CSE 548: Analysis of Algorithms

Lecture 31
(Analyzing I/O and Cache Performance)

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Iterative Matrix-Multiply Variants

double $Z[n][n], X[n][n], Y[n][n];$

I-J-K

```
for ( int i = 0; i < n; i++ )
  for ( int j = 0; j < n; j++ )
    for ( int k = 0; k < n; k++ )
      Z[ i ][ j ] += X[ i ][ k ] * Y[ k ][ j ];
```

I-K-J

```
for ( int i = 0; i < n; i++ )
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J-I-K

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J-K-I

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K-I-J

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K-J-I

```
for ( int k = 0; k < n; k++ )
  for ( int j = 0; j < n; j++ )
    for ( int i = 0; i < n; i++ )
      Z[ i ][ j ] += X[ i ][ k ] * Y[ k ][ j ];
```

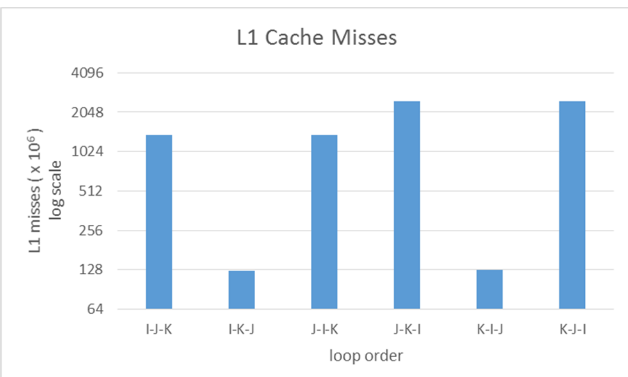
Performance of Iterative Matrix-Multiply Variants

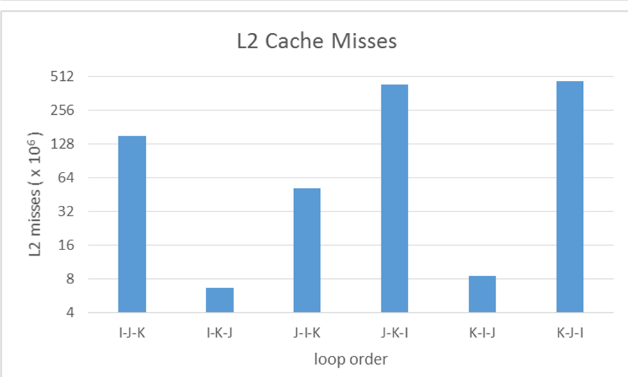
Processor: 2.7 GHz Intel Xeon E5-2680 (used only one core)

Caches & RAM: private 32KB L1, private 256KB L2, shared 20MB L3, 32 GB RAM

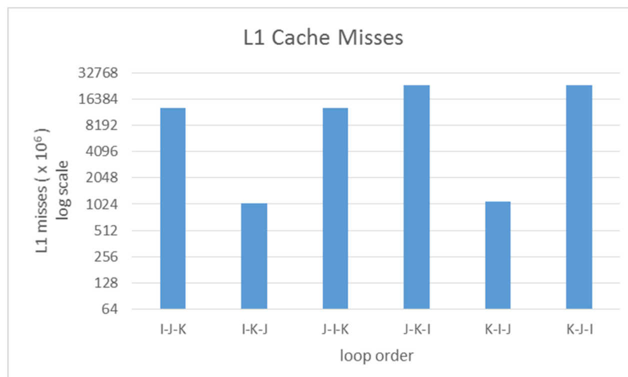
Optimizations: none (icc 13.0 with –O0)

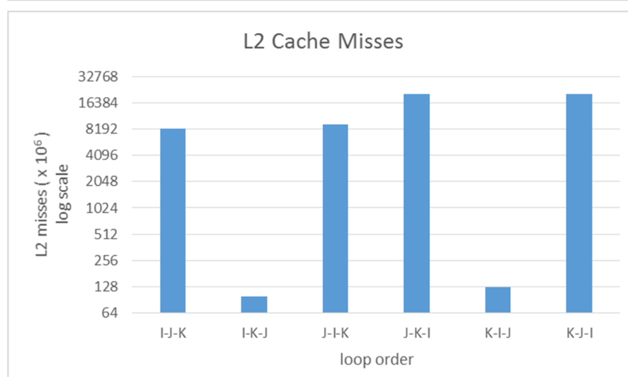
$n = 1000$



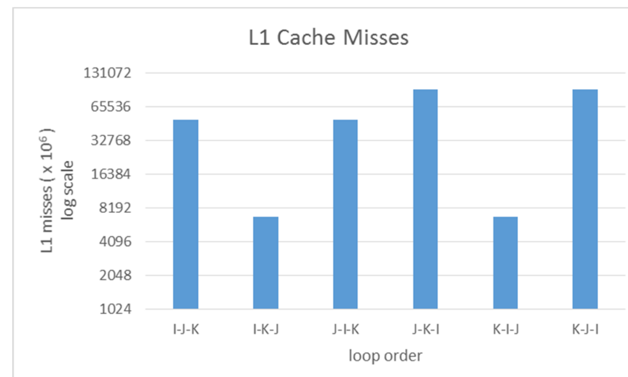


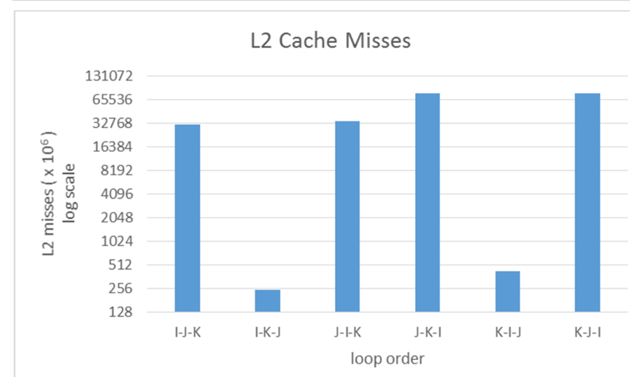
$n = 2000$





$n = 3000$





Memory: Fast, Large & Cheap!

