

CSE 548: Analysis of Algorithms

Lecture 6

(Divide-and-Conquer Algorithms: Akra-Bazzi Recurrences)

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Deterministic Select

Input: An array $A[q : r]$ of distinct elements, and integer $k \in [1, r - q + 1]$.

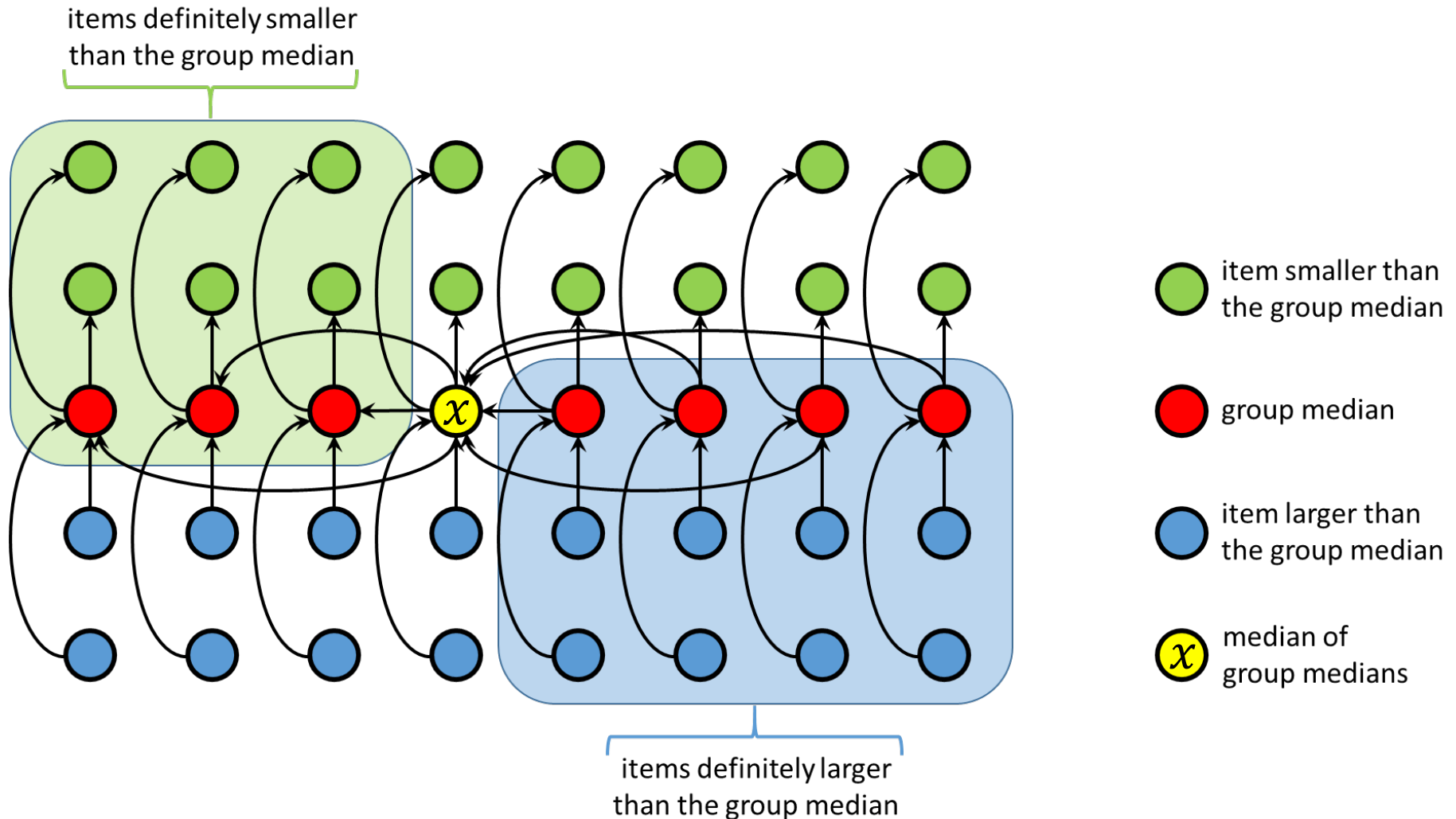
Output: An element x of $A[q : r]$ such that $rank(x, A[q : r]) = k$.

