CSE 548: Analysis of Algorithms

Lecture 7
(Divide-and-Conquer Algorithms: Akra-Bazzi Recurrences)

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Spring 2014

Deterministic Select

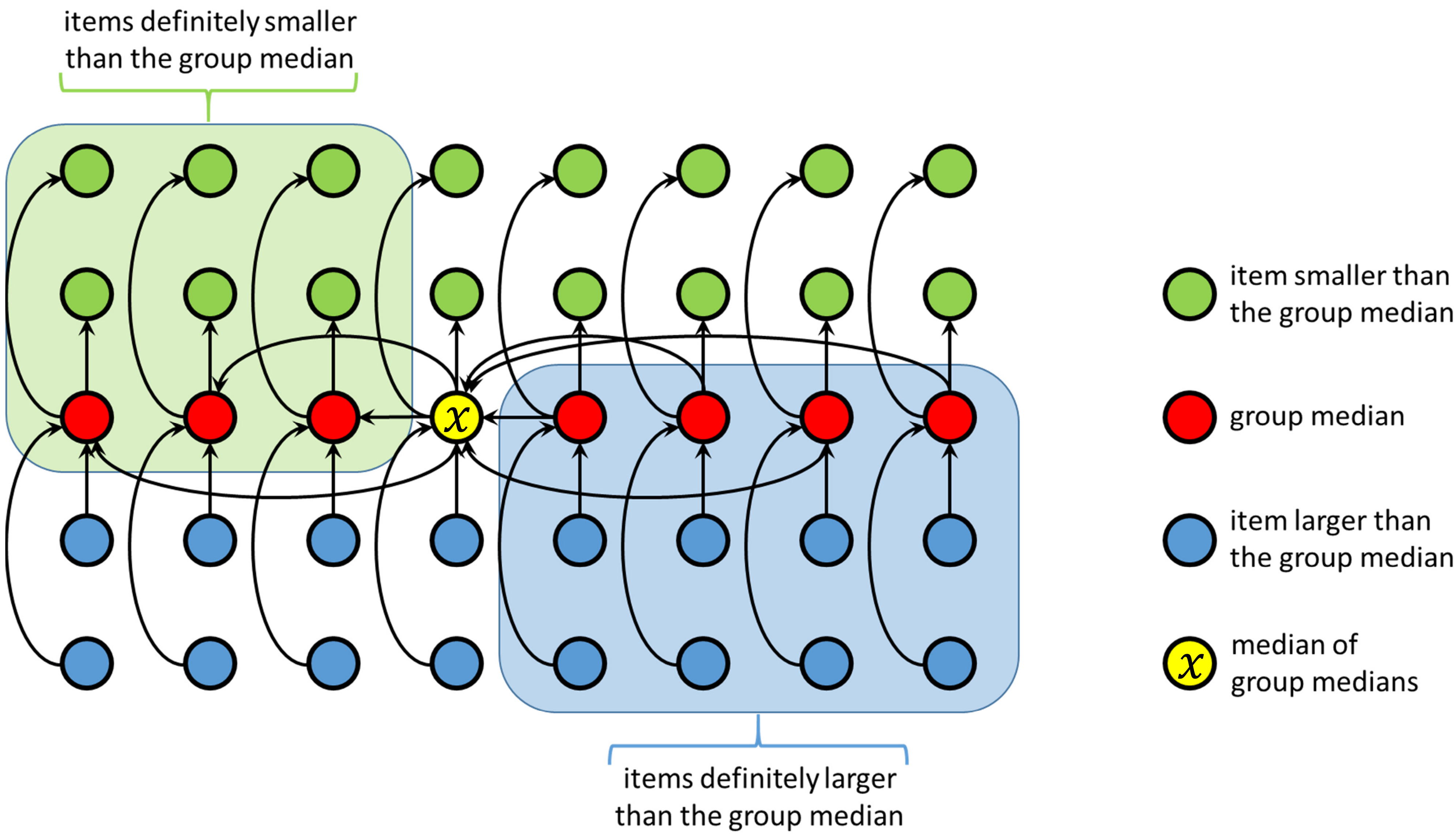
Input: An array $A[\ q : r\ ]$ of distinct elements, and integer $k \in [1, r - q + 1]$.

