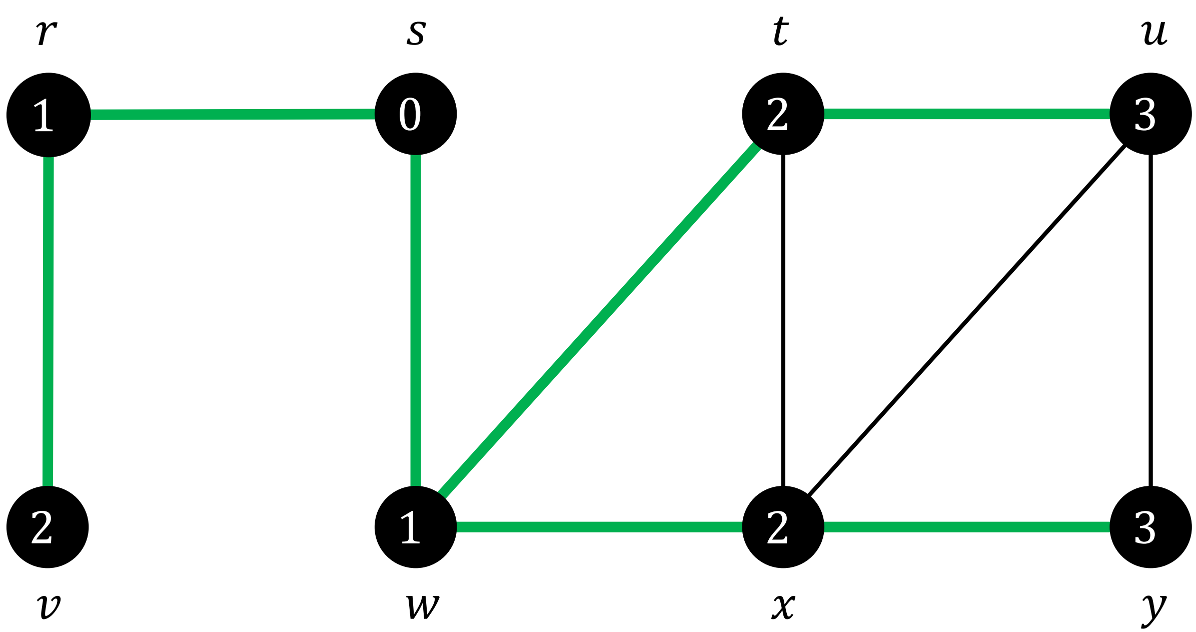
Breadth-First Search (BFS)

DEQUEUE (Q) → $\boldsymbol{u}$



Breadth-First Search (BFS)

DEQUEUE (Q) $\rightarrow y$



$Q \quad \emptyset$

Breadth-First Search (BFS)

BFS ($G,\ s$)

for each vertex $u \in G.V \setminus \{s\}$ *do*
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    for each $v \in G.Adj[u]$ *do*
        if $v.color =$ WHITE *then*
            $v.color \leftarrow$ GRAY, $v.d \leftarrow u.d + 1$, $v.\pi \leftarrow u$
            ENQUEUE($Q,\ v$)
    $u.color \leftarrow$ BLACK

Let $n = |G.V|$ and $m = |G.E|$

Time spent
- initializing $= \Theta(n)$
- enqueuing / dequeuing $= \Theta(n)$
- scanning the adjacency lists $= \Theta\left(\sum_{v \in G.V} |G.Adj[v]|\right) = \Theta(m)$

$\therefore$ Total cost $= \Theta(m + n)$

Depth-First Search (DFS)

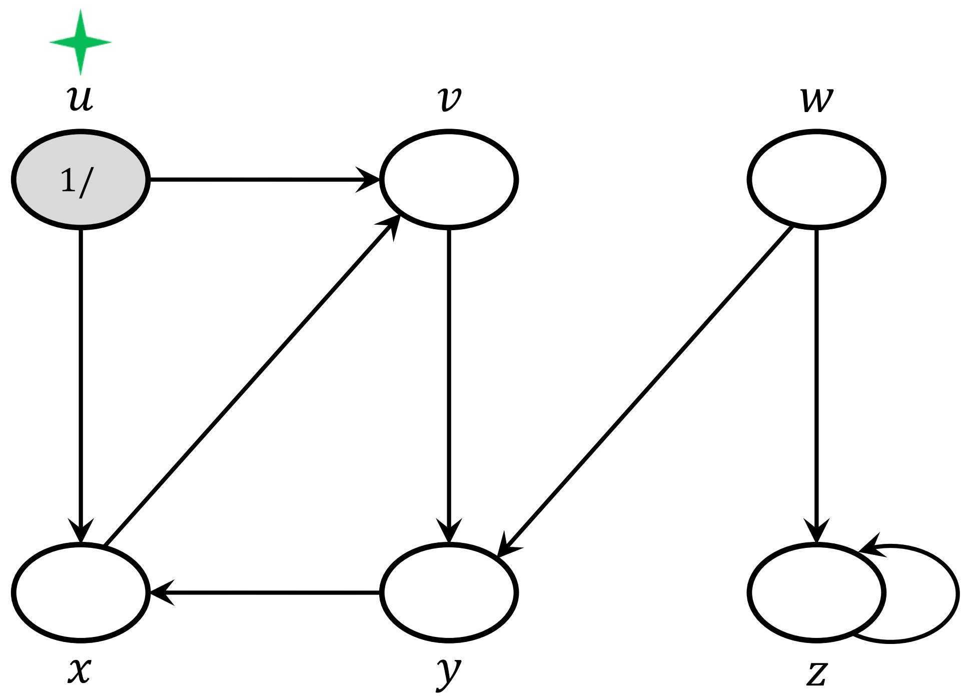
Input: Unweighted directed or undirected graph $G = (V, E)$ with vertex set V and edge set E. For each $v \in V$, the adjacency list of v is $G.Adj[v]$.

Output: For each $v \in G[V]$, $v.d$ is set to the time when v was first discovered and $v.f$ is set to the time when v's adjacency list has been examined completely. Also, $v.\pi$ pointers form a breadth-first tree rooted at s that contains all vertices reachable from s.

```
DFS ( G )
    for each vertex u ∈ G.V do
        u.color ← WHITE,  u.π ← NIL
    time ← 0
    for each u ∈ G.V do
        if u.color = WHITE then
            DFS-VISIT( G, u )
```

```
DFS-VISIT ( G, u )
    time ← time + 1
    u.d ← time
    u.color ← GRAY
    for each v ∈ G.Adj[u] do
        if v.color = WHITE then
            v.π ← u
            DFS-VISIT( G, v )
    u.color ← BLACK
    time ← time + 1
    u.f ← time
```

Depth-First Search (DFS)



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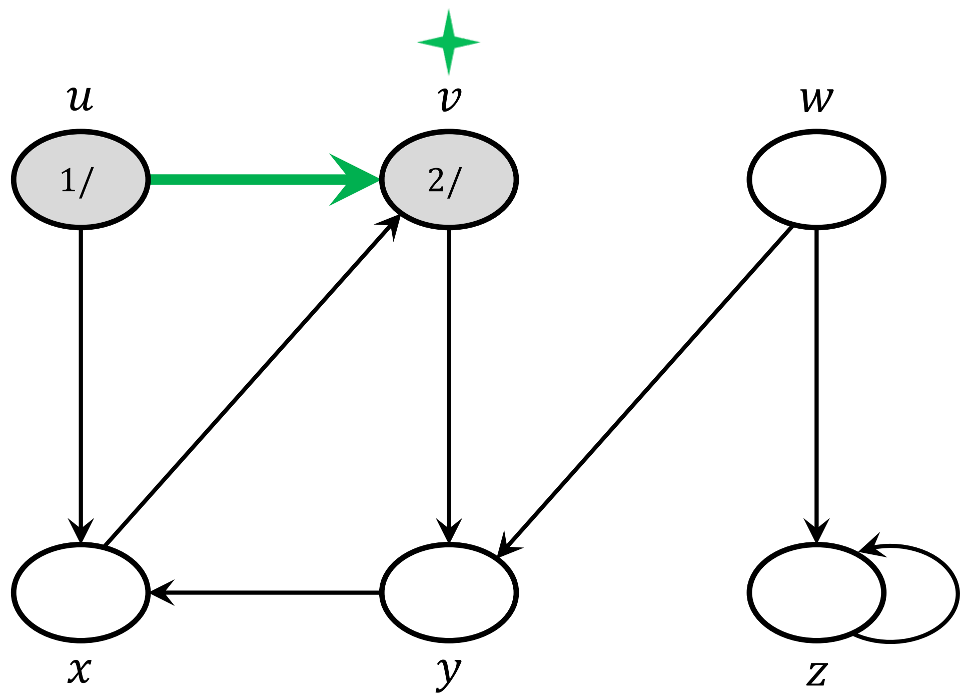
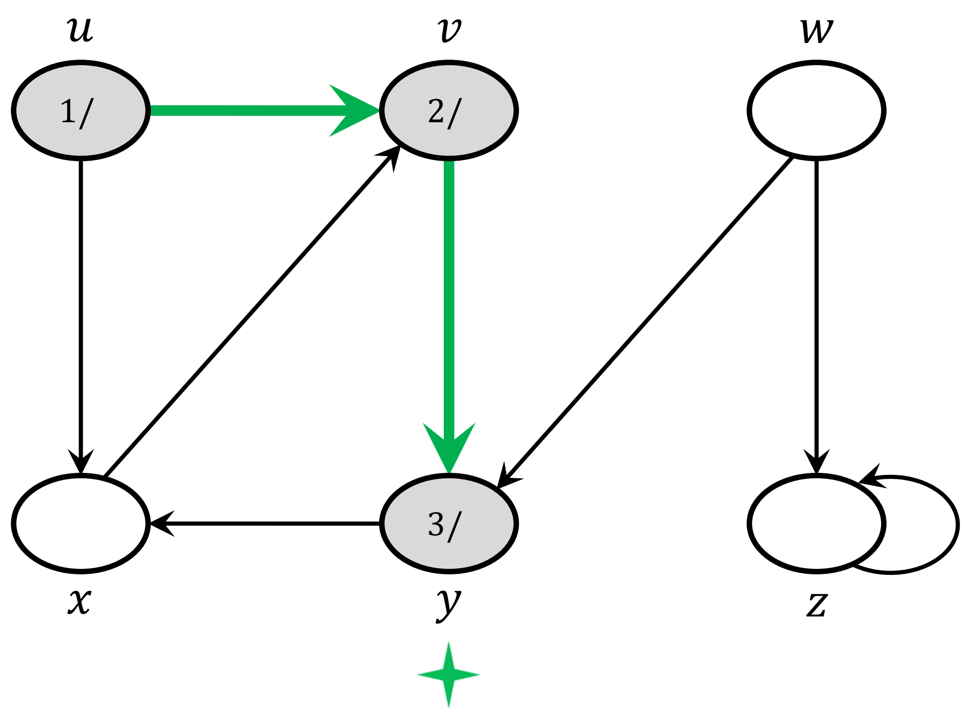


Tree Edge (T): These are edges in the depth-first forest G_π. Edge (u, v) is a tree edge if v was first discovered by exploring that edge. In the example above, we will make all tree edges green and thick.

Depth-First Search (DFS)



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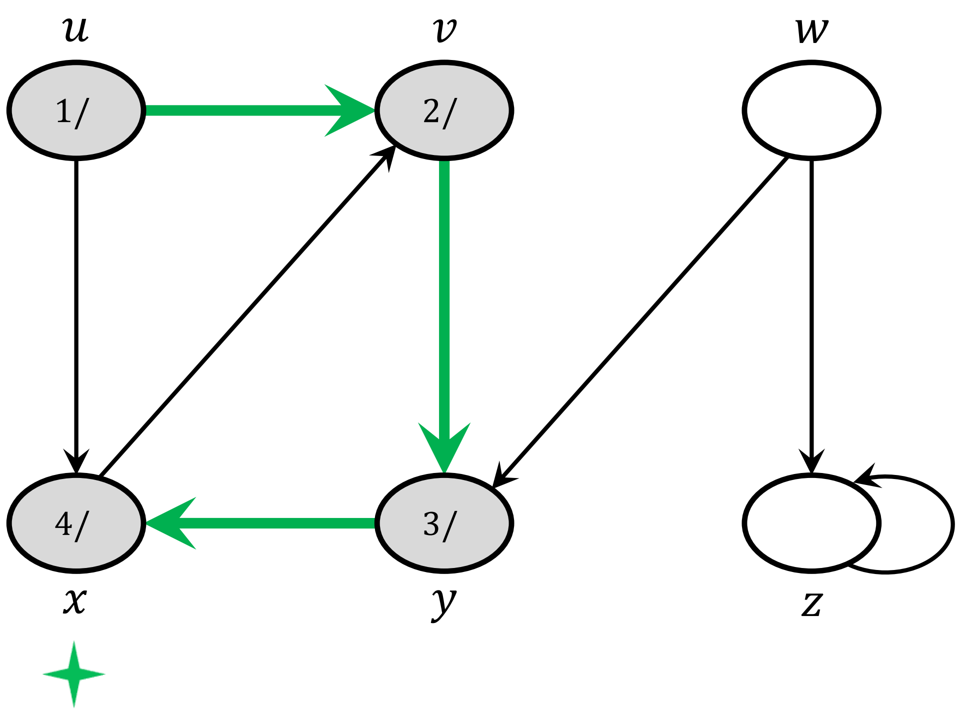

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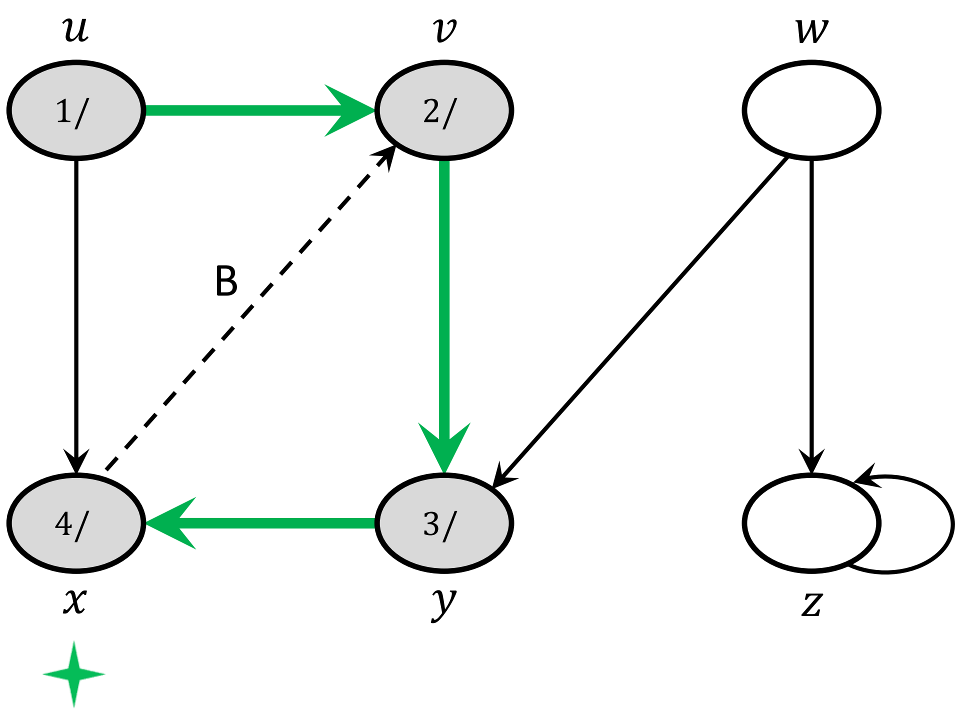
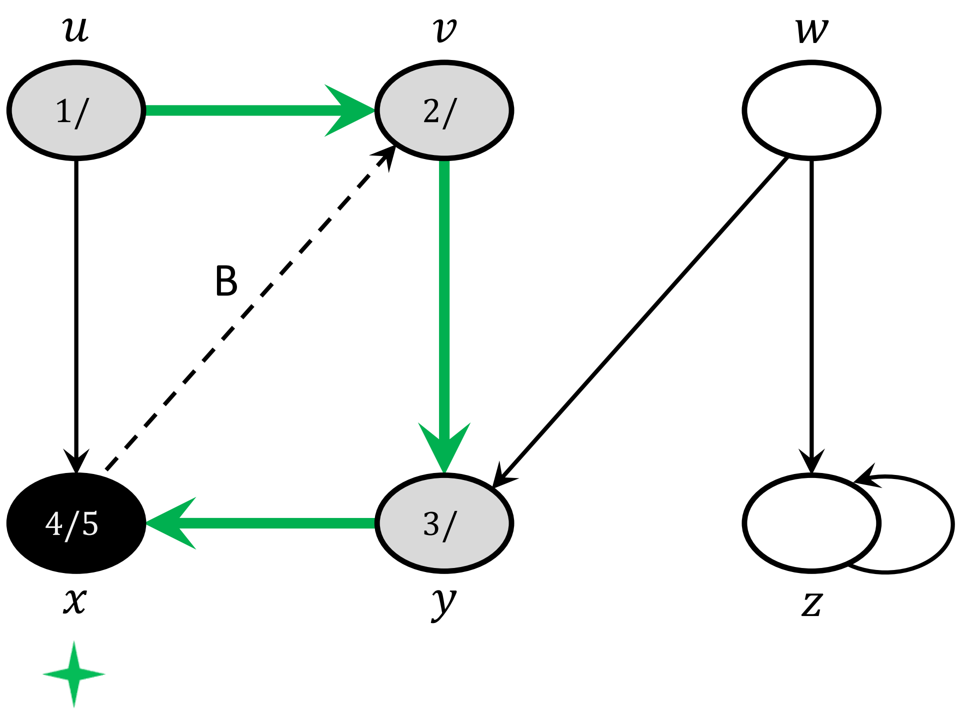
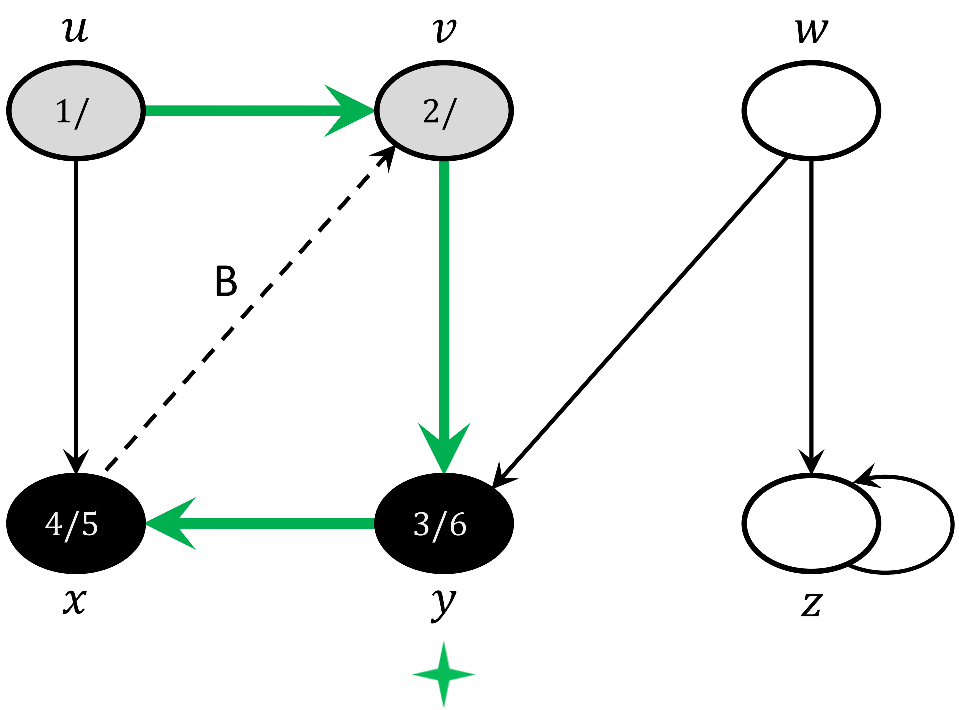


