CSE 613: Parallel Programming

Lecture 13
(Graph Algorithms: Maximal Independent Set)

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Independent Sets

Let $G = (V, E)$ be an undirected graph.

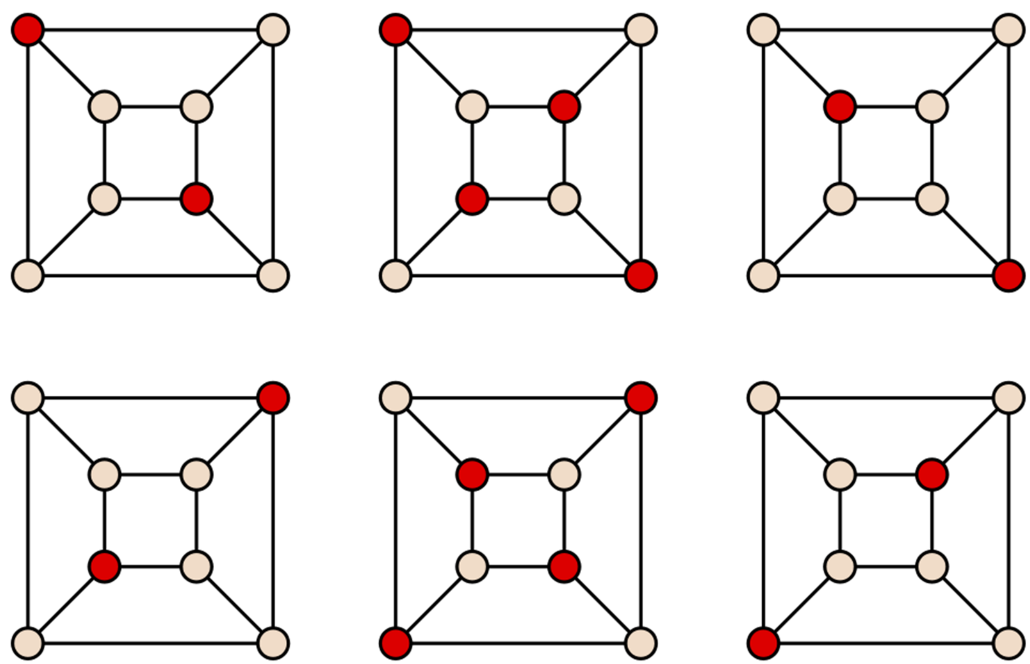
Independent Set: A subset $I \subseteq V$ is said to be *independent* provided for each $v \in I$ none of its neighbors in G belongs to I.

Maximal Independent Set: An independent set of G is *maximal* if it is not properly contained in any other independent set in G.

Maximum Independent Set: A maximal independent set of the largest size.

Finding a maximum independent set is NP-hard.

But finding a maximal independent set is trivial in the sequential setting.



Maximal Independent Sets (red vertices) of the Cube Graph
Source: Wikipedia

Finding a Maximal Independent Set Sequentially

Input: *V* is the set of vertices, and *E* is the set of edges. For each $v \in V$, we denote by $\Gamma(v)$ the set of neighboring vertices of *v*.

Output: A maximal independent set *MIS* of the input graph.

```
Serial-Greedy-MIS ( V, E )
1. MIS ← φ
2. for v ← 1 to |V| do
3.     if MIS ∩ Γ( v ) = φ then MIS ← MIS ∪ { v }
4. return MIS
```

This algorithm can be easily implemented to run in $\Theta(n + m)$ time, where n is the number of vertices and m is the number of edges in the input graph.

The output of this algorithm is called the *Lexicographically First MIS* (LFMIS).

Finding a Maximal Independent Set Sequentially

Input: *V* is the set of vertices, and *E* is the set of edges. For each $v \in V$, we denote by $\Gamma(v)$ the set of neighboring vertices of *v*.

Output: A maximal independent set *MIS* of the input graph.

Serial-Greedy-MIS-2 (*V*, *E*)

1. $MIS \leftarrow \phi$
2. *while* $|V| > 0$ *do*
3. pick an arbitrary vertex $v \in V$
4. $MIS \leftarrow MIS \cup \{ v \}$
5. $R \leftarrow \{ v \} \cup \Gamma(v)$
6. $V \leftarrow V \setminus R$
7. $E \leftarrow E \setminus \{ (v_1, v_2) \mid v_1 \in R \text{ or } v_2 \in R \}$
8. *return* *MIS*

Always choosing the vertex with the smallest id in the current graph will produce exactly the same MIS as in *Serial-Greedy-MIS*.