For efficient computation we need

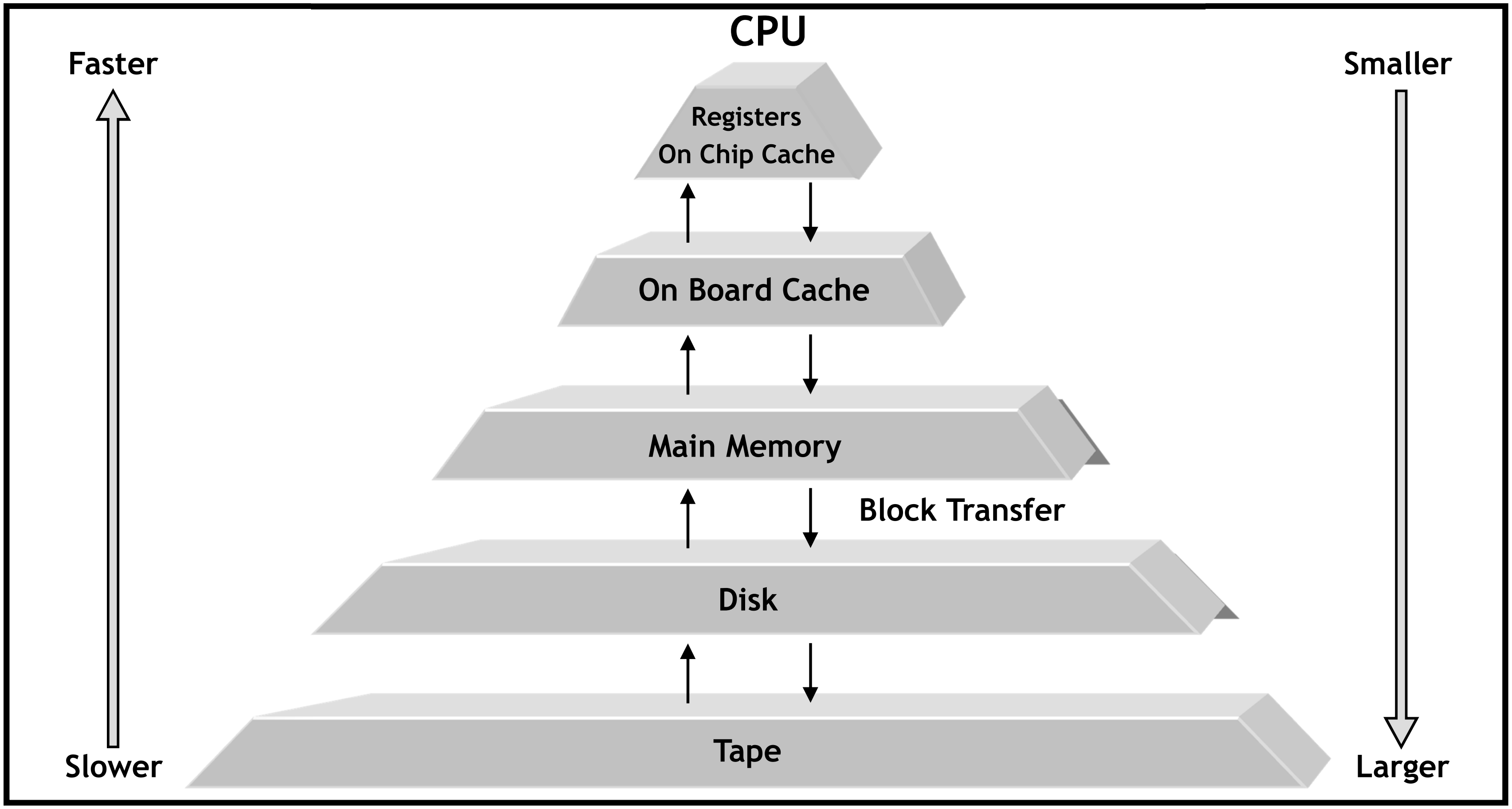
- fast processors
- fast and large (but not so expensive) memory

But memory cannot be cheap, large and fast at the same time, because of

- finite signal speed
- lack of space to put enough connecting wires

A reasonable compromise is to use a *memory hierarchy*.