Back Edge (B): A back edge goes from a vertex to its ancestor in a depth-first tree. Self-loops are also considered back edges.

Depth-First Search (DFS)



Depth-First Search (DFS)



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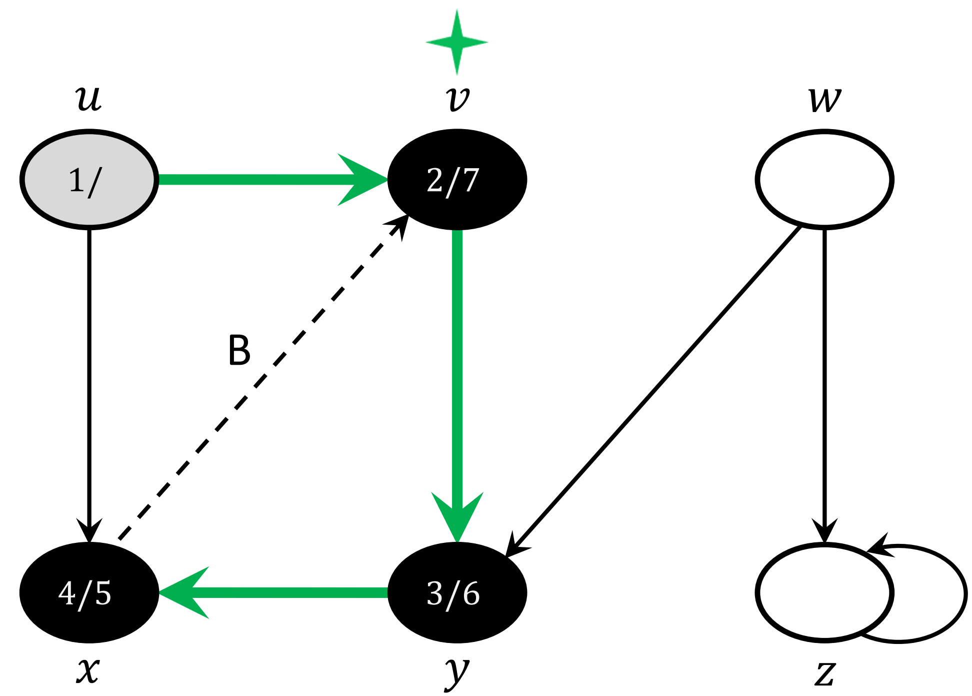

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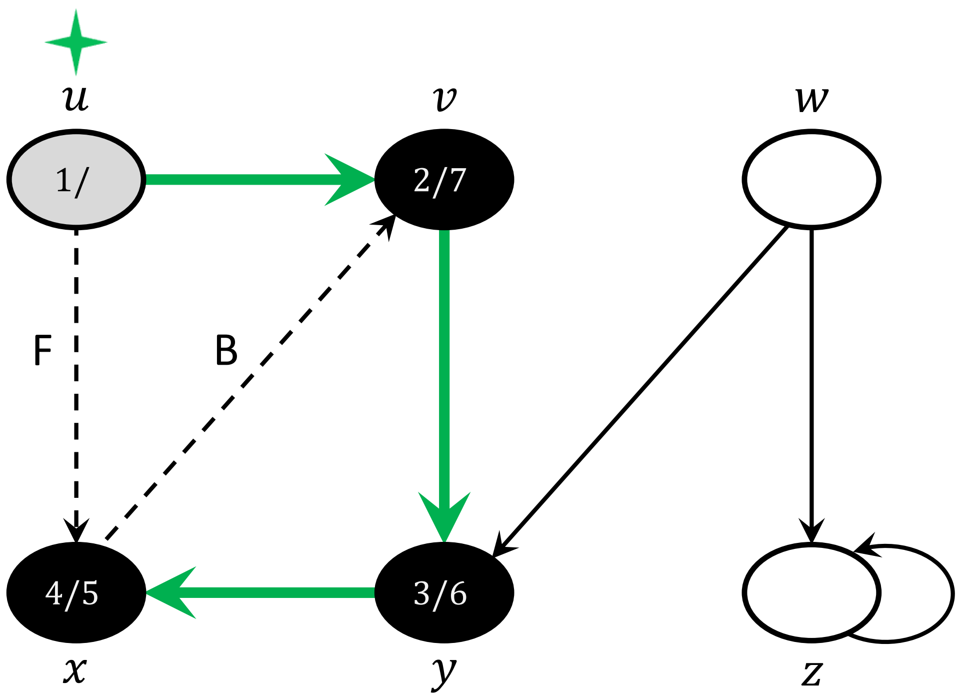
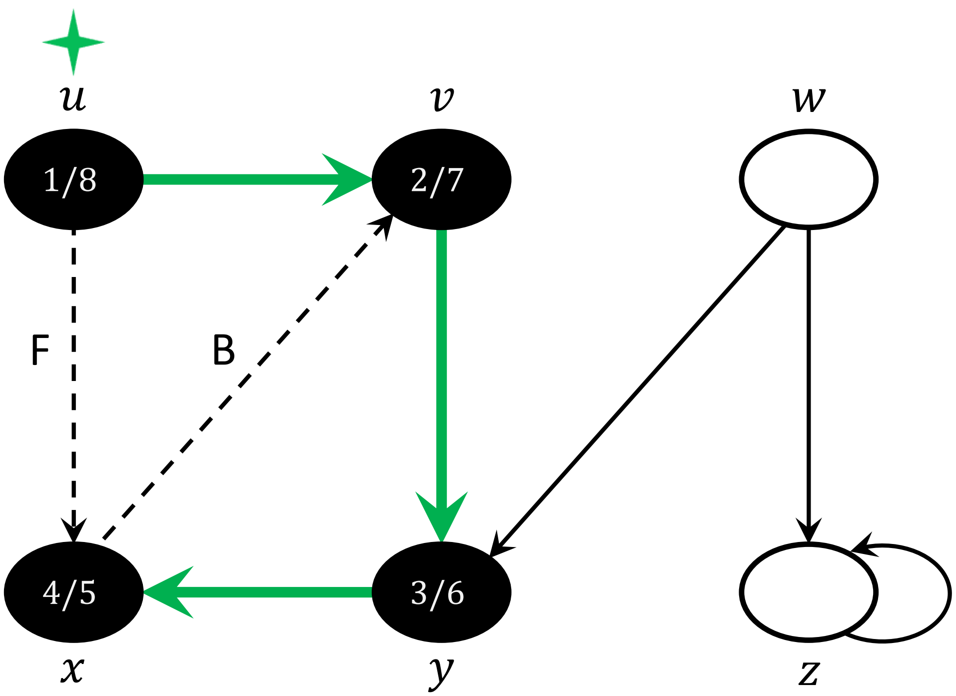
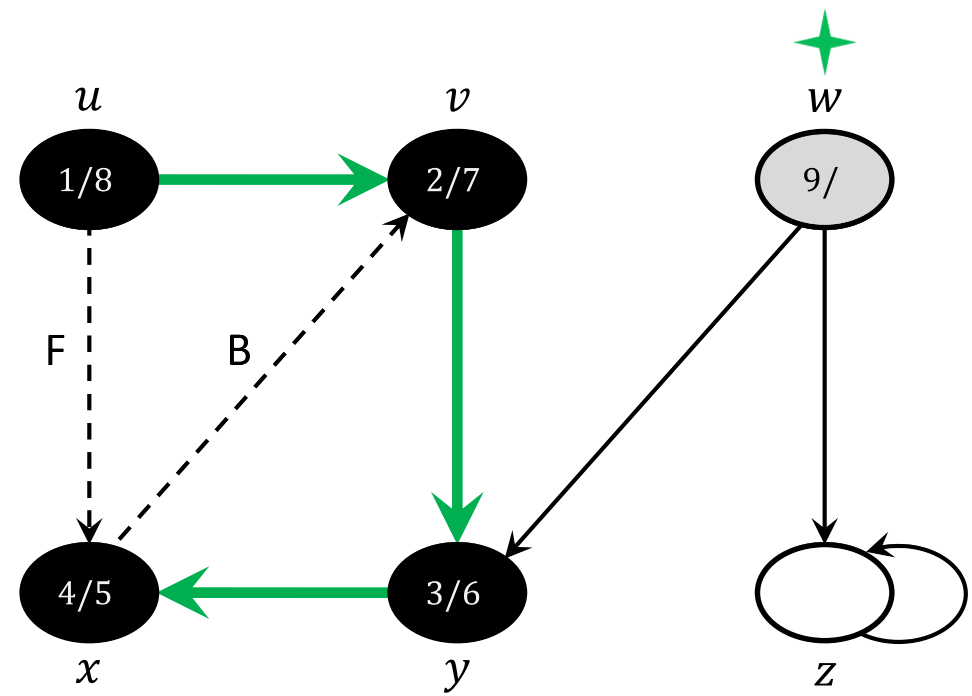


Forward Edge (F): A forward edge is a nontree edge that connects a vertex to a descendant in a depth-first tree.

Depth-First Search (DFS)



Depth-First Search (DFS)



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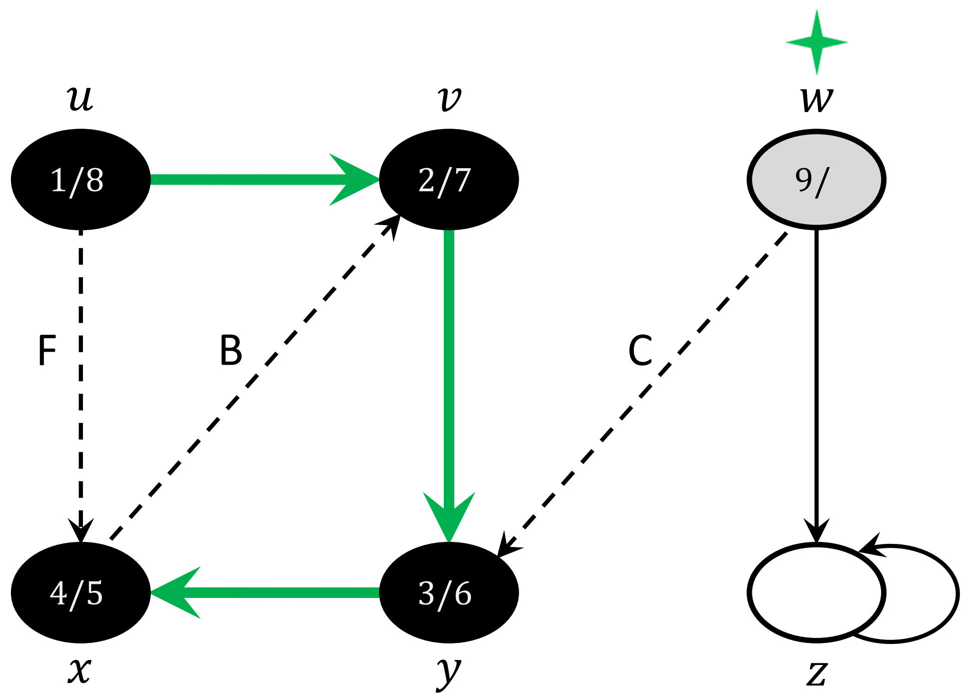
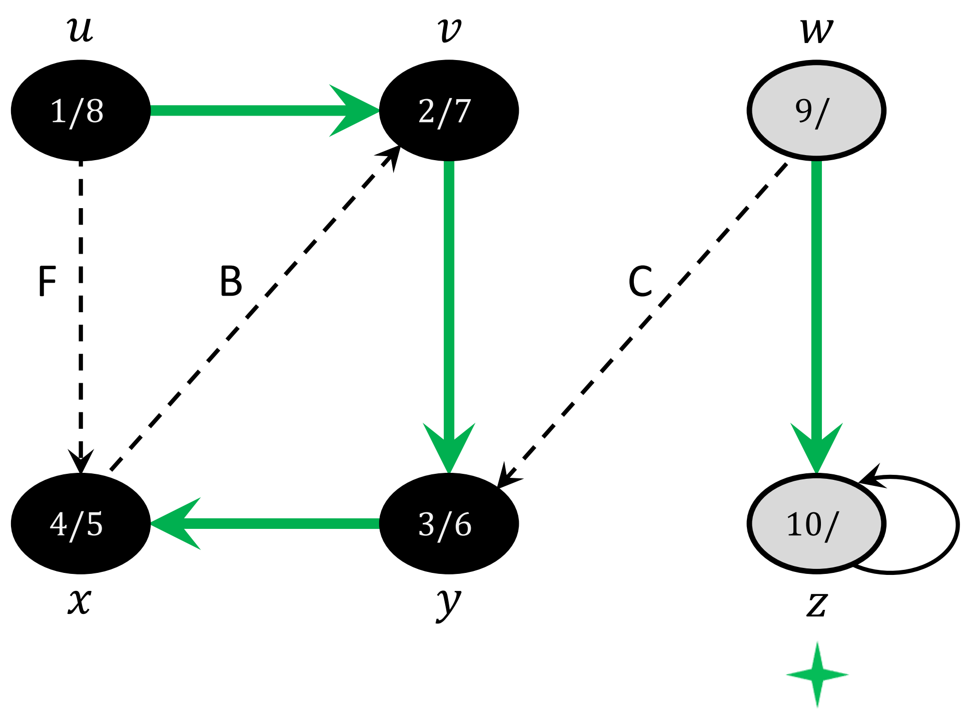


Cross Edge (C): If a non-tree edge is neither a back edge nor a forward edge then it's a cross edge. Cross edges can go between vertices in the same depth-first tree or in different depth-first trees.