Finding a Maximal Independent Set Sequentially

Input: *V* is the set of vertices, and *E* is the set of edges. For each $S \subseteq V$, we denote by $\Gamma(S)$ the set of neighboring vertices of *S*.

Output: A maximal independent set *MIS* of the input graph.

```
Serial-Greedy-MIS-3 ( V, E )
1. MIS ← φ
2. while |V| > 0 do
3.     find an independent set S ⊆ V
4.     MIS ← MIS ∪ S
5.     R ← S ∪ Γ( S )
6.     V ← V \ R
7.     E ← E \ { ( v1, v2 ) | v1 ∈ R or v2 ∈ R }
8. return MIS
```

Parallelizing Serial-Greedy-MIS-3

- Number of iterations can be kept small by finding in each iteration an S with large $S \cup \Gamma(S)$. But this is difficult to do.
- Instead in each iteration we choose an S such that a large fraction of current edges are incident on $S \cup \Gamma(S)$.

```
Serial-Greedy-MIS-3 ( V, E )
MIS ← ϕ
while |V| > 0 do
    find an independent set S ⊆ V
    MIS ← MIS ∪ S
    R ← S ∪ Γ( S )
    V ← V \ R
    E ← E \ { ( v1, v2 ) | v1 ∈ R or v2 ∈ R }
return MIS
```

- To select S we start with a random $S' \subseteq V$.
  - By choosing lower degree vertices with higher probability we are likely to have very few edges with both end-points in S'.
  - We check each edge with both end-points in S', and drop the end-point with lower degree from S'. Our intention is to keep $\Gamma(S')$ as large as we can.
  - After removing all edges as above we are left with an independent set. This is our S.
  - We will prove that if we remove $S \cup \Gamma(S)$ from the current graph a large fraction of current edges will also get removed.

Randomized Maximal Independent Set (MIS)

Input: n is the number of vertices, and for each vertex $u \in [\ 1, n\ ]$, $V[\ u\ ]$ is set to u. E is the set of edges sorted in non-decreasing order of the first vertex. For every edge $(\ u, v\ )$ both $(\ u, v\ )$ and $(\ v, u\ )$ are included in E.

Output: For all $u \in [\ 1, n\ ]$, $MIS[\ u\ ]$ is set to 1 if vertex u is in the MIS.

```
Par-Randomized-MIS ( n, V, E, MIS )
while |V| > 0 do
    array d[ 1 : |V| ], c[ 1 : |V| ] = { 0 }, M[ 1 : |V| ] = { 0 }
    parallel for i ← 1 to |E| do
        if i = |E| or E[ i ].u ≠ E[ i + 1 ].u  then c[ E[ i ].u ] ← i
    parallel for u ← 1 to |V| do
        if u = 1 then d[ u ] ← c[ u ] else d[ u ] ← c[ u ] - c[ u - 1 ]
        if d[ u ] = 0 then M[ u ] ← 1
        else M[ u ] ← 1 ( with probability 1 / ( 2d[ u ] ) )
    parallel for each ( u, v ) ∈ E do
        if M[ u ] = 1 and M[ v ] = 1 then
            if d[ u ] ≤ d[ v ] then M[ u ] ← 0 else M[ v ] ← 0
    parallel for u ← 1 to |V| do
        if M[ u ] = 1 then MIS[ V[ u ] ] ← 1
    ( V, E ) ← Par-Compress ( V, E, M )
```

for each u find the edge with the largest index i such that $E[\ i\ ].u = u$, and store that i in $c[\ u\ ]$

$d[\ u\ ]$ (i.e., degree of vertex u) can now be computed easily by subtracting $c[\ u - 1\ ]$ from $c[\ u\ ]$

mark lower-degree vertices with higher probability

if both end-points of an edge is marked, unmark the one with the lower degree

add all marked vertices to MIS

remove marked vertices along with their neighbors as well as the corresponding edges

Removing Marked Vertices and Their Neighbors

Input: Arrays V and E, and bit array $M[\ 1 : |V|\ ]$. Each entry of E is of the form $(\ u, v\ )$, where $1 \le u, v \le |V|$. If for some u, $M[\ u\ ] = 1$, then u and all v such that $(\ u, v\ ) \in E$ must be removed from V along with all edges $(\ u, v\ )$ from E.