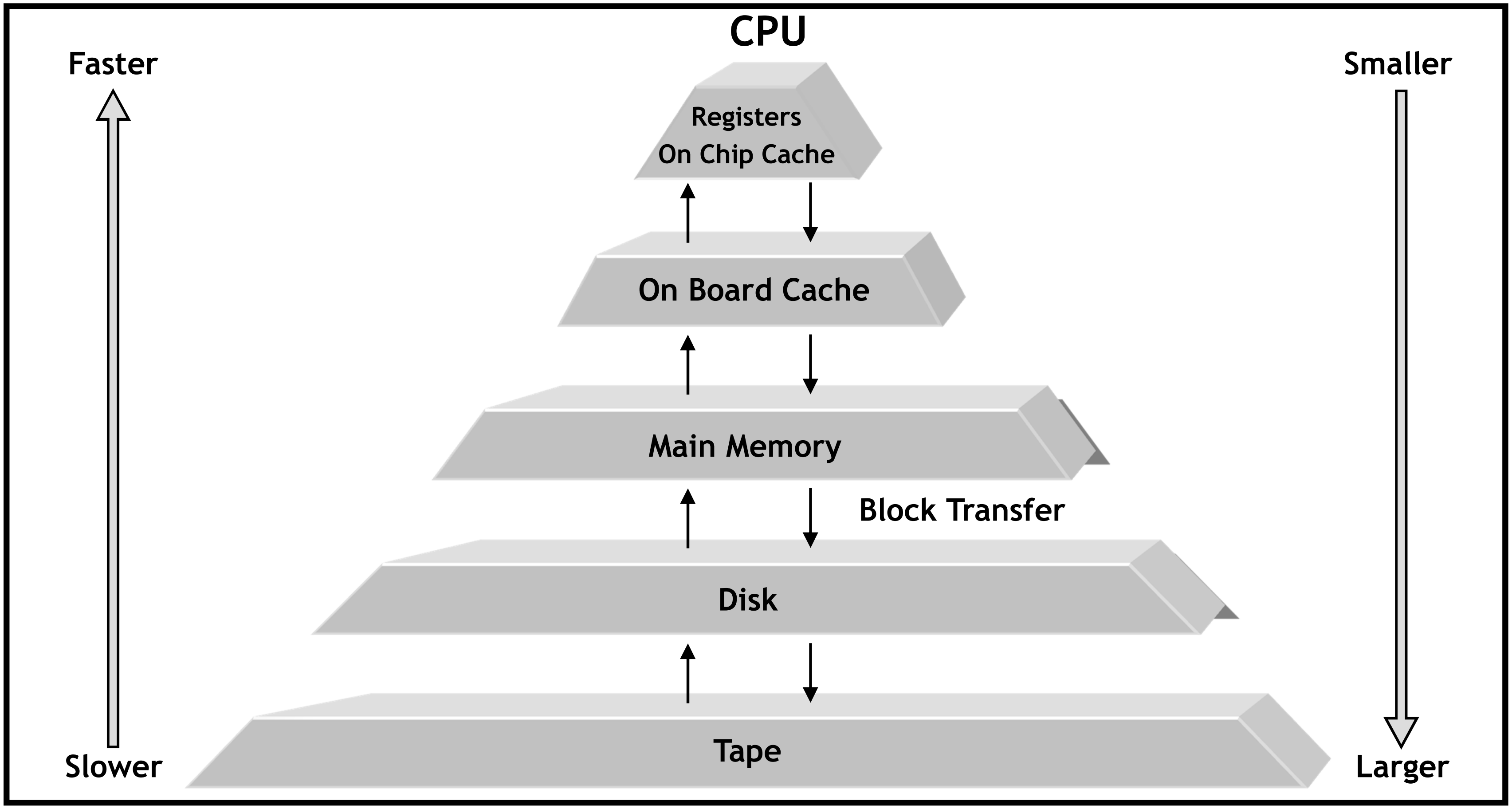
The Memory Hierarchy



A *memory hierarchy* is

- almost as fast as its fastest level
- almost as large as its largest level
- inexpensive

The Memory Hierarchy



To perform well on a memory hierarchy algorithms must have high locality in their memory access patterns.

Locality of Reference

Spatial Locality: When a block of data is brought into the cache it should contain as much useful data as possible.

Temporal Locality: Once a data point is in the cache as much useful work as possible should be done on it before evicting it from the cache.