Depth-First Search (DFS)



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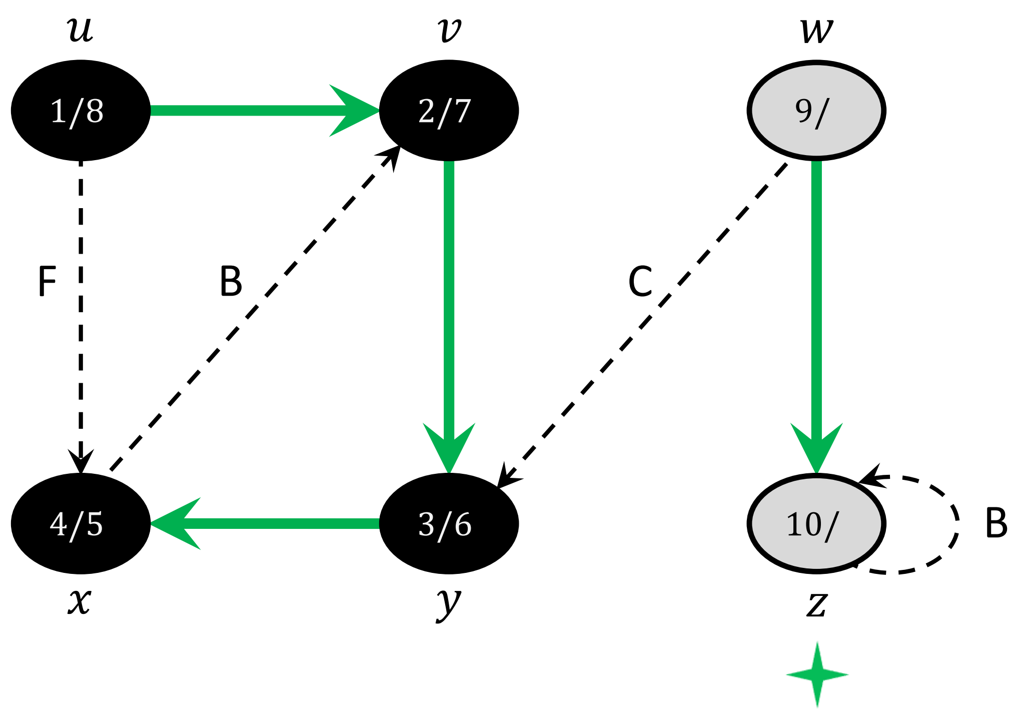

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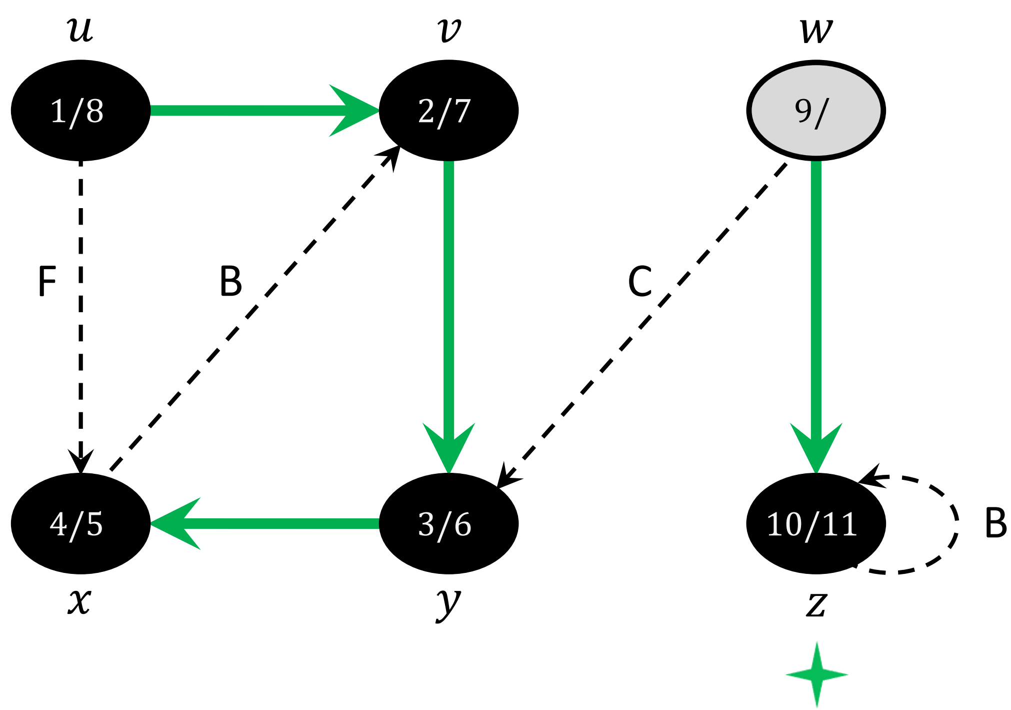

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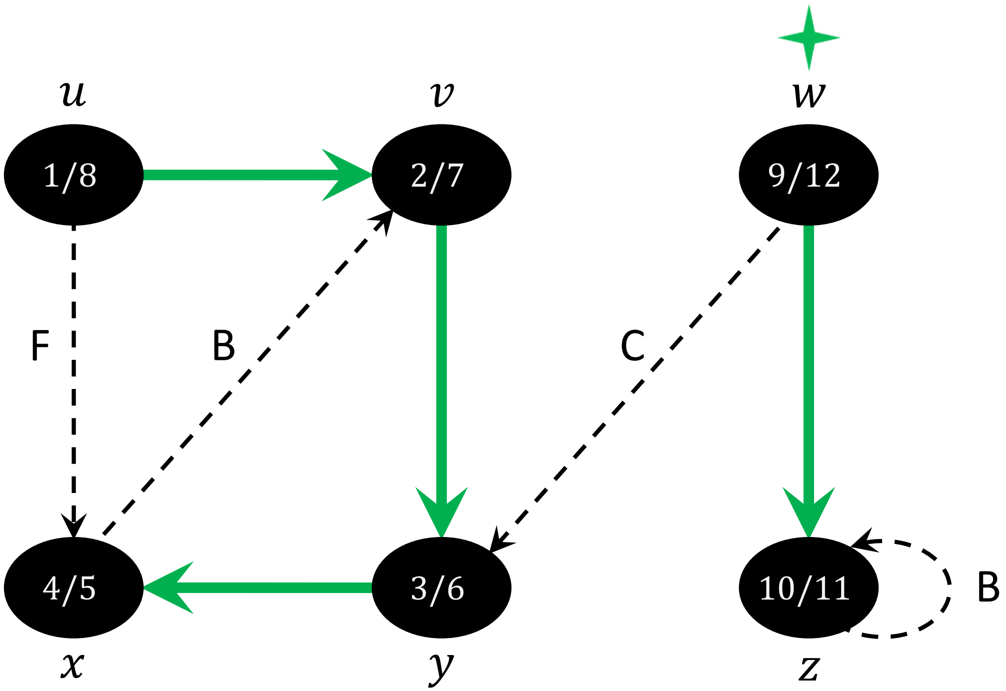


Depth-First Search (DFS)

```
DFS ( G )
    for each vertex u ∈ G.V do
        u.color ← WHITE,  u.π ← NIL
    time ← 0
    for each u ∈ G.V do
        if u.color = WHITE then
            DFS-VISIT( G, u )
```

```
DFS-VISIT ( G, u )
    time ← time + 1
    u.d ← time
    u.color ← GRAY
    for each v ∈ G.Adj[u] do
        if v.color = WHITE then
            v.π ← u
            DFS-VISIT( G, v )
    u.color ← BLACK
    time ← time + 1
    u.f ← time
```

Let $n = |G.V|$ and $m = |G.E|$

Time spent

- in *DFS* (exclusive of calls to *DFS-VISIT*) $= \Theta(n)$
- in *DFS-VISIT* scanning the adjacency lists $= \Theta\left(\sum_{v \in G.V} |G.Adj[v]|\right)$ $= \Theta(m)$

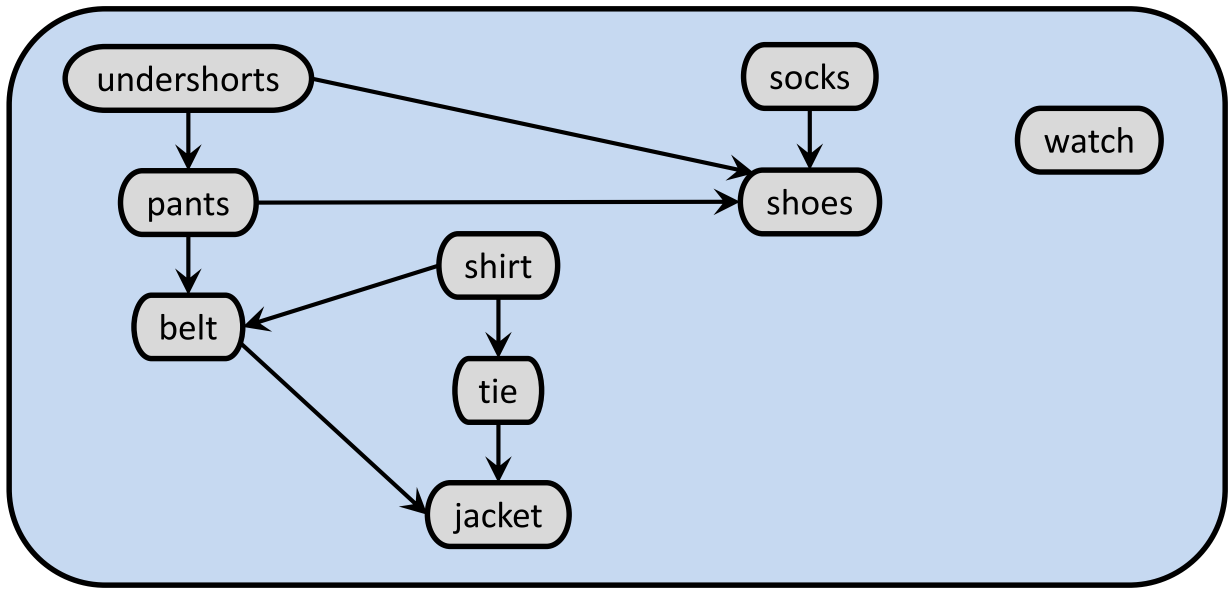
$\therefore$ Total cost $= \Theta(m + n)$

Topological Sort

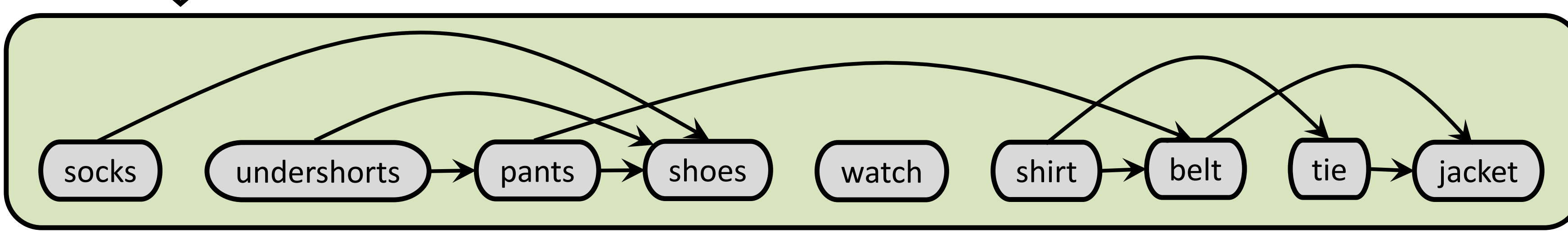
A ***topological sort*** of a DAG (i.e., directed acyclic graph) $G = (V, E)$ is a linear ordering of all its vertices such that if G contains an edge (u, v), then u appears before v in the ordering.

We can view a topological sort of a graph as an ordering of its vertices along a horizontal line so that all directed edges go from left to right.

A Directed Acyclic Graph (DAG)



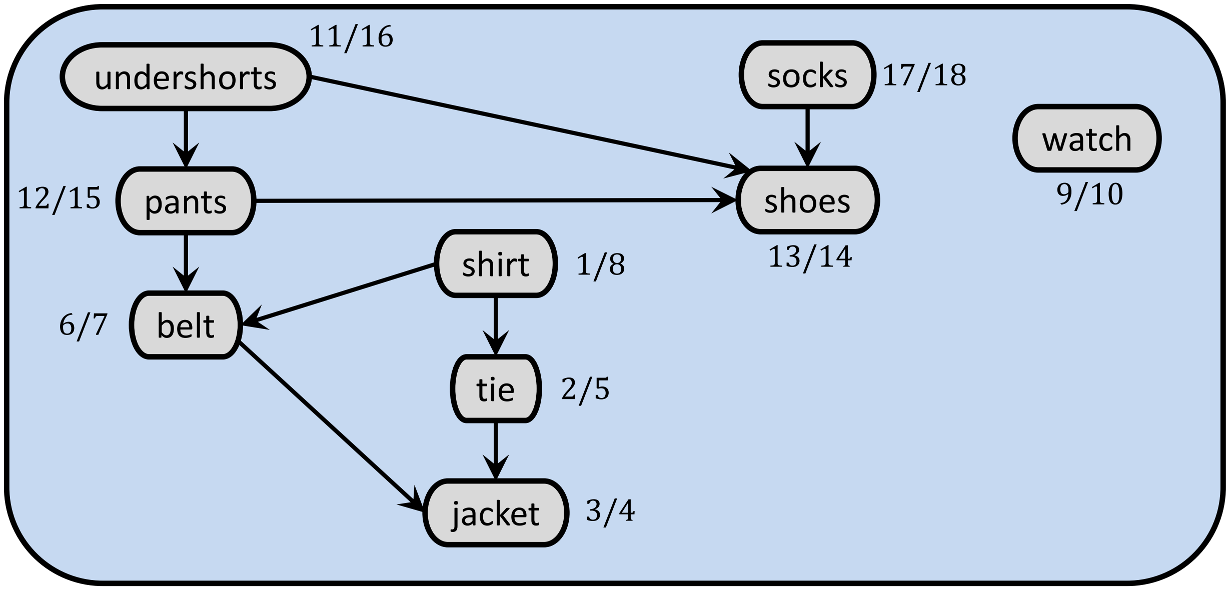
A topological sort of the DAG nodes

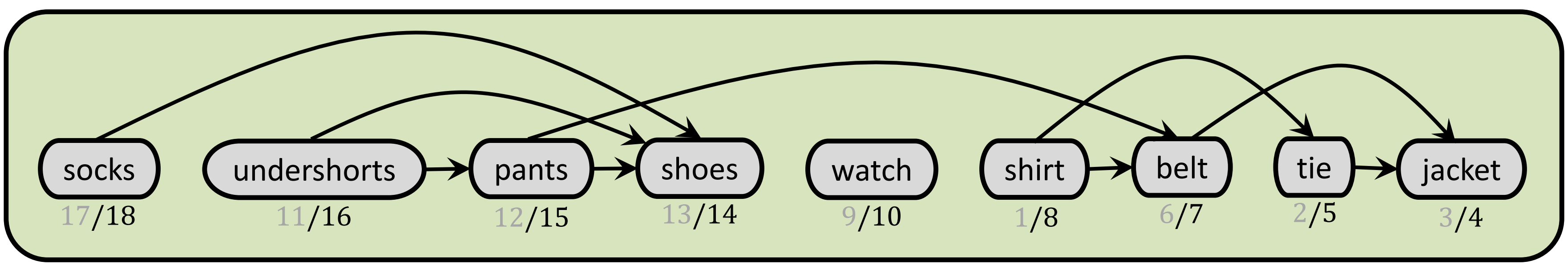


Topological Sort

TOPOLOGICAL-SORT (G)

1. call *DFS* (G) to compute the finish times $v.f$ for each vertex $v \in G.V$
2. as each vertex is finished, insert it into the front of a linked list
3. *return* the linked list of vertices





Strongly Connected Components

A ***strongly connected component*** of a directed graph $G = (V, E)$ is a maximal set of vertices $C \subseteq V$ such that for every pair of vertices u and v in C, we have both $u \rightsquigarrow v$ and $v \rightsquigarrow u$; that is, vertices u and v are reachable from each other.