Output: Updated V and E.

```
Par-Compress ( V, E, M )
array S_V[ 1 : |V| ] = { 1 }, S'_V[ 1 : |V| ], S_E[ 1 : |E| ] = { 1 }, S'_E[ 1 : |E| ]
parallel for u ← 1 to |V| do
    if M[ u ] = 1 then S_V[ u ] ← 0
parallel for i ← 1 to |E| do
    u ← E[ i ].u, v ← E[ i ].v
    if M[ u ] = 1 or M[ v ] = 1 then S_V[ u ] ← 0, S_V[ v ] ← 0, S_E[ i ] ← 0
S'_V ← Par-Prefix-Sum ( S_V, + ), S'_E ← Par-Prefix-Sum ( S_E, + )
array U[ 1 : S'_V[ |V| ] ], F[ 1 : S'_E[ |E| ] ]
parallel for u ← 1 to |V| do
    if S_V[ u ] = 1 then U[ S'_V[ u ] ] ← V[ u ]
parallel for i ← 1 to |E| do
    if S_E[ i ] = 1 then F[ S'_E[ i ] ] ← E[ i ]
parallel for i ← 1 to |F| do
    u ← F[ i ].u, v ← F[ i ].v
    F[ i ].u ← S'_V[ u ], F[ i ].v ← S'_V[ v ]
return ( U, F )
```

- initialize
- marked vertices will be removed
- neighbors of marked vertices & corresponding edges must go
- find new indices for surviving vertices & edges
- move surviving vertices to the smaller array U
- move surviving edges to the smaller array F
- update the end-points of the surviving edges to new vertex indices

Removing Marked Vertices and Their Neighbors

```
Par-Compress ( V, E, M )
1.  array S_V[ 1 : |V| ] = { 1 }, S′_V[ 1 : |V| ],
          S_E[ 1 : |E| ] = { 1 }, S′_E[ 1 : |E| ]
2.  parallel for u ← 1 to |V| do
3.      if M[ u ] = 1 then S_V[ u ] ← 0
4.  parallel for i ← 1 to |E| do
5.      u ← E[ i ].u, v ← E[ i ].v
6.      if M[ u ] = 1 or M[ v ] = 1 then
            S_V[ u ] ← 0, S_V[ v ] ← 0, S_E[ i ] ← 0
7.  S′_V ← Par-Prefix-Sum ( S_V, + ),
    S′_E ← Par-Prefix-Sum ( S_E, + )
8.  array U[ 1 : S′_V[ |V| ] ], F[ 1 : S′_E[ |E| ] ]
9.  parallel for u ← 1 to |V| do
10.     if S_V[ u ] = 1 then U[ S′_V[ u ] ] ← V[ u ]
11. parallel for i ← 1 to |E| do
12.     if S_E[ i ] = 1 then F[ S′_E[ i ] ] ← E[ i ]
13. parallel for i ← 1 to |F| do
14.     u ← F[ i ].u, v ← F[ i ].v
15.     F[ i ].u ← S′_V[ u ], F[ i ].v ← S′_V[ v ]
16. return ( U, F )
```

The prefix sums in line 7 perform $\Theta(|V| + |E|)$ work and have $\Theta(\log^2|V| + \log^2|E|)$ depth. The rest of the algorithm also perform $\Theta(|V| + |E|)$ work but in $\Theta(\log|V| + \log|E|)$ depth. Hence,

Work: $\Theta(|V| + |E|)$

Span: $\Theta(\log^2|V| + \log^2|E|)$

Randomized Maximal Independent Set (MIS)

```
Par-Randomized-MIS ( n, V, E, MIS )
 1.  while |V| > 0 do
 2.     array d[ 1 : |V| ], c[ 1 : |V| ] = { 0 },
              M[ 1 : |V| ] = { 0 }
 3.     parallel for i ← 1 to |E| do
 4.        if i = |E| or E[ i ].u ≠ E[ i + 1 ].u  then
              c[ E[ i ].u ] ← i
 5.     parallel for u ← 1 to |V| do
 6.        if u = 1 then d[ u ] ← c[ u ]
           else d[ u ] ← c[ u ] - c[ u - 1 ]
 7.        if d[ u ] = 0 then M[ u ] ← 1
 8.        else M[ u ] ← 1 ( with prob 1 / ( 2d[ u ] ) )
 9.     parallel for each ( u, v ) ∈ E do
10.        if M[ u ] = 1 and M[ v ] = 1 then
11.           if d[ u ] ≤ d[ v ] then M[ u ] ← 0
              else M[ v ] ← 0
12.     parallel for u ← 1 to |V| do
13.        if M[ u ] = 1 then MIS[ V[ u ] ] ← 1
14.     ( V, E ) ← Par-Compress ( V, E, M )
```

Let n = #vertices, and m = #edges initially.