CPU-bound vs. Memory-bound Algorithms

The Op-Space Ratio: Ratio of the number of operations performed by an algorithm to the amount of space (input + output) it uses.

Intuitively, this gives an upper bound on the average number of operations performed for every memory location accessed.

CPU-bound Algorithm:

- high op-space ratio
- more time spent in computing than transferring data
- a faster CPU results in a faster running time

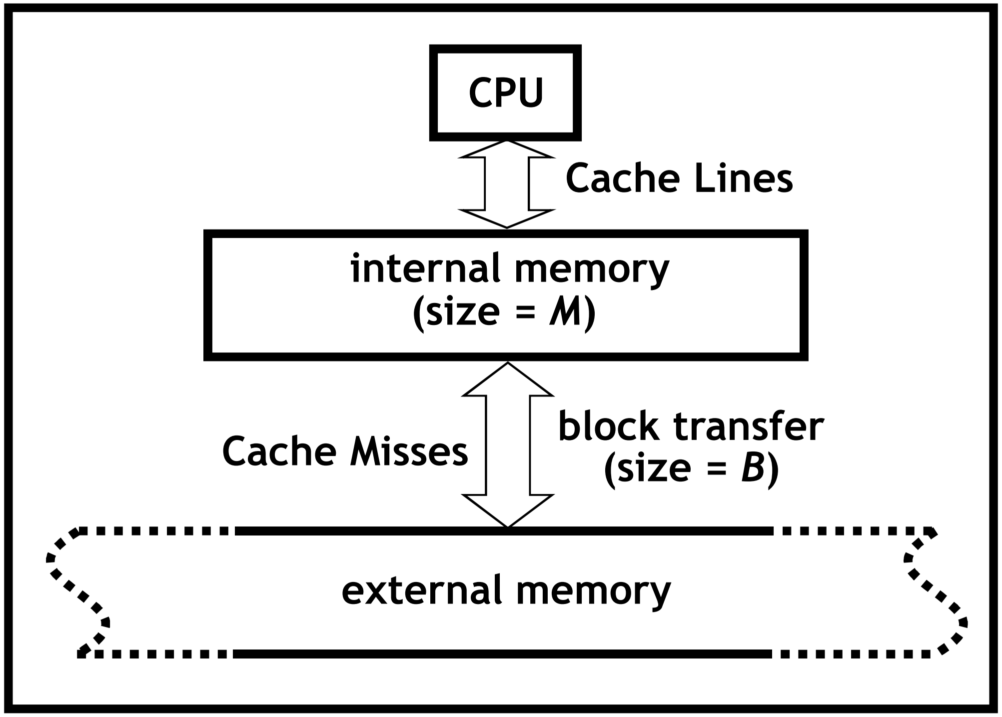
Memory-bound Algorithm:

- low op-space ratio
- more time spent in transferring data than computing
- a faster memory system leads to a faster running time

The Two-level I/O Model

The *two-level I/O model* [Aggarwal & Vitter, CACM'88] consists of:

- an *internal memory* of size M
- an arbitrarily large *external memory* partitioned into blocks of size B.



I/O complexity of an algorithm

= number of blocks transferred between these two levels

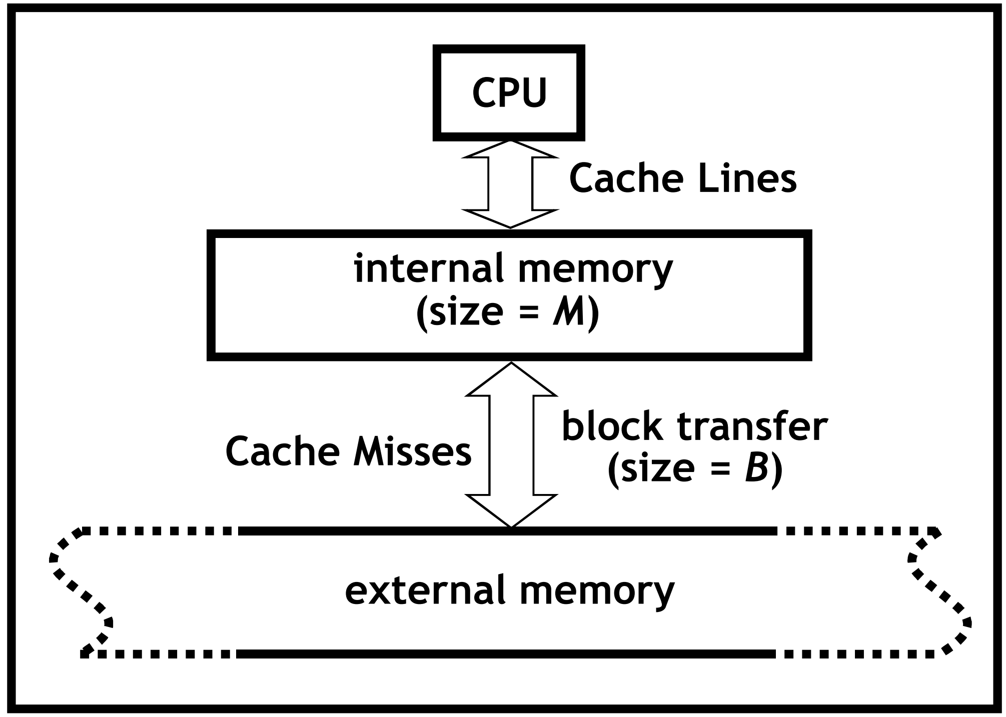
Basic I/O complexities: $scan(N) = \Theta\left(\frac{N}{B}\right)$ and $sort(N) = \Theta\left(\frac{N}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$

Algorithms often crucially depend on the knowledge of M and B

$\Rightarrow$ algorithms do not adapt well when M or B changes

The Ideal-Cache Model

The *ideal-cache model* [Frigo et al., FOCS'99] is an extension of the I/O model with the following constraint:



algorithms are not allowed to use knowledge of M and B.