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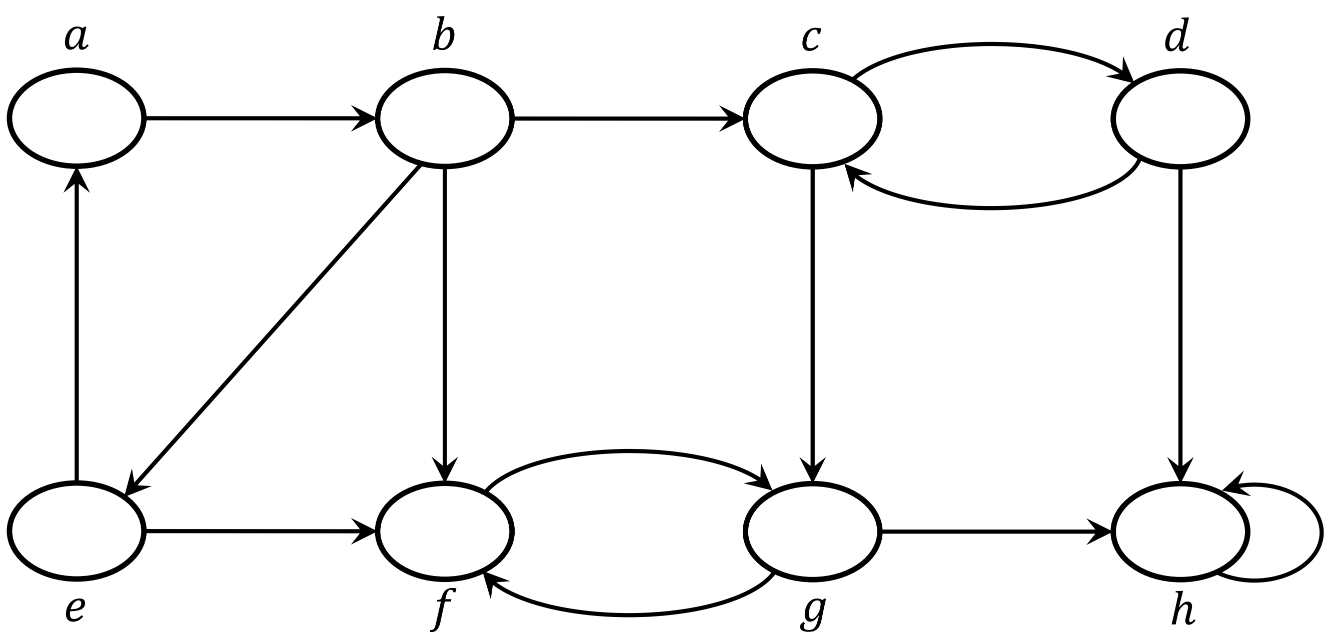

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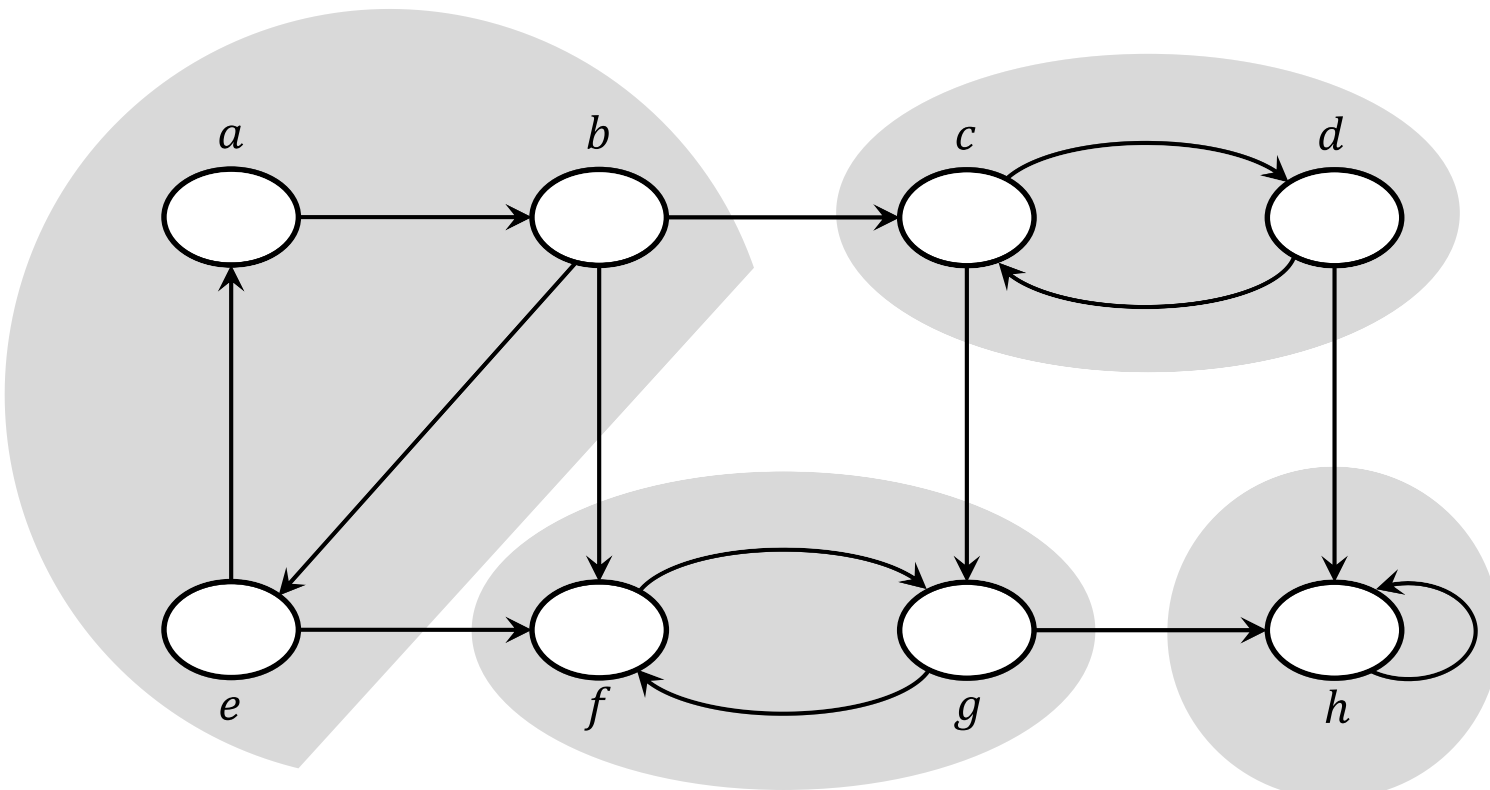

Strongly Connected Components

STRONGLY-CONNECTED-COMPONENTS (G)

1. call *DFS* (G) to compute the finish times $v.f$ for each vertex $v \in G.V$
2. compute G^T
3. call *DFS* (G^T), but in the main loop of *DFS*, consider the vertices in order of decreasing $v.f$ (as computed in line 1)
4. output the vertices of each tree in the depth-first forest formed in line 3 as a separate strongly connected component

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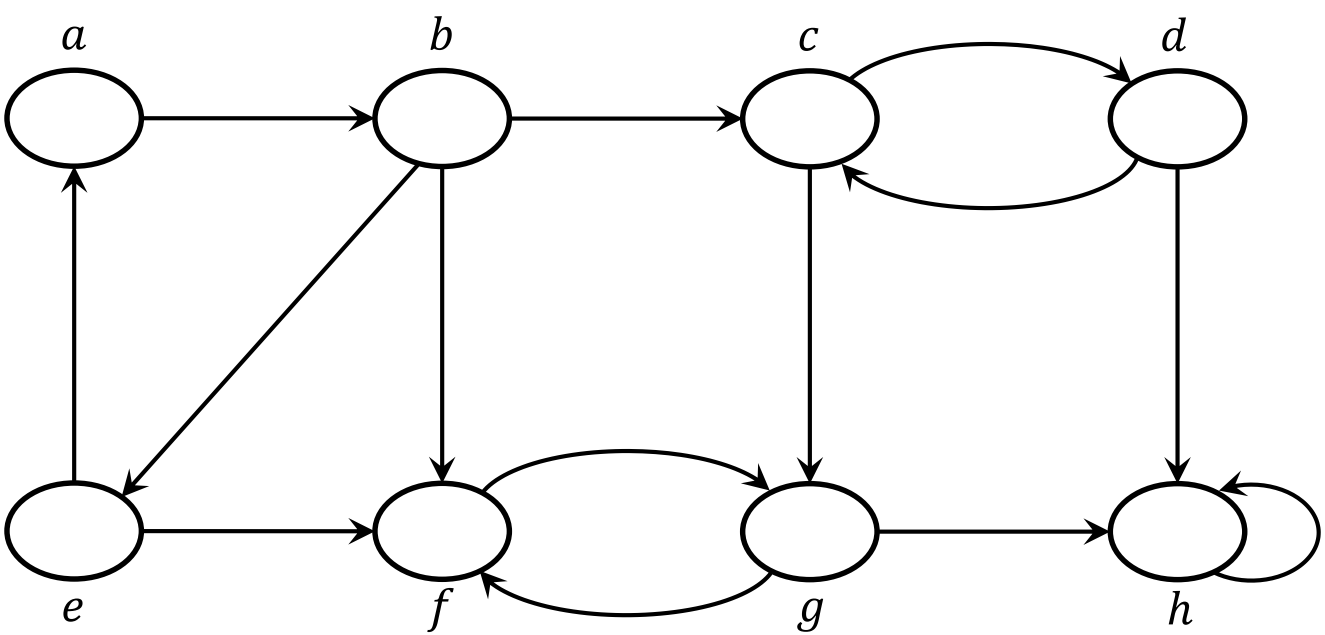

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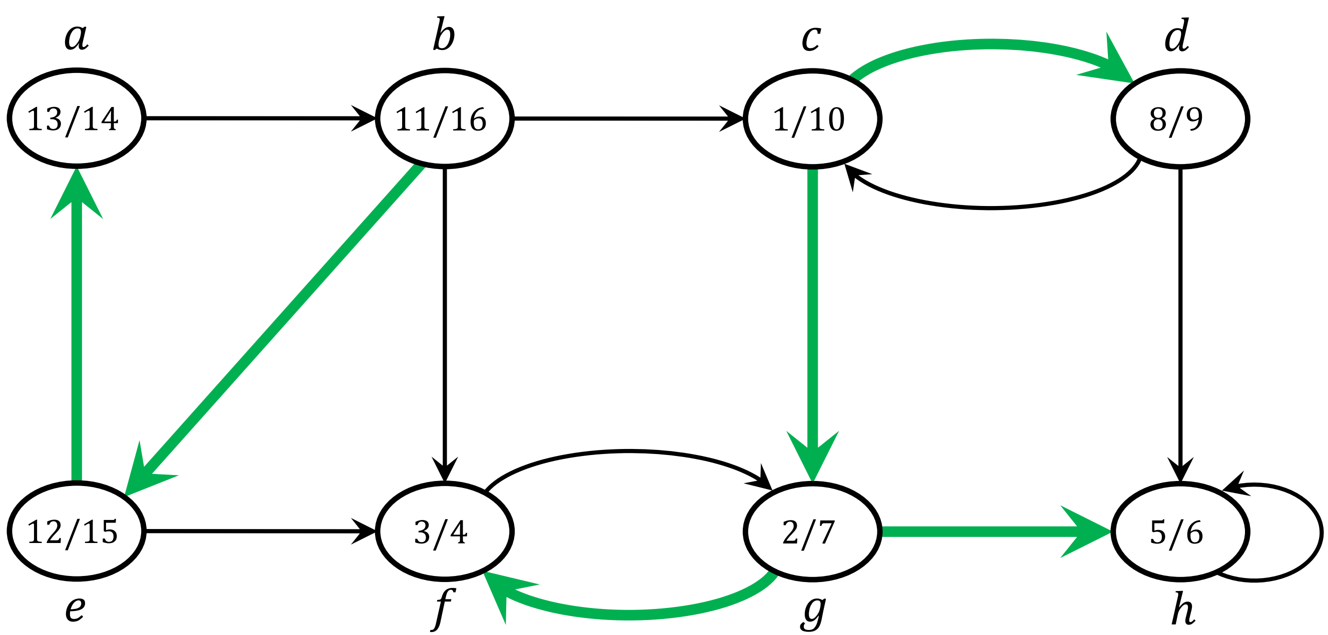

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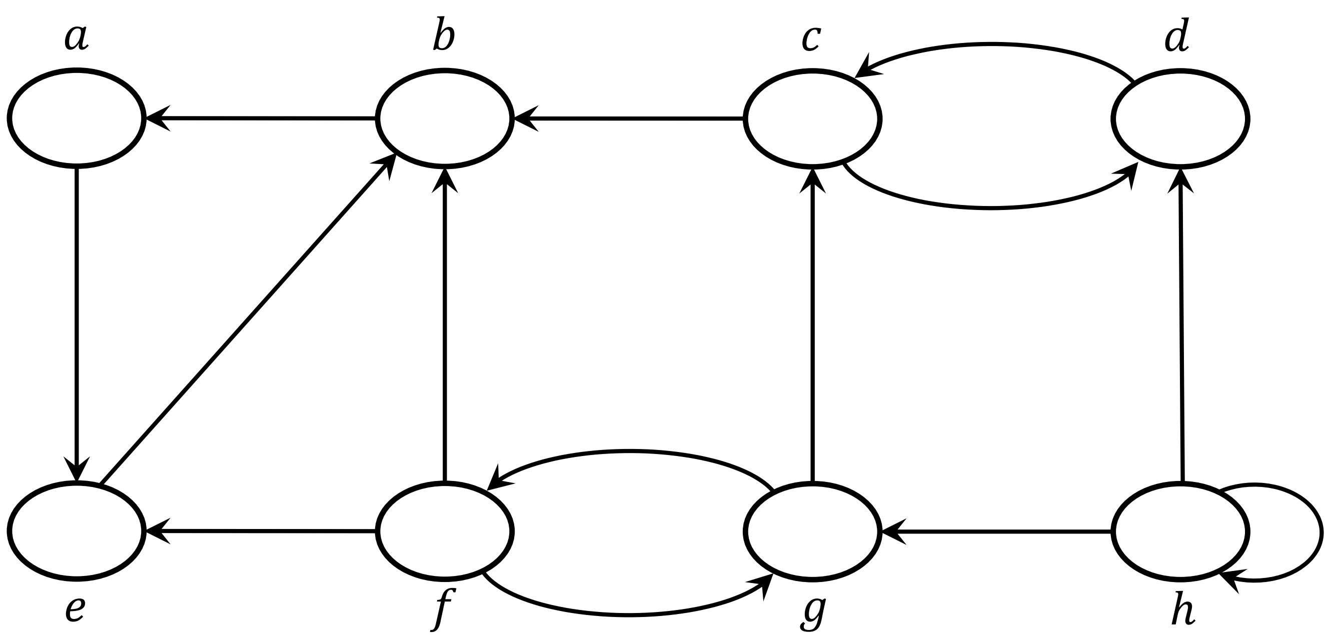

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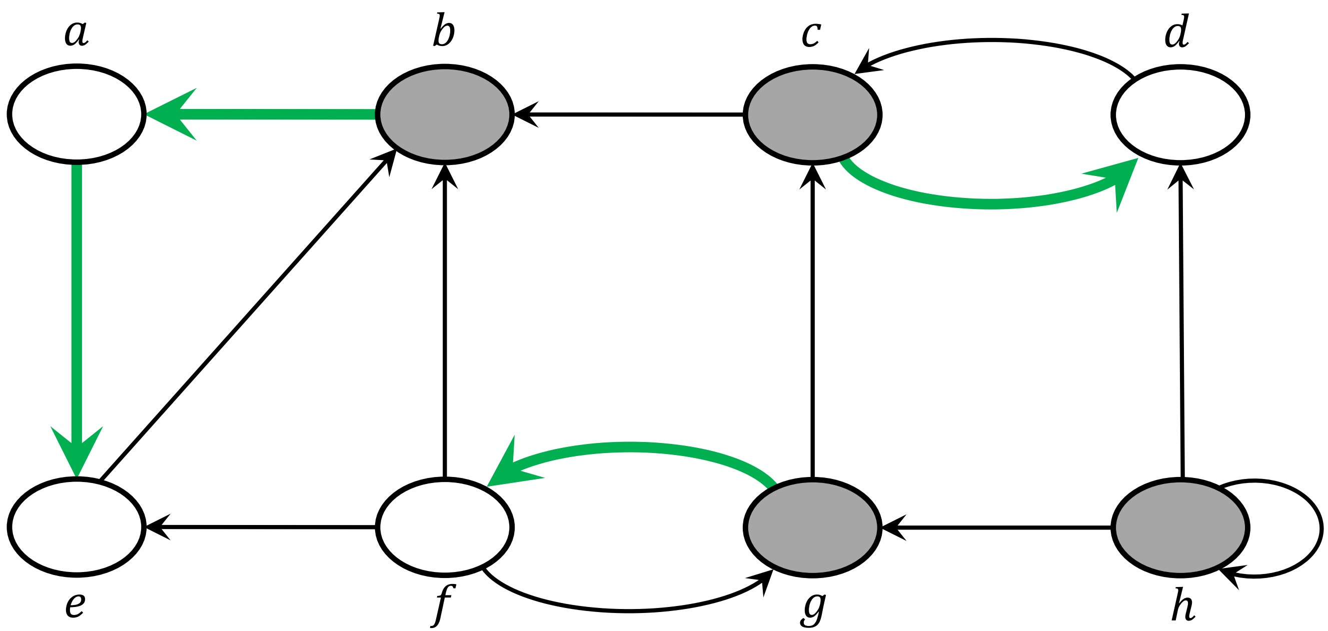

Strongly Connected Components

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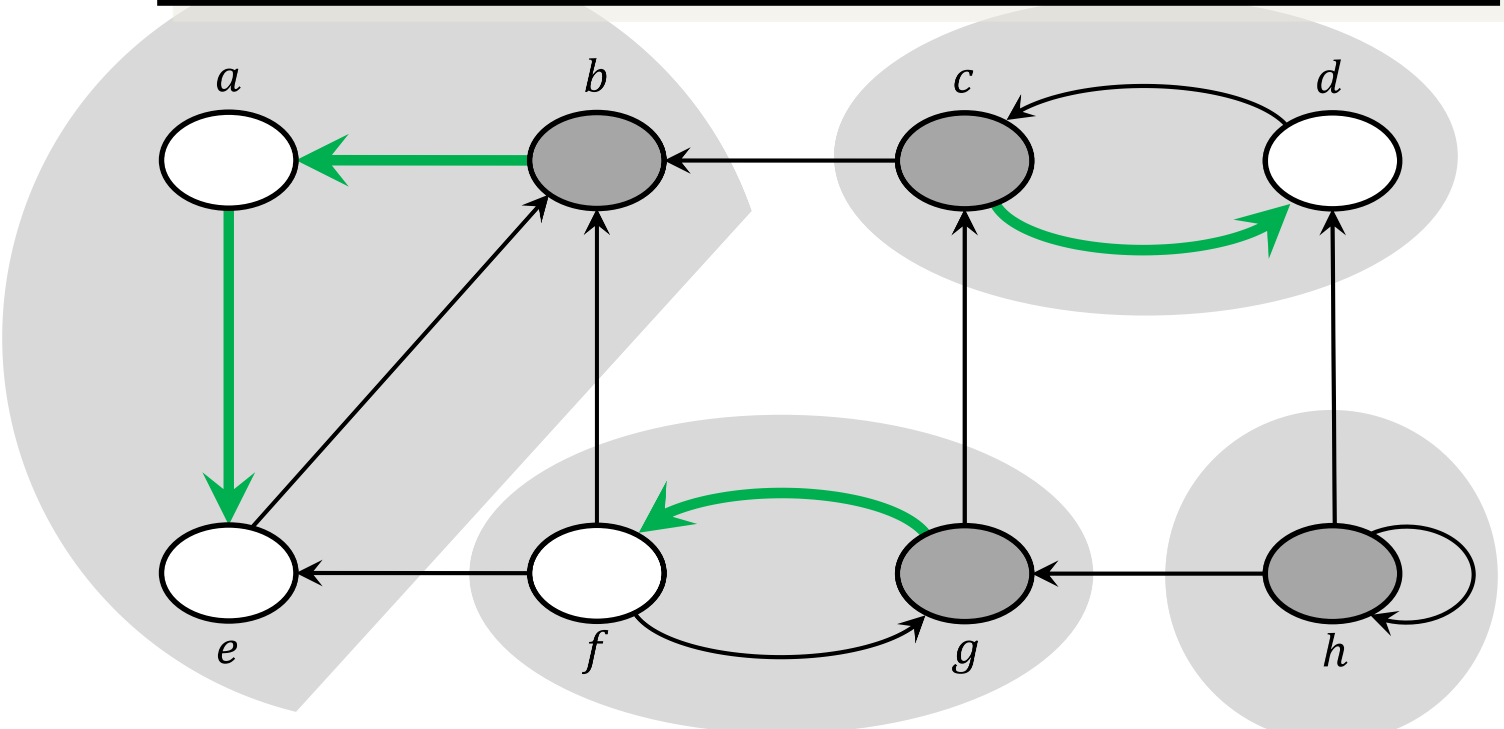