Select ($A[q : r]$, k)

1. $n \leftarrow r - q + 1$
2. *if* $n \leq 140$ *then*
3. sort $A[q : r]$ and *return* $A[q + k - 1]$
4. *else*
5. divide $A[q : r]$ into blocks B_i 's each containing 5 consecutive elements
 (last block may contain fewer than 5 elements)
6. *for* $i \leftarrow 1$ *to* $\lceil n / 5 \rceil$ *do*
7. $M[i] \leftarrow$ median of B_i using sorting
8. $x \leftarrow$ *Select* ($M[1 : \lceil n / 5 \rceil]$, $\lfloor (\lceil n / 5 \rceil + 1) / 2 \rfloor$) { median of medians }
9. $t \leftarrow$ *Partition* ($A[q : r]$, x) { partition around x which ends up at $A[t]$ }
10. *if* $k = t - q + 1$ *then return* $A[t]$
11. *else if* $k < t - q + 1$ *then return* *Select* ($A[q : t - 1]$, k)
12. *else return* *Select* ($A[t + 1 : r]$, $k - t + q - 1$)

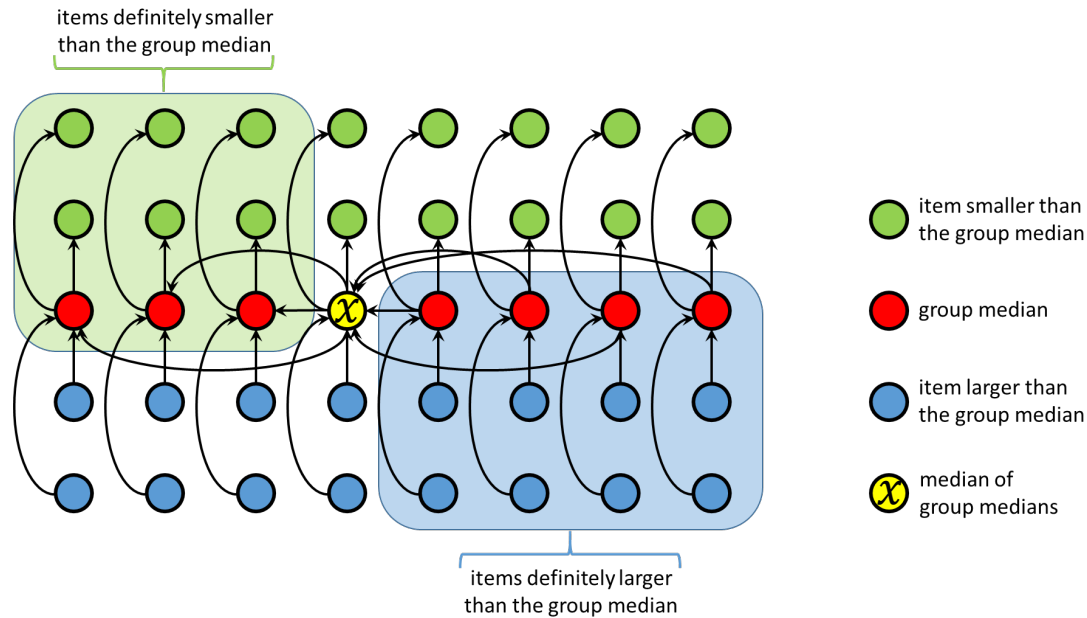
Deterministic Select

$\text{SELECT}(A, k)$: Given an unsorted set A of n ($= |A|$) items, find the k^{th} smallest item in the set