Consequences of this extension

– algorithms can simultaneously adapt to all levels of a multi-level memory hierarchy

– algorithms become more flexible and portable

Algorithms for this model are known as *cache-oblivious algorithms*.

The Ideal-Cache Model: Assumptions

The model makes the following assumptions:

- Optimal offline cache replacement policy
- Exactly two levels of memory
- Automatic replacement & full associativity

The Ideal-Cache Model: Assumptions

The model makes the following assumptions:

- Optimal offline cache replacement policy
  - LRU & FIFO allow for a constant factor approximation of optimal [Sleator & Tarjan, JACM'85]
- Exactly two levels of memory
- Automatic replacement & full associativity

The Ideal-Cache Model: Assumptions

The model makes the following assumptions:

- Optimal offline cache replacement policy
- Exactly two levels of memory
  - can be effectively removed by making several reasonable assumptions about the memory hierarchy [Frigo et al., FOCS'99]
- Automatic replacement & full associativity

The Ideal-Cache Model: Assumptions

The model makes the following assumptions:

- ❑ Optimal offline cache replacement policy
- ❑ Exactly two levels of memory
- ❑ Automatic replacement & full associativity
  - in practice, cache replacement is automatic (by OS or hardware)
  - fully associative LRU caches can be simulated in software with only a constant factor loss in expected performance [Frigo et al., FOCS’99]

The Ideal-Cache Model: Assumptions

The model makes the following assumptions:

- Optimal offline cache replacement policy
- Exactly two levels of memory
- Automatic replacement & full associativity

Often makes the following assumption, too:

- $M = \Omega\left(B^2\right)$, i.e., the cache is *tall*

The Ideal-Cache Model: Assumptions

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- Optimal offline cache replacement policy
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Often makes the following assumption, too:

- $M = \Omega\left(B^2\right)$, i.e., the cache is *tall*
  - most practical caches are tall

The Ideal-Cache Model: I/O Bounds

Cache-oblivious vs. cache-aware bounds:

- Basic I/O bounds (same as the cache-aware bounds):
  - $scan(N) = \Theta\left(\frac{N}{B}\right)$
  - $sort(N) = \Theta\left(\frac{N}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$
- Most cache-oblivious results match the I/O bounds of their cache-aware counterparts
- There are few exceptions; e.g., no cache-oblivious solution to the *permutation* problem can match cache-aware I/O bounds [Brodal & Fagerberg, STOC'03]

Some Known Cache Aware / Oblivious Results

Problem	Cache-Aware Results	Cache-Oblivious Results
Array Scanning (*scan(N)*)	$O\left(\frac{N}{B}\right)$	$O\left(\frac{N}{B}\right)$
Sorting (*sort(N)*)	$O\left(\frac{N}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$	$O\left(\frac{N}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$
Selection	$O\left(scan\left(N\right)\right)$	$O\left(scan\left(N\right)\right)$
B-Trees [Am] (*Insert*, *Delete*)	$O\left(\log_B\frac{N}{B}\right)$	$O\left(\log_B\frac{N}{B}\right)$
Priority Queue [Am] (*Insert*, *Weak Delete*, *Delete-Min*)	$O\left(\frac{1}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$	$O\left(\frac{1}{B}\log_{\frac{M}{B}}\frac{N}{B}\right)$
Matrix Multiplication	$O\left(\frac{N^3}{B\sqrt{M}}\right)$	$O\left(\frac{N^3}{B\sqrt{M}}\right)$
Sequence Alignment	$O\left(\frac{N^2}{BM}\right)$	$O\left(\frac{N^2}{BM}\right)$
Single Source Shortest Paths	$O\left(\left(V+\frac{E}{B}\right)\cdot\log_2\frac{V}{B}\right)$	$O\left(\left(V+\frac{E}{B}\right)\cdot\log_2\frac{V}{B}\right)$
Minimum Spanning Forest	$O\left(\min\left(sort\left(E\right)\log_2\log_2 V,\ V+sort\left(E\right)\right)\right)$	$O\left(\min\left(sort\left(E\right)\log_2\log_2\frac{VB}{E},\ V+sort\left(E\right)\right)\right)$

Table 1: N = #elements, V = #vertices, E = #edges, Am = Amortized.