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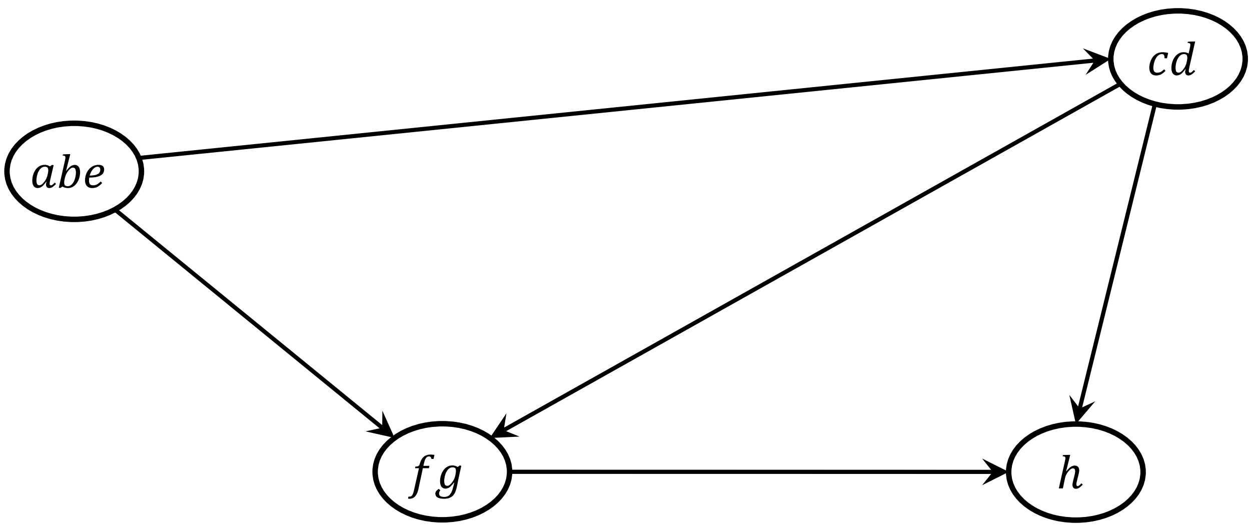

The Single-Source Shortest Paths (SSSP) Problem

We are given a weighted, directed graph $G = (V, E)$ with vertex set V and edge set E, and a weight function w such that for each edge $(u, v) \in E$, $w(u, v)$ represents its weight.

We are also given a source vertex $s \in V$.

Our goal is to find a shortest path (i.e., a path of the smallest total edge weight) from s to each vertex $v \in V$.

SSSP: Relxation

INITIALIZE-SINGLE-SOURCE ($G = (V, E)$, s)

1. *for* each vertex $v \in G.V$ *do*
2. $\quad v.d \leftarrow \infty$
3. $\quad v.\pi \leftarrow NIL$
4. $s.d \leftarrow 0$

RELAX (u, v, w)

1. *if* $u.d + w(u,v) < v.d$ *then*
2. $\quad v.d \leftarrow u.d + w(u,v)$
3. $\quad v.\pi \leftarrow u$

SSSP: Properties of Shortest Paths and Relxation

The ***weight*** $w(p)$ of path $p = \langle v_0, v_1, \dots, v_k \rangle$ is the sum of the weights of its constituent edges:

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

We define the ***shortest-path weight*** $\delta(u, v)$ from u to v by

$$\delta(u,v) = \begin{cases} \min\{w(p) : p \text{ is } u \sim v\}, & \text{if there is a path from } u \text{ to } v, \\ \infty, & \text{otherwise.} \end{cases}$$

A ***shortest path*** from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.

SSSP: Properties of Shortest Paths and Relxation

Triangle inequality (Lemma 24.10 of CLRS)

For any edge $(u, v) \in E$, we have $\delta(s, v) \leq \delta(s, u) + w(u, v)$.

Upper-bound inequality (Lemma 24.11 of CLRS)

We always have $v.d \geq \delta(s, v)$ for all vertices $v \in V$, and once $v.d$ achieves the value $\delta(u, v)$, it never changes.

No-path property (Corollary 24.12 of CLRS)

If there is no path from s to v, then we always have $v.d = \delta(s, v) = \infty$.

Convergence property (Lemma 24.14 of CLRS)

If $s \rightsquigarrow u \rightarrow v$ is a shortest path in G for some $u, v \in V$, and if $u.d = \delta(s, u)$ at any time prior to relaxing edge (u, v), then $v.d = \delta(s, v)$ at all times afterward.

SSSP: Properties of Shortest Paths and Relxation

Path-relaxation property (Lemma 24.15 of CLRS)

If $p = \langle v_0, v_1, \dots, v_k \rangle$ is a shortest path from $s = v_0$ to v_k, and we relax the edges of p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$, then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations on the edges of p.

Predecessor-subgraph property (Lemma 24.17 of CLRS)

Once $v.d = \delta(s, v)$ for all $v \in V$, the predecessor subgraph is a shortest-paths tree rooted at s.

Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

Since we already discussed Dijkstra's SSSP algorithm when we talked about greedy algorithms, we will skip over it in this lecture.

Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

Input: Weighted graph $G = (V, E)$ with vertex set V and edge set E, a non-negative weight function w, and a source vertex $s \in G[V]$.

Output: For all $v \in G[V]$, $v.d$ is set to the shortest distance from s to v.

```
Dijkstra-SSSP ( G = (V,E), w, s )
    for each vertex v ∈ G.V do
        v.d ← ∞
        v.π ← NIL
    s.d ← 0
    Min-Heap Q ← ∅
    for each vertex v ∈ G.V do
        INSERT( Q, v )
    while Q ≠ ∅ do
        u ← EXTRACT-MIN( Q )
        for each (u,v) ∈ G.E do
            if u.d + w(u,v) < v.d then
                v.d ← u.d + w(u,v)
                v.π ← u
                DECREASE-KEY( Q, v, u.d + w(u,v) )
```

Let $n = |G[V]|$ and $m = |G[E]|$

Worst-case running time:

Using a binary min-heap
$= \mathrm{O}((m+n)\log n)$

Using a Fibonacci heap
$= \mathrm{O}(m + n\log n)$

Dijkstra's SSSP Algorithm with a Min-Heap
(SSSP: Single-Source Shortest Paths)

Input: Weighted graph $G = (V, E)$ with vertex set V and edge set E, a non-negative weight function w, and a source vertex $s \in G[V]$.

Output: For all $v \in G[V]$, $v.d$ is set to the shortest distance from s to v.

```
Dijkstra-SSSP ( G = (V,E), w, s )

    for each vertex v ∈ G.V do
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    Min-Heap Q ← ∅
    for each vertex v ∈ G.V do
        INSERT( Q, v )
    while Q ≠ ∅ do
        u ← EXTRACT-MIN( Q )
        for each (u, v) ∈ G.E do
            if u.d + w(u, v) < v.d then
                v.d ← u.d + w(u, v)
                v.π ← u
                DECREASE-KEY( Q, v, u.d + w(u, v) )
```

Let $n = |G[V]|$ and $m = |G[E]|$

Worst-case running time:

Using a binary min-heap
$$= O((m + n) \log n)$$

Using a Fibonacci heap
$$= O(m + n \log n)$$

The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Input: Weighted graph $G = (V, E)$ with vertex set V and edge set E, a weight function w, and a source vertex $s \in G[V]$. Negative-weight edges are allowed (unlike Dijkstra's SSSP algorithm).

Output: Returns FALSE if a negative-weight cycle is reachable from s, otherwise returns TRUE and for all $v \in G[V]$, sets $v.d$ to the shortest distance from s to v.

INITIALIZE-SINGLE-SOURCE ($G = (V, E)$, s)

for each vertex $v \in G.V$ *do*
    $v.d \leftarrow \infty$
    $v.\pi \leftarrow NIL$
$s.d \leftarrow 0$

RELAX (u, v, w)

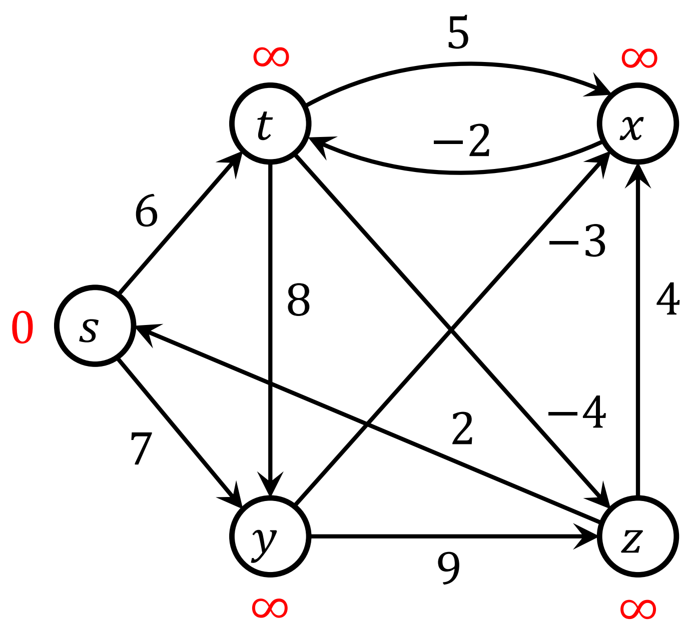
if $u.d + w(u, v) < v.d$ *then*
    $v.d \leftarrow u.d + w(u, v)$
    $v.\pi \leftarrow u$

BELLMAN-FORD ($G = (V, E)$, w, s)

INITIALIZE-SINGLE-SOURCE(G, s)
for $i \leftarrow 1$ *to* $|G.V| - 1$ *do*
    for each $(u, v) \in G.E$ *do*
        RELAX(u, v, w)
for each $(u, v) \in G.E$ *do*
    if $u.d + w(u, v) < v.d$ *then*
        return FALSE
return TRUE

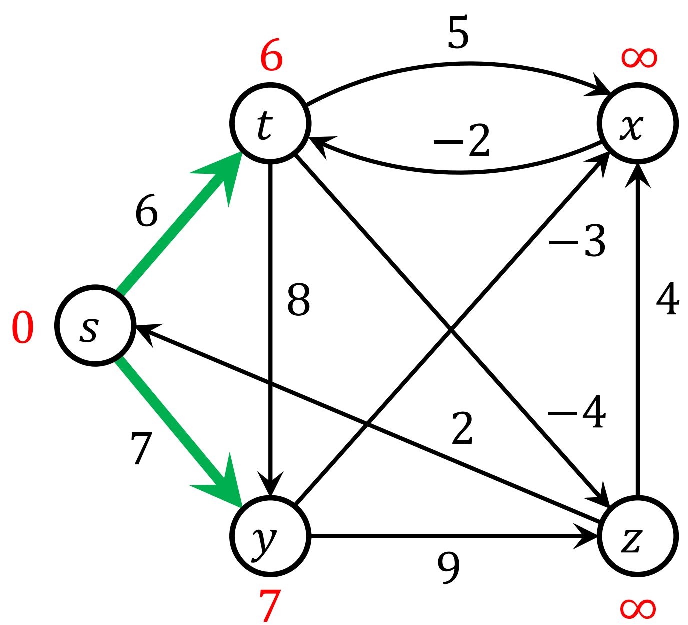
The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Initial State (with initial tentative distances)



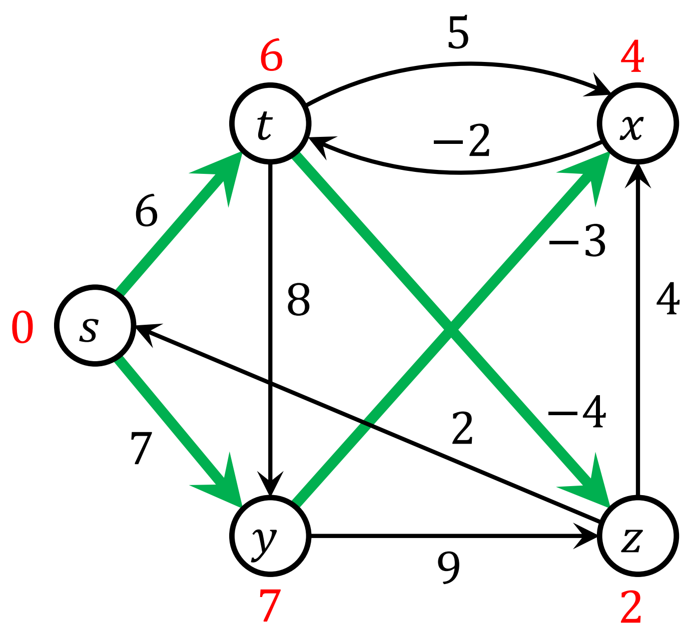
The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Iteration 1



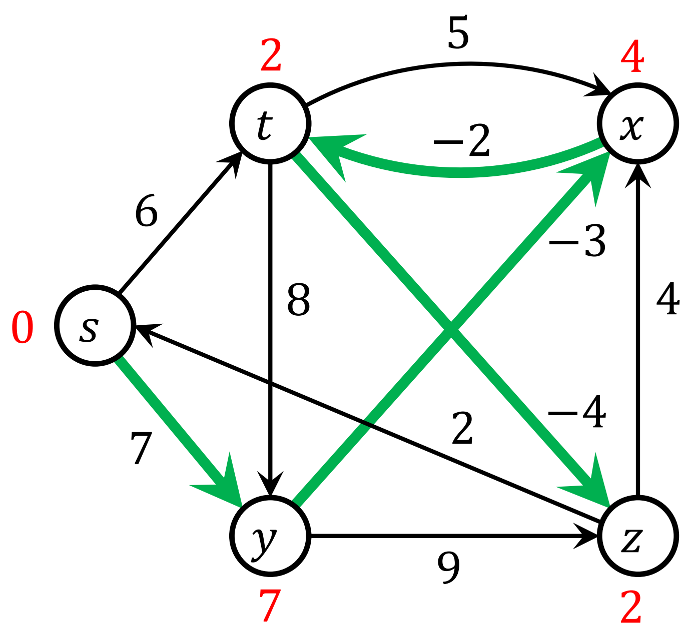
The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Iteration 2



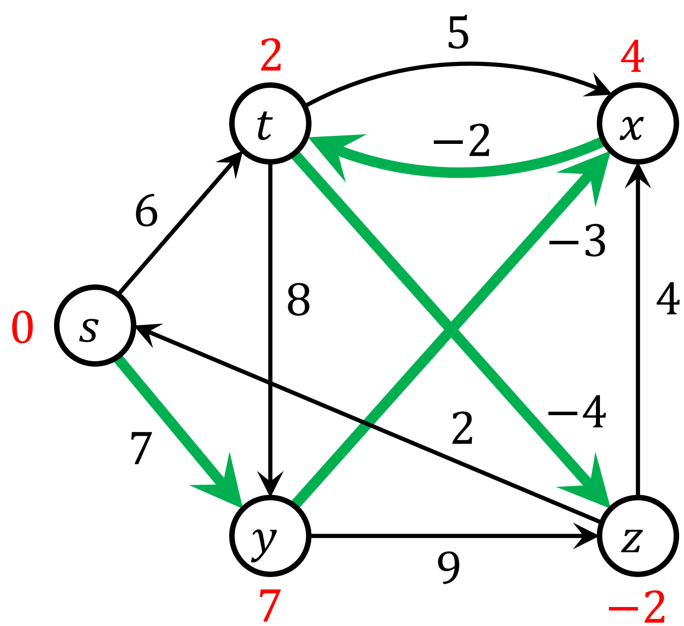
The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Iteration 3



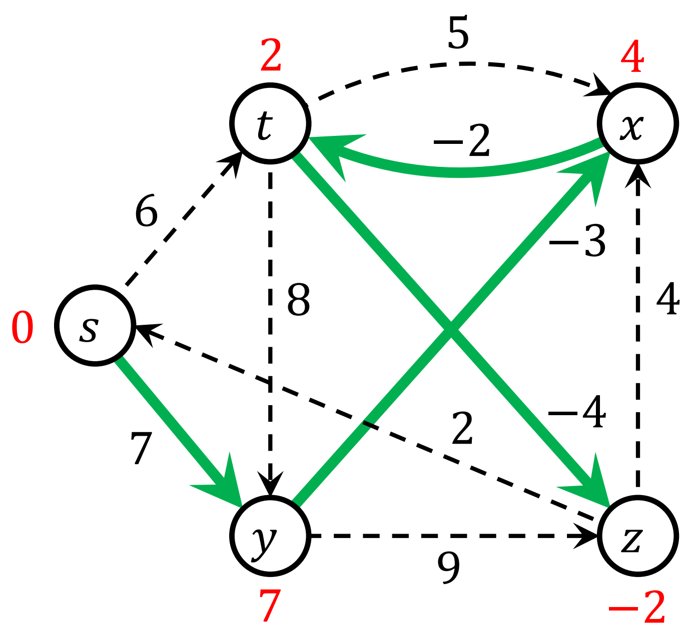
The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Iteration 4



The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

Done!