Output: An element x of $A[\ q : r\ ]$ such that $rank(x, A[\ q : r\ ]) = k$.

```
Select ( A[ q : r ], k )
    n ← r − q + 1
    if n ≤ 140 then
        sort A[ q : r ] and return A[ q + k - 1 ]
    else
        divide A[ q : r ] into blocks B_i's each containing 5 consecutive elements
                ( last block may contain fewer than 5 elements )
        for i ← 1 to ⌈ n / 5 ⌉ do
            M[ i ] ← median of B_i using sorting
        x ← Select ( M[ 1 : ⌈ n / 5 ⌉ ], ⌊( ⌈ n / 5 ⌉ + 1 ) / 2 ⌋ )  { median of medians }
        t ← Partition ( A[ q : r ], x )    { partition around x which ends up at A[ t ] }
        if k = t − q + 1 then return A[ t ]
        else if k < t − q + 1 then return Select ( A[ q : t − 1 ], k )
            else return Select ( A[ t + 1 : r ], k − t + q − 1 )
```

Deterministic Select

$\text{SELECT}(A, k)$: Given an unsorted set A of n $(= |A|)$ items, find the k^{th} smallest item in the set



Deterministic Select

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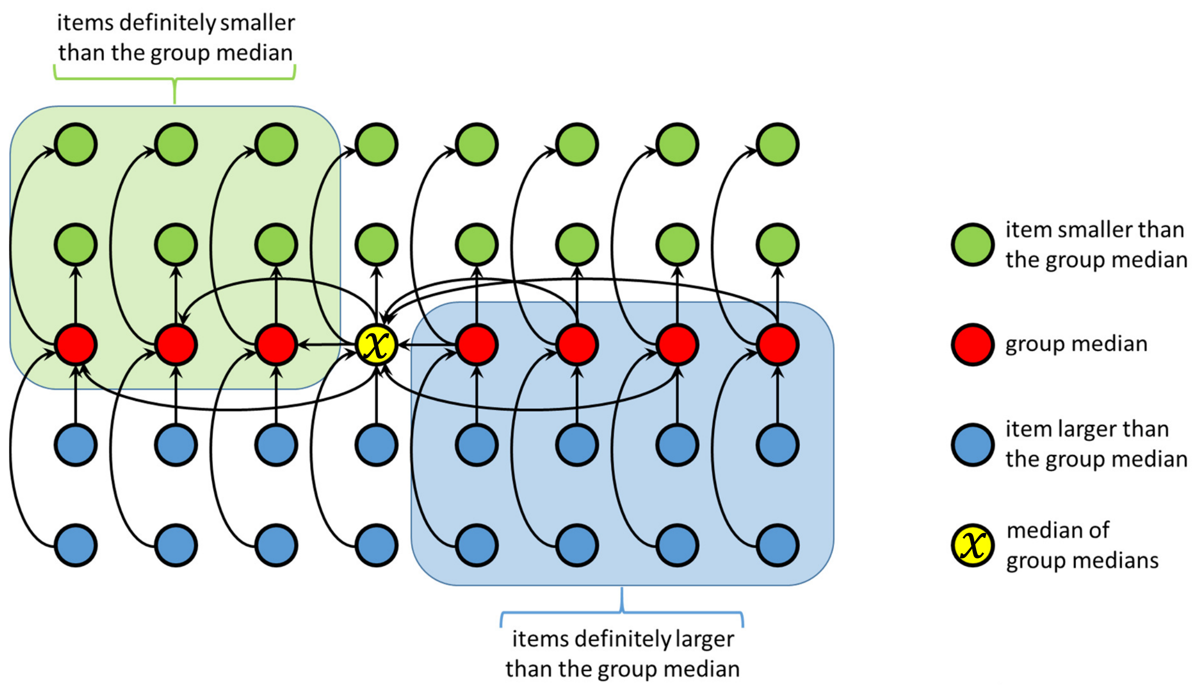

#items definitely smaller than x is $\geq 3\left(\left\lfloor \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rfloor - 1\right) \geq \frac{3n}{10} - 6$

#items definitely larger than x is $\geq 3\left(\left\lfloor \frac{1}{2}\left\lceil \frac{n}{5} \right\rceil \right\rfloor - 1\right) \geq \frac{3n}{10} - 6$

#items in any recursive call (lines 11/12) $\leq n - \left(\frac{3n}{10} - 6\right) = \frac{7n}{10} + 6$

Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm (given in Section 9.3 of CLRS):

$$T(n) \le \begin{cases} \Theta(1), & if\ n < 140, \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

Dropping the ceiling for simplicity, and observing that $\frac{7n}{10} + 6 \le \frac{8n}{10}$ when $n \ge 60$, we obtain the following upper bound on $T(n)$.

$$T'(n) = \begin{cases} \Theta(1), & if\ n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

How do you solve for $T'(n)$?