Matrix Multiplication

Iterative Matrix Multiplication

$$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$$

$$\begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{bmatrix} \times \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}$$

Iter-MM (X, Y, Z, n)

1. *for* $i \leftarrow 1$ *to* n *do*
2.   *for* $j \leftarrow 1$ *to* n *do*
3.   *for* $k \leftarrow 1$ *to* n *do*
4. $z_{ij} \leftarrow z_{ij} + x_{ik} \times y_{kj}$

Iterative Matrix Multiplication

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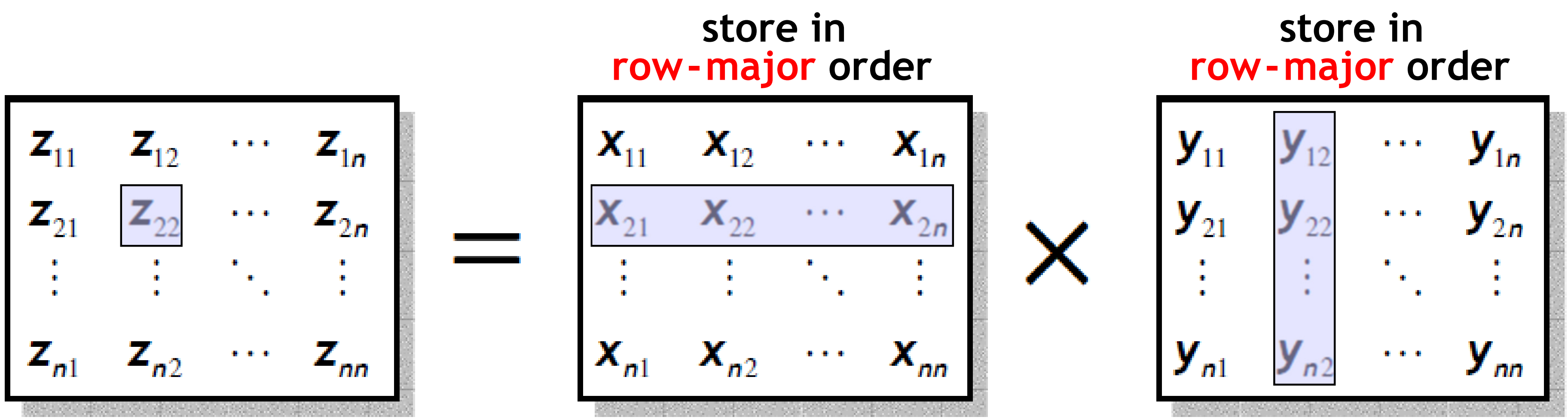


Each iteration of the *for* loop in line 3 incurs $O(n)$ cache misses.

I/O-complexity of *Iter-MM*, $Q(n) = O(n^3)$

Iterative Matrix Multiplication

```
Iter-MM ( X, Y, Z, n )
1.  for i ← 1 to n do
2.      for j ← 1 to n do
3.          for k ← 1 to n do
4.              z_ij ← z_ij + x_ik × y_kj
```

store in **row-major** order

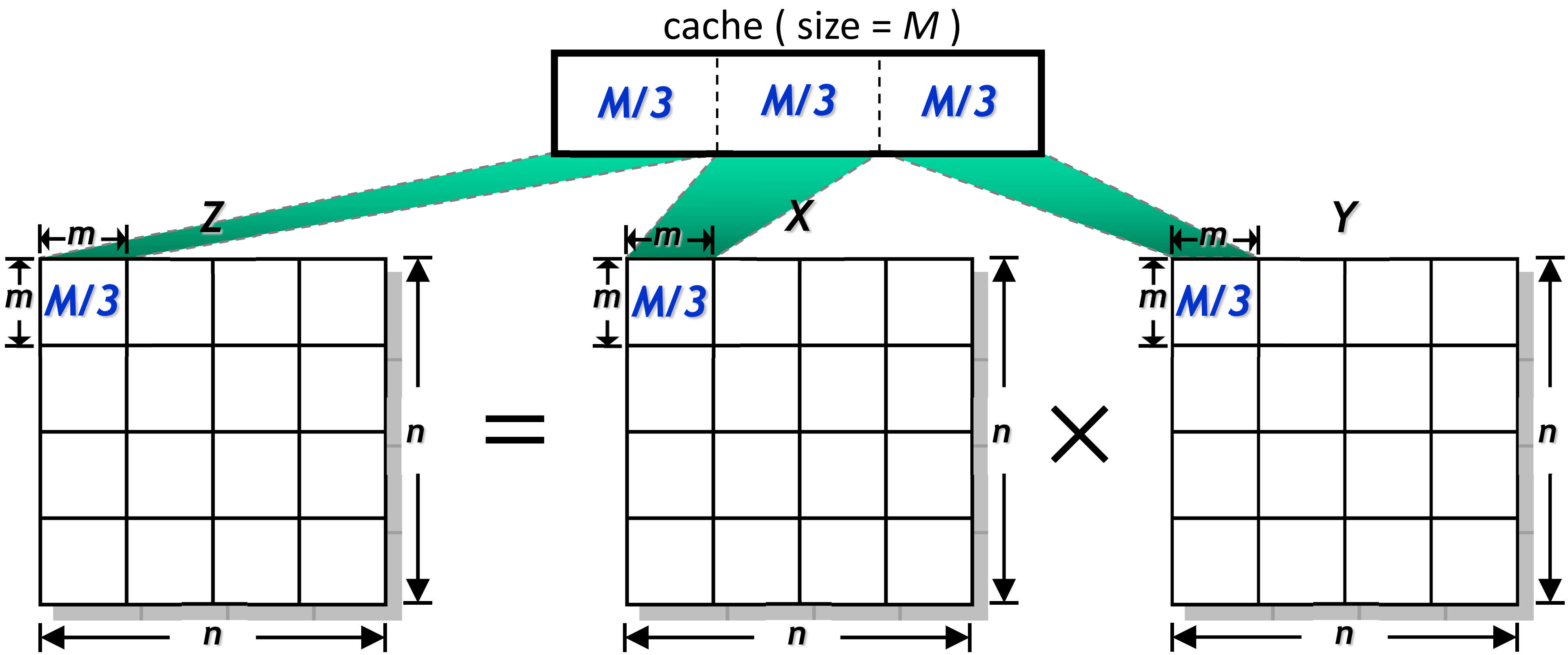
store in **column-major** order