Deterministic Select

SELECT(A, k): Given an unsorted set A of n ($= |A|$) items,
find the k^{th} smallest item in the set



$$\# \text{items definitely smaller than } x \text{ is} \geq 3 \left(\left\lfloor \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rfloor - 1 \right) \geq \frac{3n}{10} - 6$$

$$\# \text{items definitely larger than } x \text{ is} \geq 3 \left(\left\lfloor \frac{1}{2} \left\lceil \frac{n}{5} \right\rceil \right\rfloor - 1 \right) \geq \frac{3n}{10} - 6$$

$$\# \text{items in any recursive call (lines 11/12)} \leq n - \left(\frac{3n}{10} - 6 \right) = \frac{7n}{10} + 6$$

Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm (given in Section 9.3 of CLRS):

$$T(n) \leq \begin{cases} \Theta(1), & \text{if } n < 140, \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

Dropping the ceiling for simplicity, and observing that $\frac{7n}{10} + 6 \leq \frac{8n}{10}$ when $n \geq 60$, we obtain the following upper bound on $T(n)$.

$$T'(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

How do you solve for $T'(n)$?

Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm (given in Section 9.3 of CLRS):

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Dropping the ceiling for simplicity, and observing that $\frac{7n}{10} + 6 \leq \frac{7.5n}{10}$ when $n \geq 120$, we obtain the following upper bound on $T(n)$.

$$T''(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

How do you solve for $T''(n)$?

Akra-Bazzi Recurrences

Consider the following recurrence:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum_{i=1}^k a_i T(b_i x) + g(x), & \text{if } x > x_0; \end{cases}$$

where,

1. $k \geq 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 \leq i \leq k$
3. $b_i \in (0,1)$ is a constant for $1 \leq i \leq k$
4. $x \geq 1$ is a real number
5. $x_0 \geq \max \left\{ \frac{1}{b_i}, \frac{1}{1-b_i} \right\}$ is a constant for $1 \leq i \leq k$
6. $g(x)$ is a nonnegative function that satisfies a *polynomial-growth condition* (to be specified soon)

Polynomial-Growth Condition

We say that $g(x)$ satisfies the *polynomial-growth condition* if there exist positive constants c_1 and c_2 such that for all $x \geq 1$, for all $1 \leq i \leq k$, and for all $u \in [b_i x, x]$,

$$c_1 g(x) \leq g(u) \leq c_2 g(x),$$

where x , k , b_i and $g(x)$ are as defined in the previous slide.

The Akra-Bazzi Solution

Consider the recurrence given in the previous two slides under the conditions specified there:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum_{i=1}^k a_i T(b_i x) + g(x), & \text{if } x > x_0. \end{cases}$$

Let p be the unique real number for which $\sum_{i=1}^k a_i b_i^p = 1$. Then

$$T(x) = \Theta \left(x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right) \right)$$

Deterministic Select

$$T'(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

From $\left(\frac{1}{5}\right)^p + \left(\frac{4}{5}\right)^p = 1$ we get $p = 1$.

$$\begin{aligned} \text{Hence, } T'(n) &= \Theta\left(n^p \left(1 + \int_1^n \frac{u}{u^{p+1}} du\right)\right) \\ &= \Theta\left(n \left(1 + \int_1^n \frac{du}{u}\right)\right) \\ &= \Theta(n \ln n) \end{aligned}$$

Deterministic Select

$$T''(n) = \begin{cases} \Theta(1), & \text{if } n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & \text{if } n \geq 140. \end{cases}$$

From $\left(\frac{1}{5}\right)^p + \left(\frac{3}{4}\right)^p = 1$ we get $p < 1$.

$$\begin{aligned} \text{Hence, } T''(n) &= \Theta\left(n^p \left(1 + \int_1^n \frac{u}{u^{p+1}} du\right)\right) \\ &= \Theta\left(n^p \left(1 + \int_1^n \frac{du}{u^p}\right)\right) \\ &= \Theta\left(\left(\frac{1}{1-p}\right)n - \left(\frac{p}{1-p}\right)n^p\right) \\ &= \Theta(n) \end{aligned}$$