Let us assume for the time being that at least a constant fraction of the edges are removed in each iteration of the *while* loop (we will prove this shortly). Let this fraction be f (< 1).

This implies that the *while* loop iterates $\Theta\left(\log_{1/(1-f)} m\right) = \Theta(\log m)$ times. (how?)

Each iteration performs $\Theta(|V| + |E|)$ work and has $\Theta(\log^2|V| + \log^2|E|)$ depth. Hence,

Work: $T_1(n,m) = \Theta\left((n+m)\sum_{i=0}^{k}(1-f)^i\right)$

$= \Theta(n+m)$

Span: $T_\infty(n,m) = \Theta((\log^2 n + \log^2 m)\log m)$

$= \Theta(\log^3 n)$

Parallelism: $\frac{T_1(n,m)}{T_\infty(n,m)} = \Theta\left(\frac{n+m}{\log^3 n}\right)$

Analysis of Randomized MIS

Let, $d(v)$ be the degree of vertex v, and $\Gamma(v)$ be its set of neighbors.

Good Vertex: A vertex v is *good* provided $|L(v)| \geq \frac{d(v)}{3}$, where,
$L(v) = \{\, u \mid \big(u \in \Gamma(v)\big) \wedge \big(d(u) \leq d(v)\big) \,\}$.

Bad Vertex: A vertex is *bad* if it is not good.

Good Edge: An edge (u, v) is *good* if at least one of u and v is good.

Bad Edge: An edge (u, v) is *bad* if both u and v are bad.

Analysis of Randomized MIS

Lemma 1: In some iteration of the *while* loop, let v be a good vertex with $d(v) > 0$, and let M be the set of vertices that got marked (in lines 7-8). Then

$$\Pr\{\, \Gamma(v) \cap M \neq \emptyset \,\} \geq 1 - e^{-1/6}.$$

Proof: We have, $\Pr\{\, \Gamma(v) \cap M \neq \emptyset \,\} = 1 - \Pr\{\, \Gamma(v) \cap M = \emptyset \,\}$

$$= 1 - \prod_{u \in \Gamma(v)} \Pr\{\, u \notin M \,\} \geq 1 - \prod_{u \in L(v)} \Pr\{\, u \notin M \,\}$$

$$= 1 - \prod_{u \in L(v)} \left(1 - \frac{1}{2d(u)}\right) \geq 1 - \prod_{u \in L(v)} \left(1 - \frac{1}{2d(v)}\right)$$

$$= 1 - \left(1 - \frac{1}{2d(v)}\right)^{|L(v)|} \geq 1 - \left(1 - \frac{1}{2d(v)}\right)^{d(v)/3}$$

$$\geq 1 - e^{-\frac{d(v)/3}{2d(v)}} = 1 - e^{-\frac{1}{6}}$$

Analysis of Randomized MIS

Lemma 2: In any iteration of the *while* loop, let M be the set of vertices that got marked (in lines 7-8), and let S be the set of vertices that got included in the MIS (in line 13). Then

$$\Pr\{\, v \in S \mid v \in M \,\} \geq \frac{1}{2}\,.$$

Proof: We have, $\Pr\{\, v \in S \mid v \in M \,\}$

$$\geq 1 - \Pr\{\, \exists u \in \Gamma(v) \text{ s.t. } (\, d(u) \geq d(v)\,) \wedge (\, u \in M\,)\}$$

$$\geq 1 - \sum_{\substack{u \in \Gamma(v) \\ d(u) \geq d(v)}} \frac{1}{2d(u)} \geq 1 - \sum_{\substack{u \in \Gamma(v) \\ d(u) \geq d(v)}} \frac{1}{2d(v)}$$

$$\geq 1 - \sum_{u \in \Gamma(v)} \frac{1}{2d(v)} = 1 - d(v) \times \frac{1}{2d(v)} = \frac{1}{2}$$