The Bellman-Ford (SSSP) Algorithm
(SSSP: Single-Source Shortest Paths)

INITIALIZE-SINGLE-SOURCE ($G = (V, E)$, s)

1. **for** each vertex $v \in G.V$ **do**
2. $\quad v.d \leftarrow \infty$
3. $\quad v.\pi \leftarrow NIL$
4. $s.d \leftarrow 0$

RELAX (u, v, w)

1. **if** $u.d + w(u, v) < v.d$ **then**
2. $\quad v.d \leftarrow u.d + w(u, v)$
3. $\quad v.\pi \leftarrow u$

BELLMAN-FORD ($G = (V, E)$, w, s)

1. INITIALIZE-SINGLE-SOURCE(G, s)
2. **for** $i \leftarrow 1$ **to** $|G.V| - 1$ **do**
3. $\quad$ **for** each $(u, v) \in G.E$ **do**
4. $\quad\quad$ RELAX(u, v, w)
5. **for** each $(u, v) \in G.E$ **do**
6. $\quad$ **if** $u.d + w(u, v) < v.d$ **then**
7. $\quad\quad$ **return** FALSE
8. **return** TRUE

Let $n = |V|$ and $m = |E|$

Time taken by: Line 1: $\Theta(n)$
Lines $2 - 4$: $\Theta(mn)$
Lines $5 - 7$: $\Theta(m)$

Total time: $\Theta(mn)$

Correctness of the Bellman-Ford Algorithm

LEMMA 24.2 (CLRS): Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w: E \rightarrow \mathbb{R}$, and suppose G contains no negative-weight cycles reachable from s. Then, after the $|V| - 1$ iterations of the for loop of lines 2– 4 of *BELLMAN-FORD*, we have $v.d = \delta(s, v)$ for all vertices v that are reachable from s.

PROOF: The proof is based on the ***path-relaxation property***.

Consider any $v \in G.V$ reachable from s, and let $p = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v.

Because shortest paths are simple, p has at most $|V| - 1$ edges, and so $k \leq |V| - 1$. Each of the $|V| - 1$ iterations of the for loop of lines 2– 4 relaxes all $|E|$ edges. Among the edges relaxed in the i^{th} iteration, for $i = 1,2, \dots, k$, is (v_{i-1}, v_i). By the path-relaxation property, therefore, $v.d = v_k.d = \delta(s, v_k) = \delta(s, v)$.

Correctness of the Bellman-Ford Algorithm

COROLLARY 24.3 (CLRS): Let $G = (V, E)$ be a weighted, directed graph with source s and weight function $w: E \to \mathbb{R}$, and suppose G contains no negative-weight cycles reachable from s. Then, for each $v \in V$, there is a path from s to v if and only if *BELLMAN-FORD* terminates with $v.d < \infty$ when it is run on G.

Correctness of the Bellman-Ford Algorithm

THEOREM 24.4 (CLRS): Let *BELLMAN-FORD* be run on a weighted, directed graph $G = (V, E)$ with source s and weight function $w: E \to \mathbb{R}$. If G contains no negative-weight cycles reachable from s, then the algorithm returns TRUE, we have $v.d = \delta(s, v)$ for all $v \in V$, and the predecessor subgraph G_π is a shortest-paths tree rooted at s. If G does contain a negative-weight cycle reachable from s, then the algorithm returns FALSE.

Correctness of the Bellman-Ford Algorithm

PROOF OF THEOREM 24.4: Two cases:

G **contains no negative-weight cycles reachable from** ***s*****:**

If $v \in G.V$ is reachable from s then according to Lemma 24.2 we have $v.d = \delta(s, v)$ at termination. Otherwise, $v.d = \delta(s, v) = \infty$ follows from the ***no-path property***.

The ***predecessor-subgraph property***, along with $v.d = \delta(s, v)$, implies that G_π is a shortest-paths tree.

Now, since at termination, for all edges $(u, v) \in G.E$, we have, $v.d = \delta(s, v)$ and $u.d = \delta(s, u)$, then by ***triangle inequality***:

$$v.d = \delta(s, v) \leq \delta(s, u) + w(u, v) = u.d + w(u, v).$$

So, none of the tests in line 6 causes *BELLMAN-FORD* to return FALSE. Therefore, it returns TRUE.

Correctness of the Bellman-Ford Algorithm

PROOF OF THEOREM 24.4 (CONTINUED):

G contains a negative-weight cycle reachable from s:

Let $c = \langle v_0, v_1, \dots, v_k \rangle$ be the cycle, where $v_0 = v_k$. Then

$$\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0.$$

Assume for the sake of contradiction that *BELLMAN-FORD* returns TRUE.

Then $v_i.d \leq v_{i-1}.d + w(v_{i-1}, v_i)$ for $i = 1,2, \dots, k$. Thus,

$$\sum_{i=1}^{k} v_i.d \leq \sum_{i=1}^{k} \left(v_{i-1}.d + w(v_{i-1}, v_i)\right) = \sum_{i=1}^{k} v_{i-1}.d + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

But $\sum_{i=1}^{k} v_i.d = \sum_{i=1}^{k} v_{i-1}.d$, and by Corollary 24.3, each $v_i.d$ is finite.

Thus, $\sum_{i=1}^{k} w(v_{i-1}, v_i) \geq 0$, which contradicts our initial assumption that $c = \langle v_0, v_1, \dots, v_k \rangle$ is a negative-weight cycle.

SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

Input: Weighted DAG $G = (V, E)$ with vertex set V and edge set E, a weight function w, and a source vertex $s \in G[V]$. Negative-weight edges are allowed (unlike Dijkstra's SSSP algorithm).

Output: For all $v \in G[V]$, sets $v.d$ to the shortest distance from s to v.

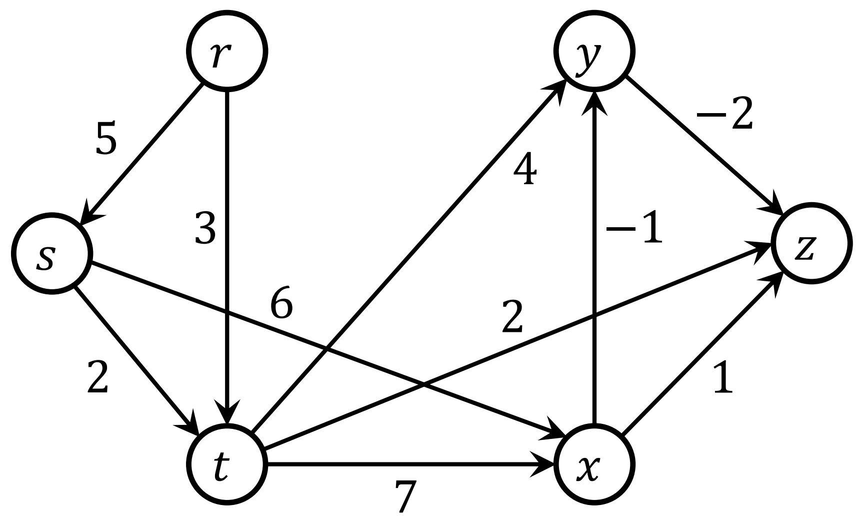
```
INITIALIZE-SINGLE-SOURCE ( G = (V, E),  s )
1.   for each vertex v ∈ G.V do
2.       v.d ← ∞
3.       v.π ← NIL
4.   s.d ← 0
```

```
RELAX ( u,  v,  w )
1.   if u.d + w(u, v) < v.d then
2.       v.d ← u.d + w(u, v)
3.       v.π ← u
```

```
DAG-SHORTEST-PATHS ( G = (V, E),  w,  s )
1.   topologically sort the vertices of G
2.   INITIALIZE-SINGLE-SOURCE( G, s )
3.   for each v ∈ V.G taken in topologically sorted order do
4.       for each (u, v) ∈ G.E do
5.           RELAX( u, v, w )
```

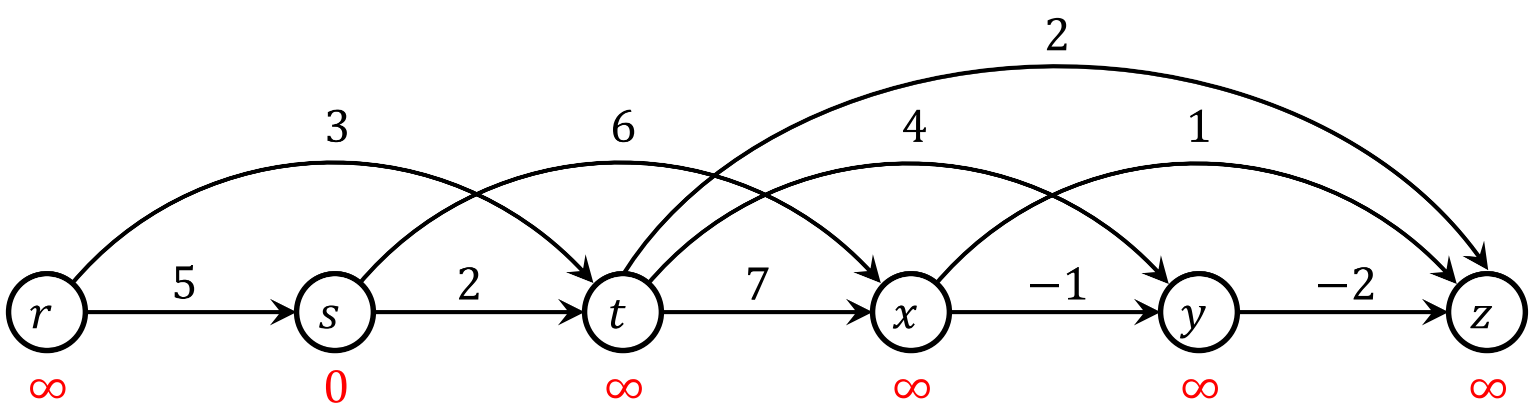
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

Given DAG



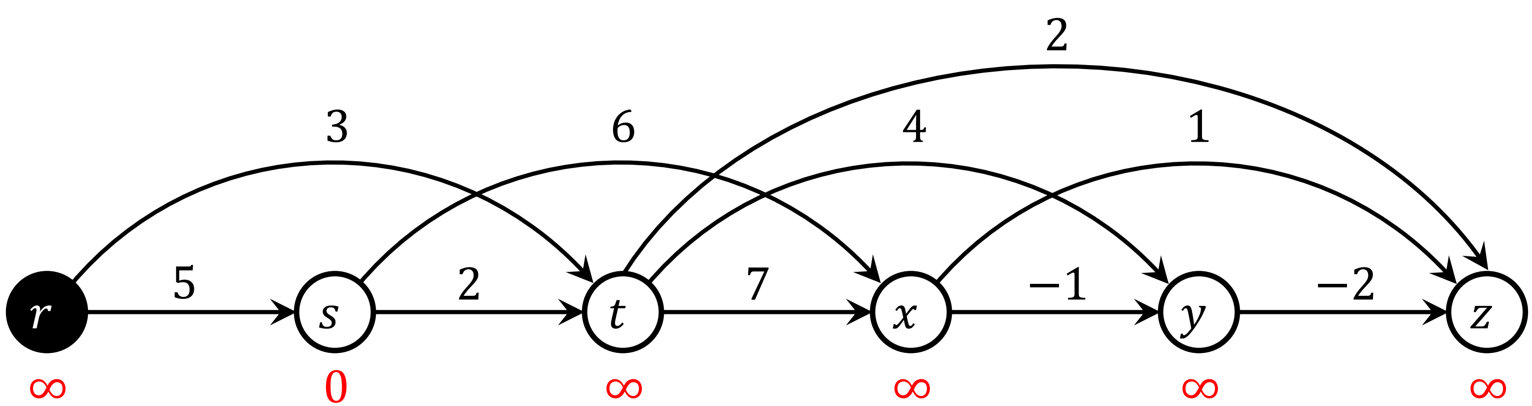
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

After Topological Sorting (with initial tentative distances)



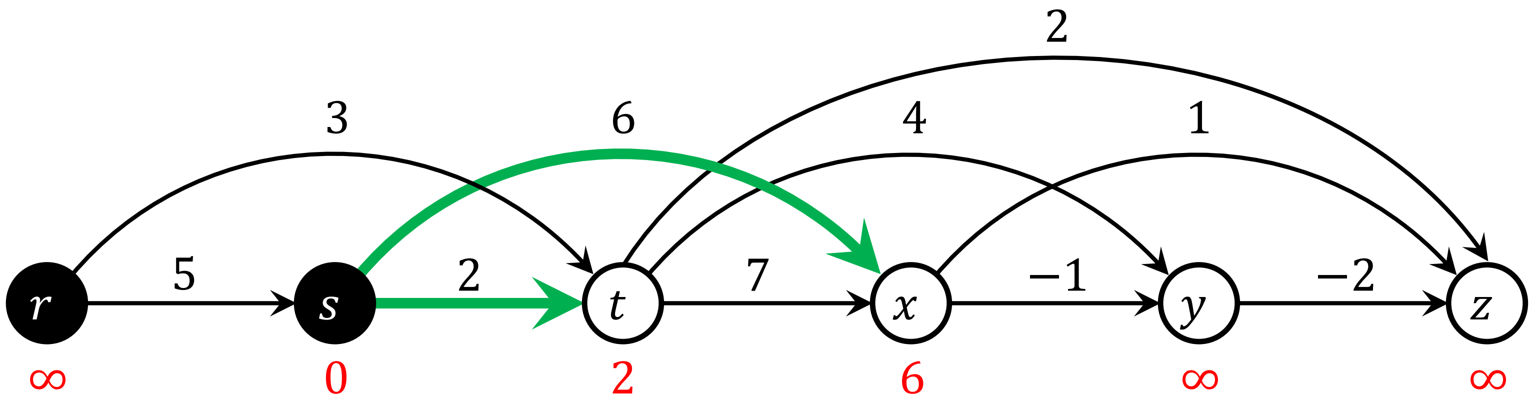
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

After Iteration 1



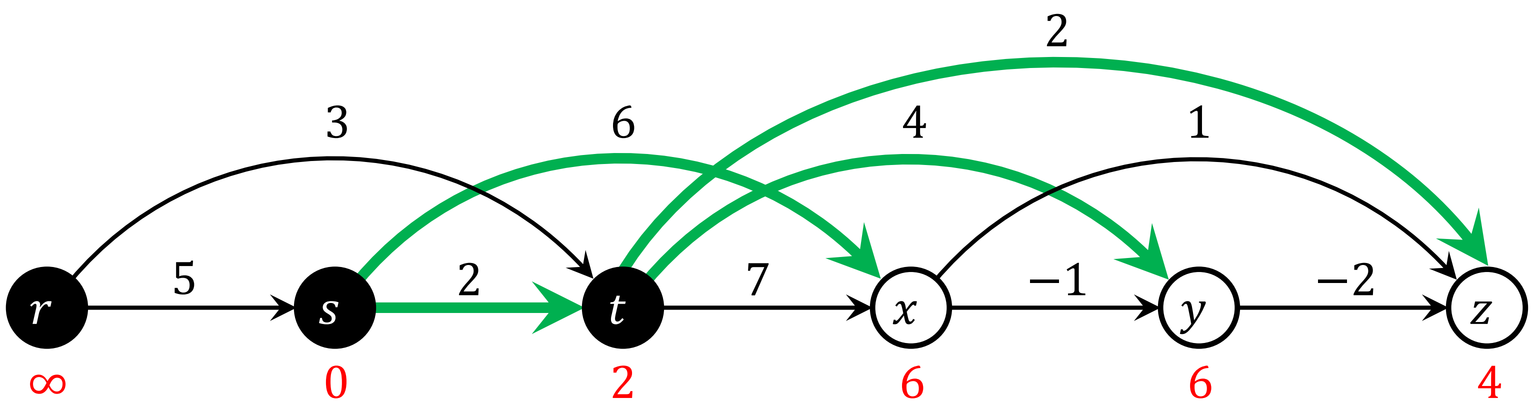
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