$$\begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nn} \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \times \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}$$

Each iteration of the *for* loop in line 3 incurs $O\left(1+\frac{n}{B}\right)$ cache misses.

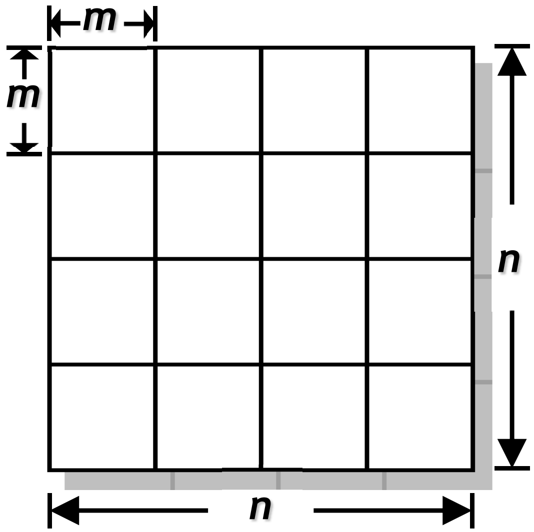
I/O-complexity of *Iter-MM*, $Q(n) = O\left(n^2\left(1+\frac{n}{B}\right)\right) = O\left(\frac{n^3}{B}+n^2\right)$

Block Matrix Multiplication




```
Block-MM ( X, Y, Z, n )
    for i ← 1 to n / m do
        for j ← 1 to n / m do
            for k ← 1 to n / m do
                Iter-MM ( X_ik, Y_kj, Z_ij )
```

Block Matrix Multiplication




```
Block-MM ( X, Y, Z, n )
1.  for i ← 1 to n / m do
2.      for j ← 1 to n / m do
3.          for k ← 1 to n / m do
4.              Iter-MM ( X_ik, Y_kj, Z_ij )
```

Choose $m = \sqrt{M/3}$, so that X_{ik}, Y_{kj} and Z_{ij} just fit into the cache.

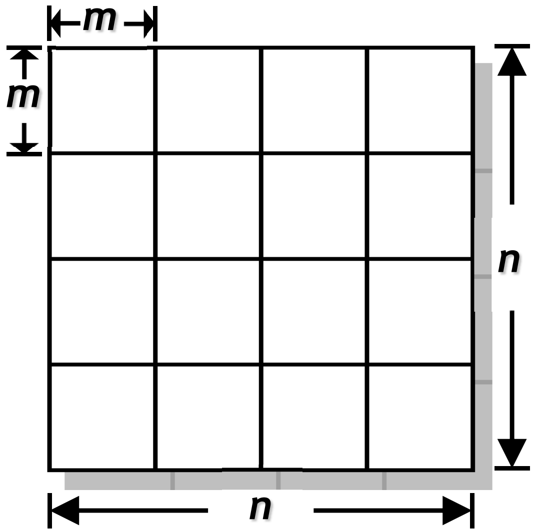
Then line 4 incurs $\Theta\left(m\left(1 + \frac{m}{B}\right)\right)$ cache misses.

