

CSE 548 / AMS 542: Analysis of Algorithms

Prerequisites Review 4 (Linear Recurrences with Constant Coefficients)

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Linear Homogeneous Recurrence

A *linear homogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

where c_1, c_2, \dots, c_k are real constants, and $c_k \neq 0$.

For constant r , $a_n = r^n$ is a solution of the recurrence relation iff:

$$\begin{aligned} r^n &= c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} \\ \Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k &= 0 \end{aligned}$$

The equation above is called the *characteristic equation* of the recurrence, and its roots are called *characteristic roots*.

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$,

Characteristic Equation: $r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0$

If the characteristic equation has k distinct roots r_1, r_2, \dots, r_k , then a sequence $\{a_n\}$ is a solution of the recurrence relation iff

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n \text{ for integers } n \geq 0,$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Characteristic Equation: $r^2 - c_1 r - c_2 = 0$

$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n \Rightarrow \{a_n\}$ is a solution to the recurrence:

$$r_1^2 = c_1 r_1 + c_2 \text{ and } r_2^2 = c_1 r_2 + c_2$$

$$\begin{aligned} c_1 a_{n-1} + c_2 a_{n-2} &= c_1 (\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2 (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}) \\ &= \alpha_1 r_1^{n-2} (c_1 r_1 + c_2) + \alpha_2 r_2^{n-2} (c_1 r_2 + c_2) \\ &= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\ &= \alpha_1 r_1^n + \alpha_2 r_2^n \\ &= a_n \end{aligned}$$

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

Characteristic Equation: $r^2 - c_1 r - c_2 = 0$

$\{a_n\}$ is a solution to the recurrence $\Rightarrow a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$:

Assume initial conditions: $a_0 = C_0$ and $a_1 = C_1$

$$a_0 = C_0 = \alpha_1 + \alpha_2$$

$$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$$

Solving: $\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$ and $\alpha_2 = \frac{C_0 r_1 - C_1}{r_1 - r_2}$

Since the initial conditions uniquely determine the sequence, it follows that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$.

Linear Homogeneous Recurrence

Recurrence for *Fibonacci numbers*:

$$f_n = \begin{cases} 0 & \text{if } n = 0, \\ 1 & \text{if } n = 1, \\ f_{n-1} + f_{n-2} & \text{otherwise.} \end{cases}$$

Characteristic equation: $r^2 - r - 1 = 0$

Characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$

Then for constants α_1 and α_2 : $f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$

Initial conditions: $f_0 = \alpha_1 + \alpha_2 = 0$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$$

Constants: $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$

Solution: $f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$

Linear Homogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$,

Characteristic Equation: $r^k - c_1 r^{k-1} - \cdots - c_{k-1} r - c_k = 0$

If the characteristic equation has t distinct roots r_1, r_2, \dots, r_t with multiplicities m_1, m_2, \dots, m_t , respectively, so that all m_i 's are positive and $\sum_{1 \leq i \leq t} m_i = k$, then a sequence $\{a_n\}$ is a solution of the recurrence relation iff

$$\begin{aligned} a_n = & (\alpha_{1,0} + \alpha_{1,1}n + \cdots + \alpha_{1,m_1-1}n^{m_1-1})r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1}n + \cdots + \alpha_{2,m_2-1}n^{m_2-1})r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1}n + \cdots + \alpha_{t,m_t-1}n^{m_t-1})r_t^n \text{ for integers } n \geq 0, \end{aligned}$$

where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Linear Homogeneous Recurrence

$$a_n = \begin{cases} 1 & \text{if } n = 0, \\ 6 & \text{if } n = 1, \\ 6a_{n-1} - 9a_{n-2} & \text{otherwise.} \end{cases}$$

Characteristic equation: $r^2 - 6r + 9 = 0$

Characteristic root: $r = 3$

Then for constants α_1 and α_2 : $a_n = \alpha_1 3^n + \alpha_2 n 3^n$

Initial conditions: $a_0 = \alpha_1 = 1$

$$a_1 = 3\alpha_1 + 3\alpha_2 = 6$$

Constants: $\alpha_1 = 1$ and $\alpha_2 = 1$

Solution: $a_n = 3^n(n + 1)$

Linear Homogeneous Recurrence

$$a_n = \begin{cases} 2 & \text{if } n = 0, \\ 7 & \text{if } n = 1, \\ a_{n-1} + 2a_{n-2} & \text{otherwise.} \end{cases}$$

$$= 3 \cdot 2^n - (-1)^n$$

$$a_n = \begin{cases} 2 & \text{if } n = 0, \\ 5 & \text{if } n = 1, \\ 15 & \text{if } n = 2, \\ 6a_{n-1} - 11a_{n-2} + 6a_{n-3} & \text{otherwise.} \end{cases}$$

$$= 1 - 2^n + 2 \cdot 3^n$$

$$a_n = \begin{cases} 1 & \text{if } n = 0, \\ -2 & \text{if } n = 1, \\ -1 & \text{if } n = 2, \\ -3a_{n-1} - 3a_{n-2} - a_{n-3} & \text{otherwise.} \end{cases}$$

$$= (1 + 3n - 2n^2)(-1)^n$$

Linear Nonhomogeneous Recurrence

A *linear nonhomogeneous recurrence relation of degree k with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

where c_1, c_2, \dots, c_k are real constants, $c_k \neq 0$, and $F(n)$ is a function not identically zero depending only on n .

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the *associated homogeneous recurrence relation*.

Linear Nonhomogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$,

Suppose $\{a_n^{(p)}\}$ is a particular solution of the recurrence above,
and $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence.

Then every solution of the given nonhomogeneous recurrence is of
the form $\{a_n^{(p)} + a_n^{(h)}\}$.

Linear Nonhomogeneous Recurrence

Recurrence: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$,

Suppose $F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$,

where b_0, b_1, \dots, b_t and s are real numbers.

If s is not a solution of the characteristic equation of the associated homogeneous recurrence, then there is an $a_n^{(p)}$ of the form:

$$(p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

If s is a solution of the characteristic equation and its multiplicity is m , then there is an $a_n^{(p)}$ of the form:

$$n^m (p_t n^t + p_{t-1} n^{t-1} + \cdots + p_1 n + p_0) s^n.$$

Linear Nonhomogeneous Recurrence

$$a_n = \begin{cases} 3 & \text{if } n = 1, \\ 3a_{n-1} + 2n & \text{otherwise.} \end{cases}$$

Associated homogeneous equation: $a_n = 3a_{n-1}$

Homogeneous solution: $a_n^{(h)} = \alpha 3^n$

Particular solution of nonhomogeneous recurrence: $a_n^{(p)} = p_1 n + p_0$

Then $p_1 n + p_0 = 3(p_1(n-1) + p_0) + 2n$

$$\Rightarrow (2 + 2p_1)n + (2p_0 - 3p_1) = 0 \Rightarrow p_1 = -1, p_0 = -\frac{3}{2}$$

Solution: $a_n = a_n^{(p)} + a_n^{(h)} = -n - \frac{3}{2} + \alpha \cdot 3^n$

$$a_1 = 3 \Rightarrow \alpha = \frac{11}{6}$$

Hence $a_n = -n - \frac{3}{2} + \frac{11}{6} \cdot 3^n$