After Iteration 2



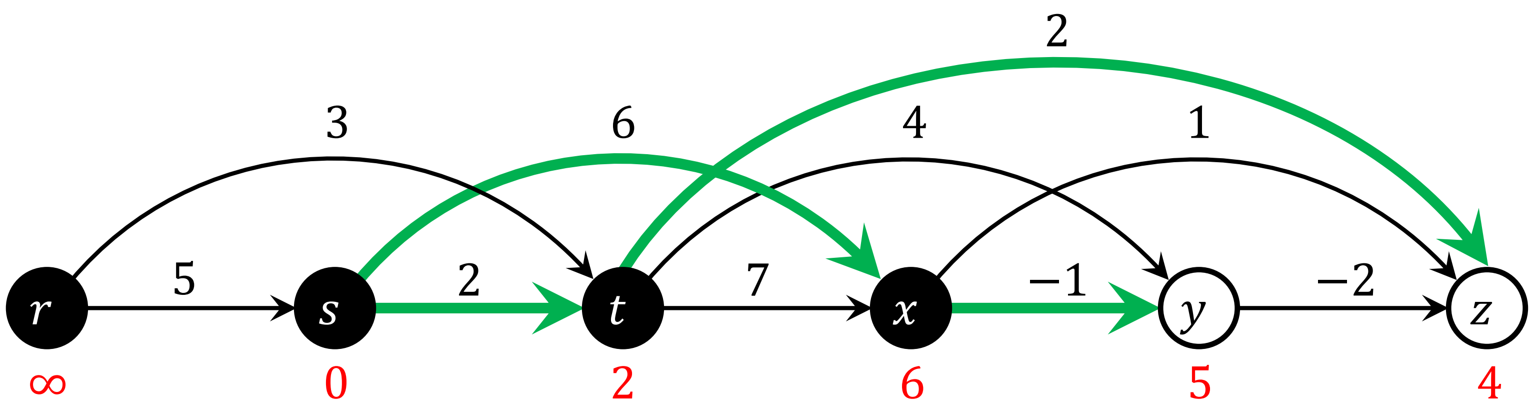
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

After Iteration 3



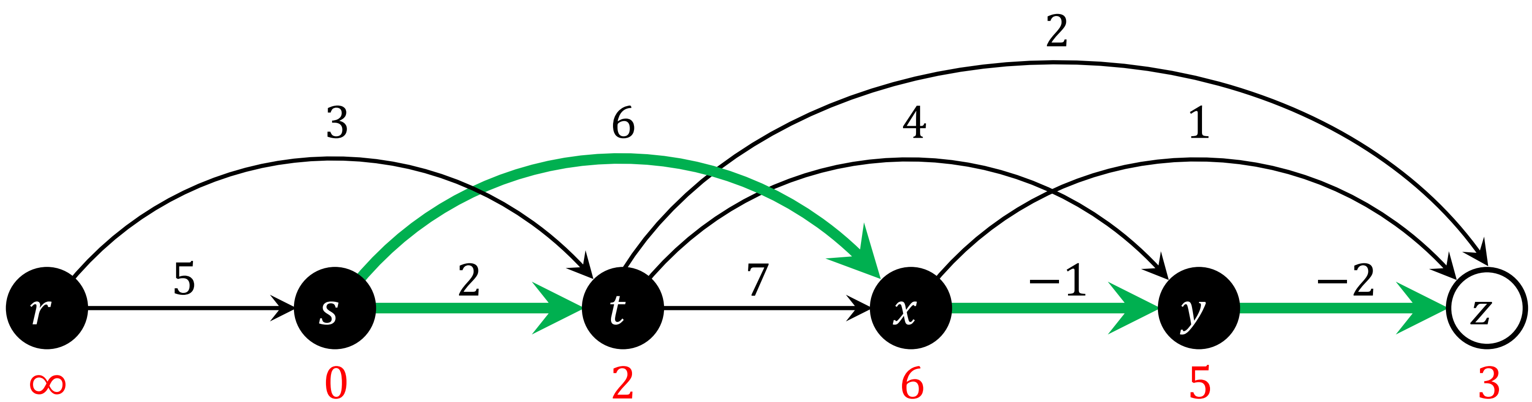
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

After Iteration 4



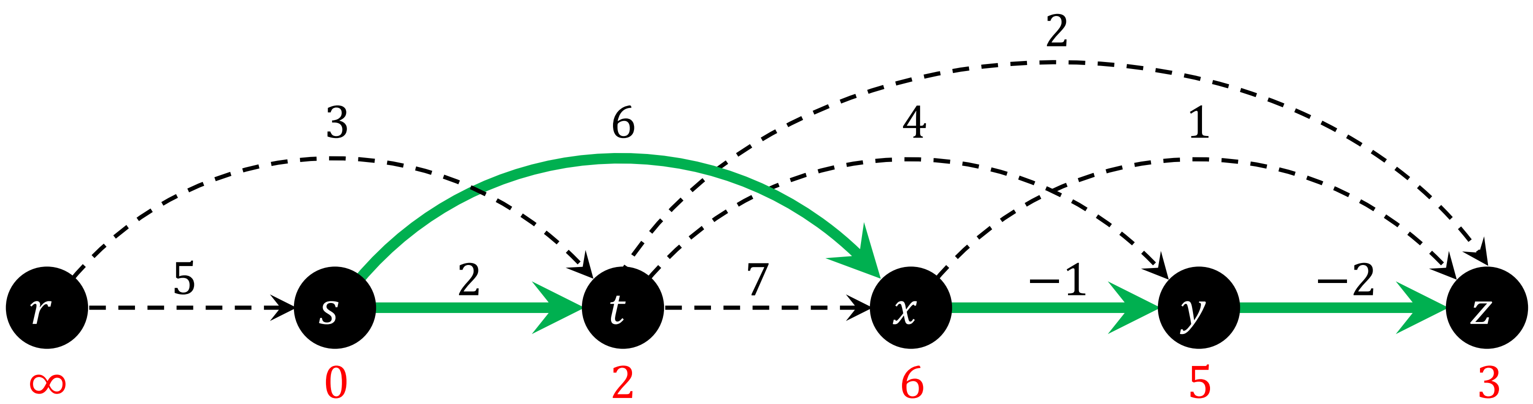
SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

After Iteration 5



SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

Done!



SSSP in Directed Acyclic Graphs (DAGs)
(SSSP: Single-Source Shortest Paths)

```
INITIALIZE-SINGLE-SOURCE ( G = (V, E), s )
    for each vertex v ∈ G.V do
        v.d ← ∞
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    s.d ← 0
```

```
RELAX ( u, v, w )
    if u.d + w(u, v) < v.d then
        v.d ← u.d + w(u, v)
        v.π ← u
```

```
DAG-SHORTEST-PATHS ( G = (V, E), w, s )
1.  topologically sort the vertices of G
2.  INITIALIZE-SINGLE-SOURCE( G, s )
3.  for each v ∈ V.G taken in topologically sorted order do
4.      for each (u, v) ∈ G.E do
5.          RELAX( u, v, w )
```

Let $n = |V|$ and
$m = |E|$

Time taken by: Line 1: $\Theta(n + m)$
Line 2: $\Theta(n)$
Lines $3 - 5$: $\Theta(m)$

Total time: $\Theta(n + m)$

Correctness of DAG-SHORTEST-PATHS

THEOREM 24.5 (CLRS): If a weighted, directed graph $G = (V, E)$ has a source vertex s and no cycles, then at the termination of the *DAG-SHORTEST-PATHS* procedure, $v.d = \delta(s, v)$ for all vertices $v \in G.V$, and the predecessor subgraph G_π is a shortest-paths tree.

PROOF: Consider any $v \in G.V$.

If v is not reachable from s then $v.d = \delta(s, v) = \infty$ follows from the ***no-path property***.

If v is reachable from s, and let $p = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v. Since we process the vertices in topological order, we relax the edges on p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$. The ***path-relaxation property*** implies that $v_i.d = \delta(s, v_i)$ at termination for $i = 1, 2, \dots, k$.

By the ***predecessor-subgraph property***, G_π is a shortest-paths tree.

Correctness of DAG-SHORTEST-PATHS

THEOREM 24.5 (CLRS): If a weighted, directed graph $G = (V, E)$ has a source vertex s and no cycles, then at the termination of the *DAG-SHORTEST-PATHS* procedure, $v.d = \delta(s, v)$ for all vertices $v \in G.V$, and the predecessor subgraph G_π is a shortest-paths tree.

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By the ***predecessor-subgraph property***, G_π is a shortest-paths tree.

Correctness of DAG-SHORTEST-PATHS

THEOREM 24.5 (CLRS): If a weighted, directed graph $G = (V, E)$ has a source vertex s and no cycles, then at the termination of the *DAG-SHORTEST-PATHS* procedure, $v.d = \delta(s, v)$ for all vertices $v \in G.V$, and the predecessor subgraph G_π is a shortest-paths tree.

PROOF: Consider any $v \in G.V$.

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If v is reachable from s, and let $p = \langle v_0, v_1, \dots, v_k \rangle$, where $v_0 = s$ and $v_k = v$, be any shortest path from s to v. Since we process the vertices in topological order, we relax the edges on p in the order $(v_0, v_1), (v_1, v_2), \dots, (v_{k-1}, v_k)$. The ***path-relaxation property*** implies that $v_i.d = \delta(s, v_i)$ at termination for $i = 1, 2, \dots, k$.

By the ***predecessor-subgraph property***, G_π is a shortest-paths tree.

The All-Pairs Shortest Paths (APSP) Problem

We are given a weighted, directed graph $G = (V, E)$ with vertex set V and edge set E, and a weight function w such that for each edge $(u, v) \in E$, $w(u, v)$ represents its weight.

Our goal is to find, for every pair of vertices $u, v \in G.V$, a shortest path (i.e., a path of the smallest total edge weight) from u to v.

The All-Pairs Shortest Paths (APSP) Problem

One can solve the APSP problem by running an SSSP algorithm $n = |G.V|$ times, once for each vertex as the source.

If all edge weights are nonnegative, one can use ***Dijkstra's SSSP algorithm***. Using a binary min-heap as the priority queue, one can solve the problem in $O(n(m+n)\log n)$ time, where $m = |G.E|$. Using a Fibonacci heap as the priority queue yields a running time of $O\left(n^2 \log n + mn\right)$.

If G has negative-weight edges, then one can use the slower ***Bellman-Ford SSSP algorithm*** resulting in a running time of $O\left(mn^2\right)$ which is $O\left(n^4\right)$ for dense graphs.

The All-Pairs Shortest Paths (APSP) Problem

We assume that the edge-weights are given as an $n \times n$ adjacency matrix $W = (w_{ij})$, where

$$w_{ij} = \begin{cases} 0, & if\ i = j, \\ weight\ of\ directed\ edge\ (i,j) & if\ i \neq j\ and\ (i,j) \in E, \\ \infty & if\ i \neq j\ and\ (i,j) \notin E. \end{cases}$$

We allow negative-weight edges, but we assume for the time being that G contains no negative-weight cycles.

APSP: Extending SPs by One Edge at a Time

Let $l_{ij}^{(m)}$ be the minimum weight of any path from vertex i to vertex j that contains at most m edges. Then

$$l_{ij}^{(m)} = \begin{cases} 0, & if\ m = 0\ and\ i = j, \\ \infty & if\ m = 0\ and\ i \neq j, \\ \min\limits_{1 \le k \le n} \left\{ l_{ik}^{(m-1)} + w_{kj} \right\}, & otherwise\ (i.e., m > 0). \end{cases}$$

If G has no negative-weight cycles, then for every pair of vertices i and j for which $\delta(i, j) < \infty$, there is a shortest path from i to j that is simple and thus contains at most $n - 1$ edges. A path from vertex i to vertex j with more than $n - 1$ edges cannot have lower weight than a shortest path from i to j. Hence,

$$\delta(i, j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = \cdots.$$

APSP: Extending SPs by One Edge at a Time

Let $l_{ij}^{(m)}$ be the minimum weight of any path from vertex i to vertex j that contains at most m edges. Then

$$l_{ij}^{(m)} = \begin{cases} 0, & if\ m = 0\ and\ i = j, \\ \infty & if\ m = 0\ and\ i \neq j, \\ \min\limits_{1 \le k \le n} \left\{ l_{ik}^{(m-1)} + w_{kj} \right\}, & otherwise\ (i.e., m > 0). \end{cases}$$

If G has no negative-weight cycles, then for every pair of vertices i and j for which $\delta(i,j) < \infty$, there is a shortest path from i to j that is simple and thus contains at most $n-1$ edges. A path from vertex i to vertex j with more than $n-1$ edges cannot have lower weight than a shortest path from i to j. Hence,

$$\delta(i,j) = l_{ij}^{(n-1)} = l_{ij}^{(n)} = l_{ij}^{(n+1)} = \cdots.$$

APSP: Extending SPs by One Edge at a Time

```
EXTEND-SHORTEST-PATHS ( L, W )
1.    n ← L.rows
2.    let L′ = (l′_ij) be a new n × n matrix
3.    for i ← 1 to n do
4.        for j ← 1 to n do
5.            l′_ij ← ∞
6.            for k ← 1 to n do
7.                l′_ij ← min( l′_ij, l′_ik + w_kj )
8.    return L′
```

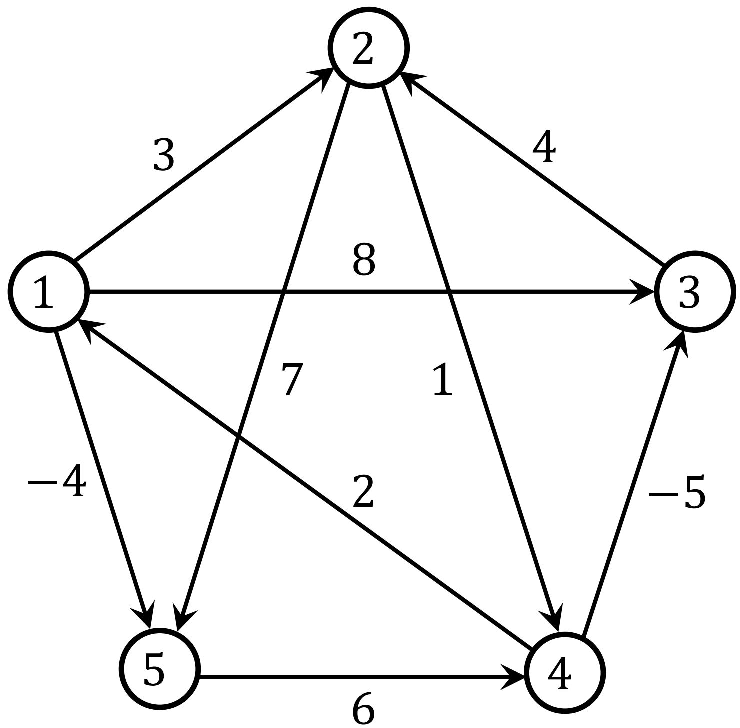
EXTEND-SHORTEST-PATHS (L, W)

1. $n \leftarrow L.rows$
2. let $L' = \left(l'_{ij} \right)$ be a new $n \times n$ matrix
3. *for* $i \leftarrow 1$ *to* n *do*
4.   *for* $j \leftarrow 1$ *to* n *do*
5. $l'_{ij} \leftarrow \infty$
6.   *for* $k \leftarrow 1$ *to* n *do*
7. $l'_{ij} \leftarrow \min\left(l'_{ij},\ l'_{ik} + w_{kj} \right)$
8. *return* L'

SLOW-ALL-PAIRS-SHORTEST-PATHS (W)

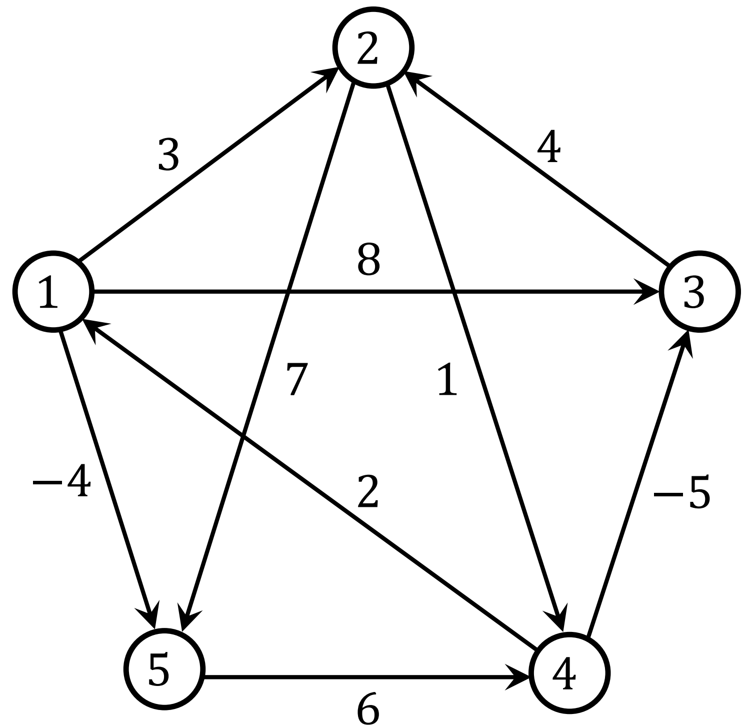
1. $n \leftarrow W.rows$
2. $L^{(1)} \leftarrow W$
3. *for* $m \leftarrow 2$ *to* $n - 1$ *do*
4.   let $L^{(m)}$ be a new $n \times n$ matrix
5. $L^{(m)} \leftarrow$ EXTEND-SHORTEST-PATHS$(L^{(m-1)}, W)$
6. *return* $L^{(n-1)}$