Examples of Akra-Bazzi Recurrences

Example 1: $T(x) = 2T\left(\frac{x}{4}\right) + 3T\left(\frac{x}{6}\right) + \Theta(x \log x)$

Then $p = 1$ and $T(x) = \Theta\left(x \left(1 + \int_1^x \frac{u \log u}{u^2} du\right)\right) = \Theta(x \log^2 x)$

Example 2: $T(x) = 2T\left(\frac{x}{2}\right) + \frac{8}{9}T\left(\frac{3x}{4}\right) + \Theta\left(\frac{x^2}{\log(x+1)}\right)$

Then $p = 2$ and $T(x) = \Theta\left(x^2 \left(1 + \int_1^x \frac{u^2 \log(u+1)}{u^3} du\right)\right) = \Theta(x^2 \log \log(x + 1))$

Example 3: $T(x) = T\left(\frac{x}{2}\right) + \Theta(\log x)$

Then $p = 0$ and $T(x) = \Theta\left(1 + \int_1^x \frac{\log u}{u} du\right) = \Theta(\log^2 x)$

Example 4: $T(x) = \frac{1}{2}T\left(\frac{x}{2}\right) + \Theta\left(\frac{1}{x}\right)$

Then $p = -1$ and $T(x) = \Theta\left(\frac{1}{x} \left(1 + \int_1^x \frac{1}{u} du\right)\right) = \Theta\left(\frac{\log x}{x}\right)$

A Helping Lemma

Lemma: If $g(x)$ is a nonnegative function that satisfies the polynomial-growth condition, then there exist positive constants c_3 and c_4 such that for $1 \leq i \leq k$ and all $x \geq 1$,

$$c_3 g(x) \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq c_4 g(x).$$

Proof:

$$b_i x \leq u \leq x$$

$$\Rightarrow \frac{1}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \leq \frac{1}{u^{p+1}} \leq \frac{1}{\min\{(b_i x)^{p+1}, x^{p+1}\}}$$

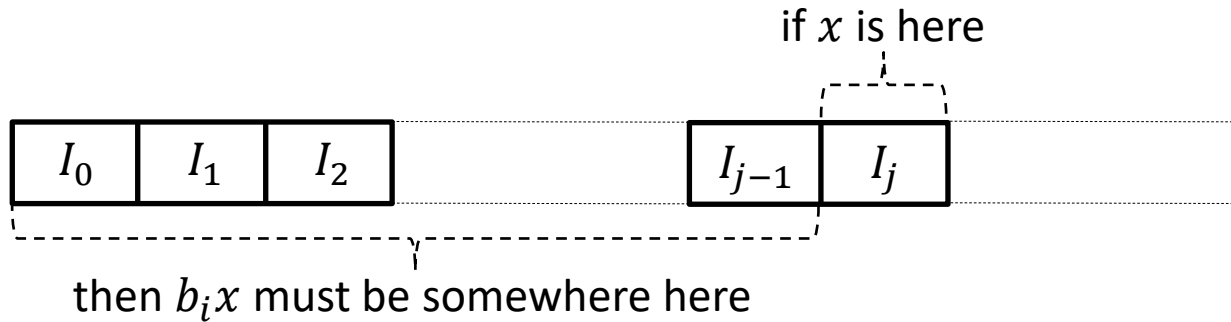
$$\Rightarrow \frac{x^p c_1 g(x)}{\max\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^x du \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq \frac{x^p c_2 g(x)}{\min\{(b_i x)^{p+1}, x^{p+1}\}} \int_{b_i x}^x du$$

$$\Rightarrow \frac{(1 - b_i) c_1}{\max\{1, b_i^{p+1}\}} g(x) \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq \frac{(1 - b_i) c_2}{\min\{1, b_i^{p+1}\}} g(x)$$

$$\Rightarrow c_3 g(x) \leq x^p \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \leq c_4 g(x)$$

Partitioning the Domain of x

Let $I_0 = [1, x_0]$ and $I_j = (x_0 + j - 1, x_0 + j]$ for $j \geq 1$.