Deterministic Select

The following recurrence describes the worst-case running time of the deterministic selection algorithm (given in Section 9.3 of CLRS):

$$T(n) \le \begin{cases} \Theta(1), & if\ n < 140, \\ T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

Dropping the ceiling for simplicity, and observing that $\frac{7n}{10} + 6 \le \frac{7.5n}{10}$ when $n \ge 120$, we obtain the following upper bound on $T(n)$.

$$T''(n) = \begin{cases} \Theta(1), & if\ n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & if\ n \ge 140. \end{cases}$$

How do you solve for $T''(n)$?

Akra-Bazzi Recurrences

Consider the following recurrence:

$$T(x) = \begin{cases} \Theta(1), & \mathit{if}\ 1 \le x \le x_0, \\ \displaystyle\sum_{i=1}^{k} a_i T(b_i x) + g(x), & \mathit{if}\ x > x_0; \end{cases}$$

where,

1. $k \ge 1$ is an integer constant
2. $a_i > 0$ is a constant for $1 \le i \le k$
3. $b_i \in (0,1)$ is a constant for $1 \le i \le k$
4. $x \ge 1$ is a real number
5. $x_0 \ge \max\left\{\frac{1}{b_i}, \frac{1}{1-b_i}\right\}$ is a constant for $1 \le i \le k$
6. $g(x)$ is a nonnegative function that satisfies a *polynomial-growth condition* (to be specified soon)

Polynomial-Growth Condition

We say that $g(x)$ satisfies the *polynomial-growth condition* if there exist positive constants c_1 and c_2 such that for all $x \geq 1$, for all $1 \leq i \leq k$, and for all $u \in [b_i x, x]$,

$$c_1 g(x) \leq g(u) \leq c_2 g(x),$$

where x, k, b_i and $g(x)$ are as defined in the previous slide.

The Akra-Bazzi Solution

Consider the recurrence given in the previous two slides under the conditions specified there:

$$T(x) = \begin{cases} \Theta(1), & if\ 1 \le x \le x_0, \\ \sum_{i=1}^{k} a_i T(b_i x) + g(x), & if\ x > x_0. \end{cases}$$

Let p be the unique real number for which $\sum_{i=1}^{k} a_i b_i^p = 1$. Then

$$T(x) = \Theta\left(x^p\left(1 + \int_1^x \frac{g(u)}{u^{p+1}}\,du\right)\right)$$

Deterministic Select

$$T'(n) = \begin{cases} \Theta(1), & if\ n < 140, \\ T'\left(\frac{n}{5}\right) + T'\left(\frac{4n}{5}\right) + \Theta(n), & if\ n \geq 140. \end{cases}$$

From $\left(\frac{1}{5}\right)^p + \left(\frac{4}{5}\right)^p = 1$ we get $p = 1$.

Hence, $T'(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{p+1}}\,du\right)\right)$

$$= \Theta\left(n\left(1 + \int_1^n \frac{du}{u}\right)\right)$$

$$= \Theta(n \ln n)$$

Deterministic Select

$$T''(n) = \begin{cases} \Theta(1), & if\ n < 140, \\ T''\left(\frac{n}{5}\right) + T''\left(\frac{3n}{4}\right) + \Theta(n), & if\ n \geq 140. \end{cases}$$

From $\left(\frac{1}{5}\right)^p + \left(\frac{3}{4}\right)^p = 1$ we get $p < 1$.

Hence, $T''(n) = \Theta\left(n^p\left(1 + \int_1^n \frac{u}{u^{p+1}}\,du\right)\right)$

$$= \Theta\left(n^p\left(1 + \int_1^n \frac{du}{u^p}\right)\right)$$

$$= \Theta\left(\left(\frac{1}{1-p}\right)n - \left(\frac{p}{1-p}\right)n^p\right)$$

$$= \Theta(n)$$

Continued to Lecture 8