APSP: Extending SPs by One Edge at a Time



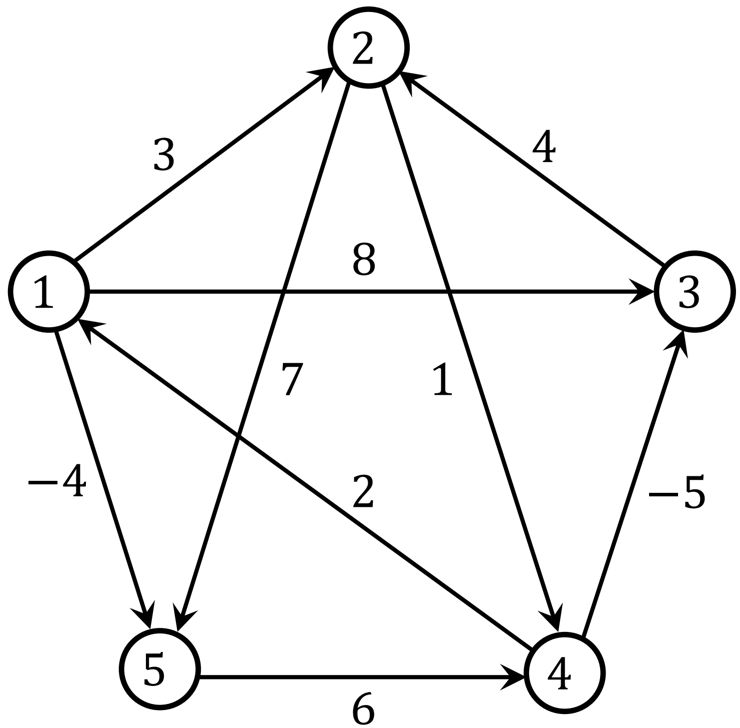
$$W = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

APSP: Extending SPs by One Edge at a Time



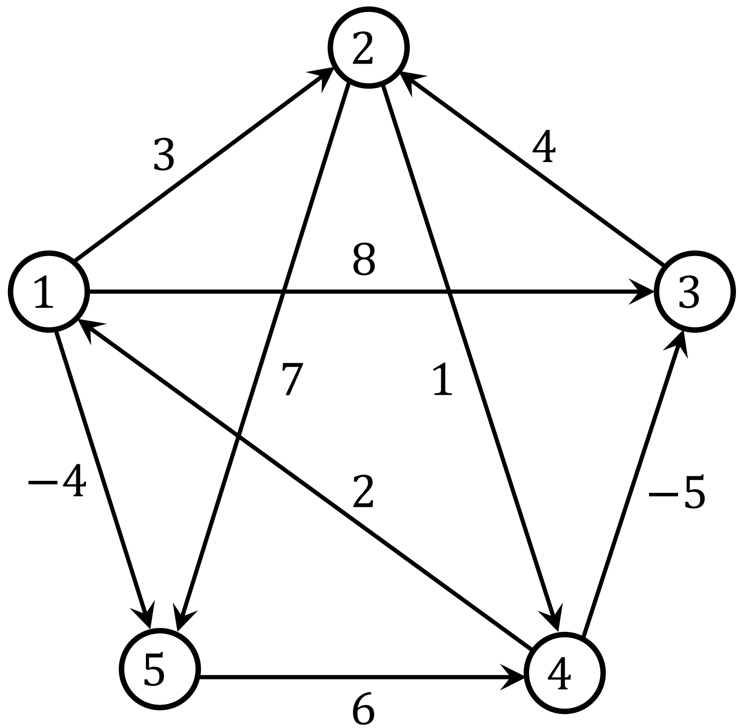
$$L^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix}$$

APSP: Extending SPs by One Edge at a Time



$$L^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 2 & -4 \\ 3 & 0 & -4 & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & \infty & 1 & 6 & 0 \end{pmatrix} \qquad L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

APSP: Extending SPs by One Edge at a Time



$$L^{(3)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad L^{(4)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

APSP: Extending SPs by One Edge at a Time

Note the similarity between *EXTEND-SHORTEST-PATHS* and *SQUARE-MATRIX-MULTIPLY*:

EXTEND-SHORTEST-PATHS ($L,\ W$)

$n \leftarrow L.rows$
let $L' = \left(l'_{ij}\right)$ be a new $n \times n$ matrix
for $i \leftarrow 1$ *to* n *do*
    for $j \leftarrow 1$ *to* n *do*
        $l'_{ij} \leftarrow \infty$
        for $k \leftarrow 1$ *to* n *do*
            $l'_{ij} \leftarrow \min\left(l'_{ij},\ l'_{ik} + w_{kj}\right)$
return L'

SQUARE-MATRIX-MULTIPLY ($A,\ B$)

$n \leftarrow A.rows$
let $C = \left(c_{ij}\right)$ be a new $n \times n$ matrix
for $i \leftarrow 1$ *to* n *do*
    for $j \leftarrow 1$ *to* n *do*
        $c_{ij} \leftarrow 0$
        for $k \leftarrow 1$ *to* n *do*
            $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$
return C

Both have the same $\Theta\left(n^3\right)$ running time.

APSP: Extending SPs by One Edge at a Time

EXTEND-SHORTEST-PATHS (L, W)

1. $n \leftarrow L.rows$
2. let $L' = \left(l'_{ij}\right)$ be a new $n \times n$ matrix
3. *for* $i \leftarrow 1$ *to* n *do*
4.   *for* $j \leftarrow 1$ *to* n *do*
5. $l'_{ij} \leftarrow \infty$
6.   *for* $k \leftarrow 1$ *to* n *do*
7. $l'_{ij} \leftarrow \min\left(l'_{ij},\ l'_{ik} + w_{kj}\right)$
8. *return* L'

Running time

$$= \Theta\left(n^3\right)$$

SLOW-ALL-PAIRS-SHORTEST-PATHS (W)

1. $n \leftarrow W.rows$
2. $L^{(1)} \leftarrow W$
3. *for* $m \leftarrow 2$ *to* $n - 1$ *do*
4.   let $L^{(m)}$ be a new $n \times n$ matrix
5. $L^{(m)} \leftarrow$ EXTEND-SHORTEST-PATHS($L^{(m-1)}, W$)
6. *return* $L^{(n-1)}$

Running time

$$= n \times \Theta(n^3)$$
$$= \Theta(n^4)$$

APSP: Extending SPs by Repeated Squaring

EXTEND-SHORTEST-PATHS (L, W)

$n \leftarrow L.rows$
let $L' = \left(l'_{ij}\right)$ be a new $n \times n$ matrix
for $i \leftarrow 1$ *to* n *do*
    for $j \leftarrow 1$ *to* n *do*
        $l'_{ij} \leftarrow \infty$
        for $k \leftarrow 1$ *to* n *do*
            $l'_{ij} \leftarrow \min\left(l'_{ij},\ l'_{ik} + w_{kj}\right)$
return L'

FASTER-ALL-PAIRS-SHORTEST-PATHS (W)

$n \leftarrow W.rows$
$L^{(1)} \leftarrow W$
$m \leftarrow 1$
while $m < n - 1$ *do*
    let $L^{(2m)}$ be a new $n \times n$ matrix
    $L^{(2m)} \leftarrow$ EXTEND-SHORTEST-PATHS($L^{(m)}, L^{(m)}$)
    $m \leftarrow 2m$
return $L^{(m)}$

APSP: Extending SPs by Repeated Squaring

```
EXTEND-SHORTEST-PATHS ( L, W )

1.  n ← L.rows
2.  let L' = (l'_ij) be a new n × n matrix
3.  for i ← 1 to n do
4.      for j ← 1 to n do
5.          l'_ij ← ∞
6.          for k ← 1 to n do
7.              l'_ij ← min( l'_ij, l'_ik + w_kj )
8.  return L'
```

Running time

$= \Theta(n^3)$

```
FASTER-ALL-PAIRS-SHORTEST-PATHS ( W )

1.  n ← W.rows
2.  L^(1) ← W
3.  m ← 1
4.  while m < n − 1 do
5.      let L^(2m) be a new n × n matrix
6.      L^(2m) ← EXTEND-SHORTEST-PATHS( L^(m), L^(m) )
7.      m ← 2m
8.  return L^(m)
```

Running time

$= \lceil \log_2(n-1) \rceil$

$\times \Theta(n^3)$

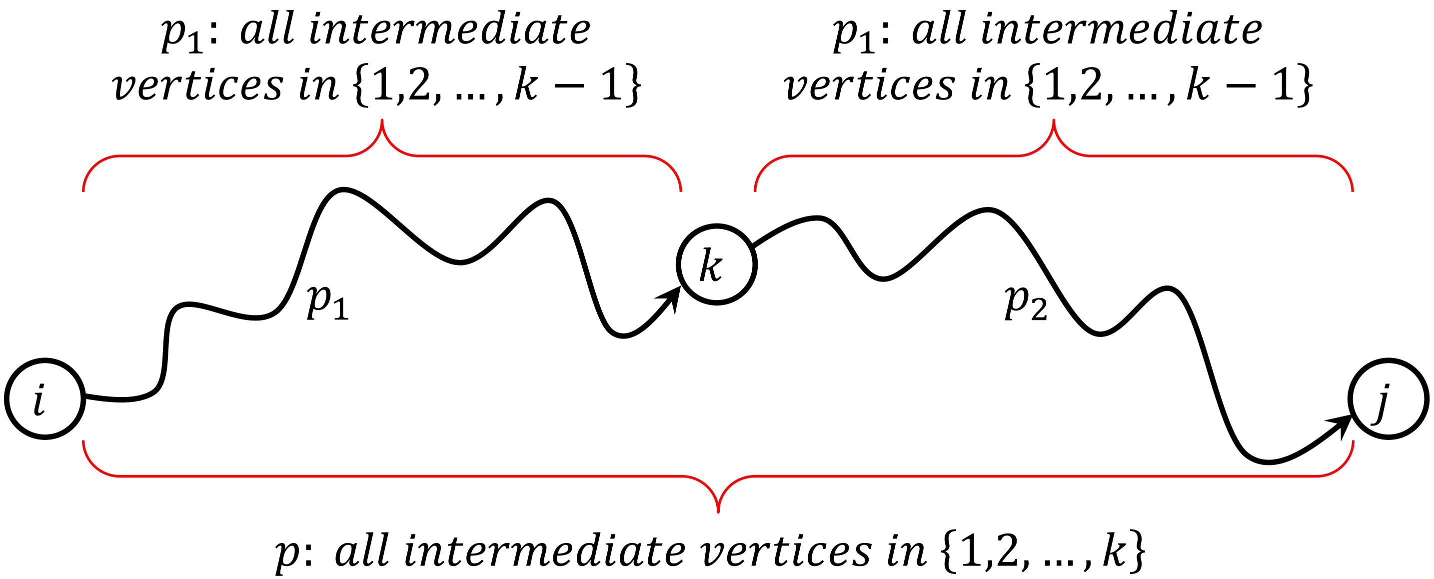
$= \Theta(n^3 \log n)$

APSP: Floyd-Warshall's Algorithm

Let $d_{ij}^{(k)}$ be the minimum weight of any path from vertex i to vertex j for which all intermediate vertices are in $\{1, 2, \ldots, k\}$. Then

$$d_{ij}^{(k)} = \begin{cases} w_{ij}, & if\ k = 0, \\ \min\{d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} & if\ k \geq 1. \end{cases}$$

Then $D^{(n)} = \left(d_{ij}^{(n)}\right)$ gives: $d_{ij}^{(n)} = \delta(i, j)$ for all $i, j \in G.V$.



APSP: Floyd-Warshall's Algorithm

FLOYD-WARSHALL (W)

$n \leftarrow W.rows$

$D^{(0)} \leftarrow W$

for $k \leftarrow 1$ *to* n *do*

  let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix

  for $i \leftarrow 1$ *to* n *do*

    for $j \leftarrow 1$ *to* n *do*

      $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$

return $D^{(n)}$

APSP: Floyd-Warshall with Predecessor Matrix

FLOYD-WARSHALL (W)

$n \leftarrow W.rows$

$D^{(0)} \leftarrow W$

let $\Pi^{(0)} = \left(\pi_{ij}^{(0)}\right)$ be a new $n \times n$ matrix

for $i \leftarrow 1$ *to* n *do*

 for $j \leftarrow 1$ *to* n *do*

  if $i = j$ *or* $w_{ij} = \infty$ *then* $\pi_{ij}^{(0)} \leftarrow NIL$

  else $\pi_{ij}^{(0)} \leftarrow i$

for $k \leftarrow 1$ *to* n *do*

 let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ and $\Pi^{(k)} = \left(\pi_{ij}^{(k)}\right)$ be new $n \times n$ matrices

 for $i \leftarrow 1$ *to* n *do*

  for $j \leftarrow 1$ *to* n *do*

   if $d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}$ *then* $\pi_{ij}^{(k)} \leftarrow \pi_{ij}^{(k-1)}$

   else $\pi_{ij}^{(k)} \leftarrow \pi_{kj}^{(k-1)}$

   $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$

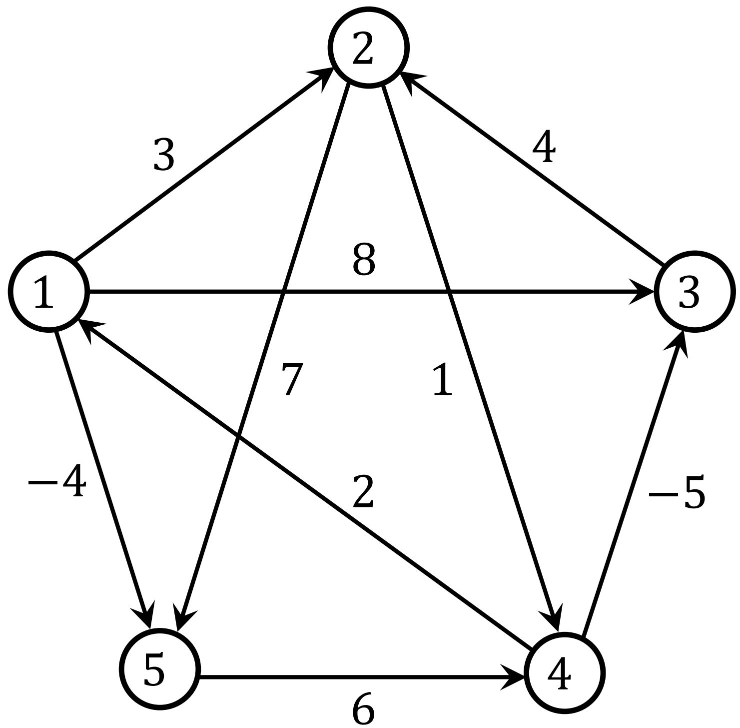
return $D^{(n)}$ and $\Pi^{(n)}$

APSP: Floyd-Warshall with Predecessor Matrix

PRINT-ALL-PAIRS-SHORTEST-PATH (Π, i, j)

if $i = j$ *then*
    print i
elseif $\pi_{ij} = NIL$ *then*
    print "no path from" i "to" j "exists"
else PRINT-ALL-PAIRS-SHORTEST-PATH (Π, i, π_{ij})
    print j

APSP: Floyd-Warshall with Predecessor Matrix



$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(0)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall with Predecessor Matrix

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(0)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall with Predecessor Matrix

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(1)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall with Predecessor Matrix

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall with Predecessor Matrix

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix} \qquad \Pi^{(3)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall with Predecessor Matrix

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(4)} = \begin{pmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix} \qquad \Pi^{(5)} = \begin{pmatrix} NIL & 3 & 4 & 5 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

APSP: Floyd-Warshall's Algorithm

FLOYD-WARSHALL (W)

1. $n \leftarrow W.rows$
2. $D^{(0)} \leftarrow W$
3. *for* $k \leftarrow 1$ *to* n *do*
4. let $D^{(k)} = \left(d_{ij}^{(k)}\right)$ be a new $n \times n$ matrix
5. *for* $i \leftarrow 1$ *to* n *do*
6. *for* $j \leftarrow 1$ *to* n *do*
7. $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$
8. *return* $D^{(n)}$

Running Time $= \Theta\left(n^3\right)$

Space Complexity $= \Theta\left(n^3\right)$

APSP: Floyd-Warshall's Algorithm

But $D^{(k)}$ depends only on $D^{(k-1)}$.

FLOYD-WARSHALL-QUADRATIC-SPACE (W)

1. $n \leftarrow W.rows$
2. let $D^{(0)} = \left(d_{ij}^{(0)}\right)$ and $D^{(1)} = \left(d_{ij}^{(1)}\right)$ be new $n \times n$ matrices
3. $D^{(0)} \leftarrow W$
4. *for* $k \leftarrow 1$ *to* n *do*
5.   *for* $i \leftarrow 1$ *to* n *do*
6.   *for* $j \leftarrow 1$ *to* n *do*
7. $d_{ij}^{(1)} \leftarrow \min\left(d_{ij}^{(0)},\ d_{ik}^{(0)} + d_{kj}^{(0)}\right)$
8. $D^{(0)} \leftarrow D^{(1)}$
9. *return* $D^{(0)}$

Running Time $= \Theta(n^3)$

Space Complexity $= \Theta(n^2)$

APSP: Floyd-Warshall's Algorithm

Can be solved in-place!

```
FLOYD-WARSHALL-IN-PLACE ( W )

1.   n ← W.rows
2.   for k ← 1 to n do
3.       for i ← 1 to n do
4.           for j ← 1 to n do
5.               w_ij ← min(w_ij, w_ik + w_kj)
6.   return W
```

Running Time $= \Theta\left(n^3\right)$

Space Complexity $= \Theta\left(n^2\right)$