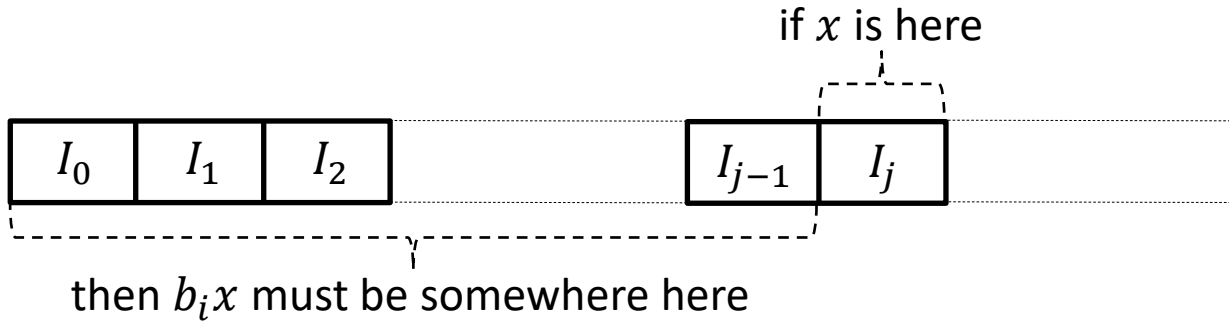


That allows us to use induction in the proof of:

$$T(x) = \begin{cases} \Theta(1), & \text{if } 1 \leq x \leq x_0, \\ \sum_{i=1}^k a_i T(b_i x) + g(x), & \text{if } x > x_0. \end{cases}$$

Partitioning the Domain of x

Let $I_0 = [1, x_0]$ and $I_j = (x_0 + j - 1, x_0 + j]$ for $j \geq 1$.



Proof:

$$x_0 + j - 1 < x \leq x_0 + j$$

$$\Rightarrow b_i(x_0 + j - 1) < b_i x \leq b_i(x_0 + j)$$

$$\Rightarrow b_i x_0 < b_i x \leq b_i x_0 + j$$

$$\Rightarrow 1 < b_i x \leq x_0 + j - (1 - b_i)x_0$$

$$\Rightarrow 1 < b_i x \leq x_0 + j - 1$$

Derivation of the Akra-Bazzi Solution

Lower Bound: There exists a constant $c_5 > 0$ such that for all $x > x_0$,

$$T(x) \geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right).$$

Proof: By induction on the interval I_j containing x .

Base case ($j = 0$) follows since $T(x) = \Theta(1)$ when $x \in I_0 = [1, x_0]$.

Induction:
$$T(x) = \sum_{i=1}^k a_i T(b_i x) + g(x) \geq \sum_{i=1}^k a_i c_5 (b_i x)^p \left(1 + \int_1^{b_i x} \frac{g(u)}{u^{p+1}} du \right) + g(x)$$

$$= c_5 x^p \sum_{i=1}^k a_i b_i^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \int_{b_i x}^x \frac{g(u)}{u^{p+1}} du \right) + g(x)$$

$$\geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du - \frac{c_4}{x^p} g(x) \right) \sum_{i=1}^k a_i b_i^p + g(x)$$

$$= c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right) + (1 - c_4 c_5) g(x) \geq c_5 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right)$$

(assuming $c_4 c_5 \leq 1$)

Derivation of the Akra-Bazzi Solution

Upper Bound: There exists a constant $c_6 > 0$ such that for all $x > x_0$,

$$T(x) \leq c_6 x^p \left(1 + \int_1^x \frac{g(u)}{u^{p+1}} du \right).$$

Proof: Similar to the lower bound proof.