Analysis of Randomized MIS

Lemma 3: In any iteration of the *while* loop, let V_G be the set of good vertices, and let S be the vertex set that got included in the MIS. Then

$$\Pr\{\, v \in S \cup \Gamma(S) \mid v \in V_G \,\} \geq \frac{1}{2}\left(1 \; - e^{-\,1/6}\right).$$

Proof: We have, $\Pr\{\, v \in S \cup \Gamma(S) \mid v \in V_G \,\}$

$$\geq \Pr\{\, v \in \Gamma(S) \mid v \in V_G \,\} = \Pr\{\, \Gamma(v) \cap S \neq \phi \mid v \in V_G \,\}$$

$$= \Pr\{\, \Gamma(v) \cap S \neq \phi \mid \Gamma(v) \cap M \neq \phi, v \in V_G \,\}$$
$$\times \Pr\{\, \Gamma(v) \cap M \neq \phi \mid v \in V_G \,\}$$

$$\geq \Pr\{\, u \in S \mid u \in \Gamma(v) \cap M, v \in V_G \,\}$$
$$\times \Pr\{\, \Gamma(v) \cap M \neq \phi \mid v \in V_G \,\}$$

$$\geq \frac{1}{2}\left(1 - e^{-1/6}\right)$$

Analysis of Randomized MIS

Lemma 3: In any iteration of the *while* loop, let V_G be the set of good vertices, and let S be the vertex set that got included in the MIS. Then

$$\Pr\{\, v \in S \cup \Gamma(S) \mid v \in V_G \,\} \geq \frac{1}{2}\left(1 \; - e^{-\,1/6}\right).$$

Corollary 1: In any iteration of the *while* loop, a good vertex gets removed (in line 14) with probability at least $\frac{1}{2}\left(1 \; - e^{-\,1/6}\right)$.

Corollary 2: In any iteration of the *while* loop, a good edge gets removed (in line 14) with probability at least $\frac{1}{2}\left(1 \; - e^{-\,1/6}\right)$.

Analysis of Randomized MIS

Lemma 4: In any iteration of the *while* loop, let E and E_G be the sets of all edges and good edges, respectively. Then $|\mathrm{E_G}| \geq |E|/2$.

Proof: For each edge $(u, v) \in E$, direct (u, v) from u to v if $d(u) \leq d(v)$, and v to u otherwise.

For every vertex v in the resulting digraph let $d_i(v)$ and $d_o(v)$ denote its in-degree and out-degree, respectively.

Let V_G and V_B be the set of good and bad vertices, respectively.

Then for each $v \in V_B$, $d_o(v) - d_i(v) \geq \frac{d(v)}{3}$.

Let m_{BB}, m_{BG}, m_{GB} and m_{GG} be the #edges directed from V_B to V_B, from V_B to V_G, from V_G to V_B, and from V_G to V_G, respectively.

Analysis of Randomized MIS

Lemma 4: In any iteration of the *while* loop, let E and E_G be the sets of all edges and good edges, respectively. Then $|E_G| \geq |E|/2$.

Proof (continued): We have,

$$2m_{BB} + m_{BG} + m_{GB}$$

$$= \sum_{v \in V_B} d(v) \leq 3 \sum_{v \in V_B} (d_o(v) - d_i(v)) = 3 \sum_{v \in V_G} (d_i(v) - d_o(v))$$

$$= 3\big((m_{BG} + m_{GG}) - (m_{GB} + m_{GG})\big) = 3(m_{BG} - m_{GB})$$

$$\leq 3(m_{BG} + m_{GB})$$

Thus $2m_{BB} + m_{BG} + m_{GB} \leq 3(m_{BG} + m_{GB})$

$$\Rightarrow m_{BB} \leq m_{BG} + m_{GB} \Rightarrow m_{BB} \leq m_{BG} + m_{GB} + m_{GG}$$

$$\Rightarrow (m_{BG} + m_{GB} + m_{GG}) + \ m_{BB} \leq 2(m_{BG} + m_{GB} + m_{GG})$$

$$\Rightarrow |E| \leq 2|E_G|$$

Analysis of Randomized MIS

Lemma 5: In any iteration of the *while* loop, let E be the set of all edges. Then the expected number of edges removed (in line 14) during this iteration is at least $\frac{1}{4}\left(1 - e^{-1/6}\right)|E|$.

Proof: Follows from Lemma 4 and Corollary 2.