I/O-complexity of *Block-MM* [assuming a *tall cache*, i.e., $M = \Omega(B^2)$]

$$= \Theta\left(\left(\frac{n}{m}\right)^3 \left(m + \frac{m^2}{B}\right)\right) = \Theta\left(\frac{n^3}{m^2} + \frac{n^3}{Bm}\right) = \Theta\left(\frac{n^3}{M} + \frac{n^3}{B\sqrt{M}}\right) = \Theta\left(\frac{n^3}{B\sqrt{M}}\right)$$

(Optimal: Hong & Kung, STOC'81)

Block Matrix Multiplication




```
Block-MM( X, Y, Z, n )
1. for i ← 1 to n / m do
2.     for j ← 1 to n / m do
3.         for k ← 1 to n / m do
4.             Iter-MM( X_ik, Y_kj, Z_ij )
```

Choose $m = \sqrt{M/2}$ so that X, Y, and Z just fit into the cache.

The … ses.

I/O- … cache, i.e., $M = \Omega(B^2)$]

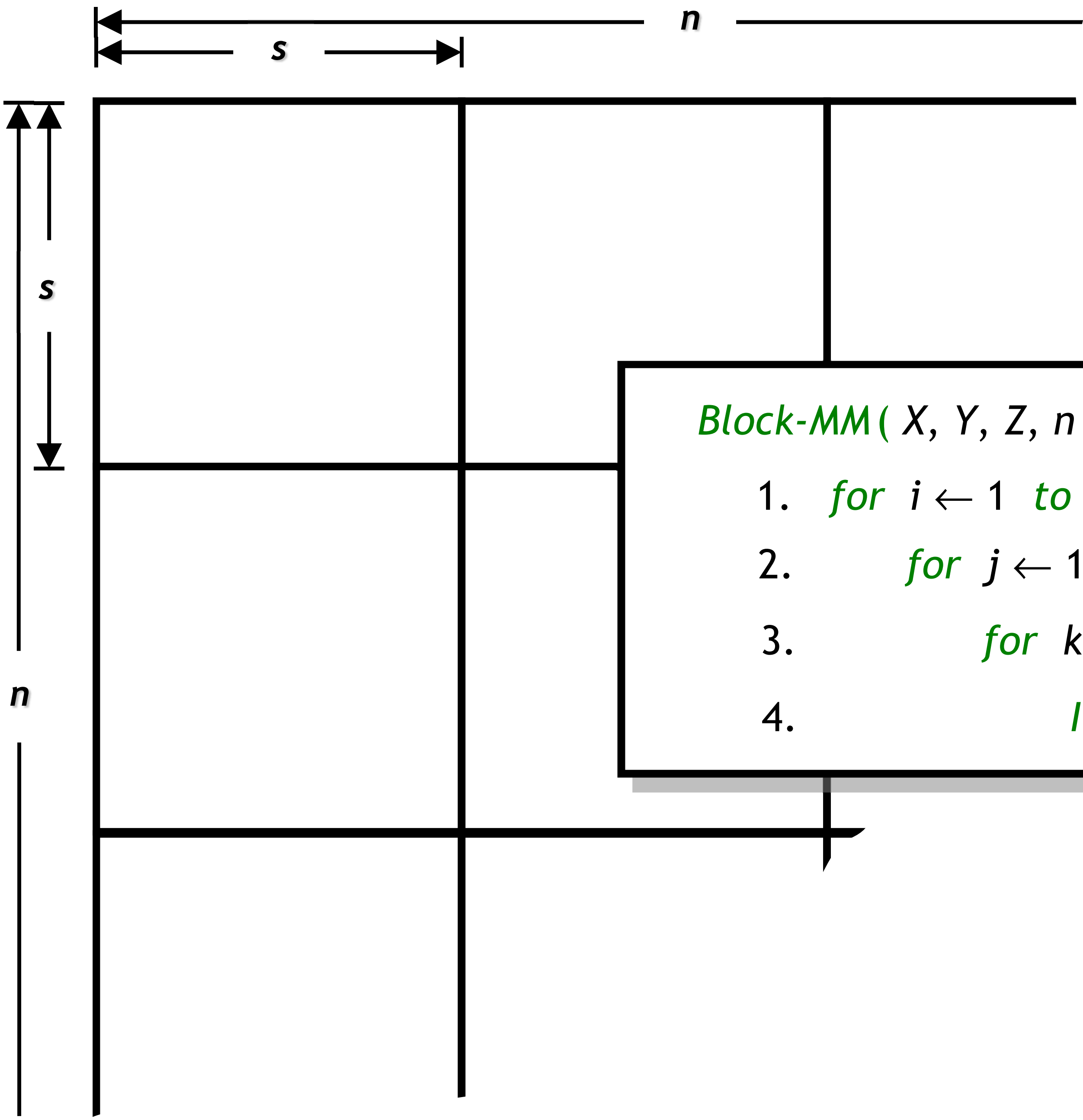
$$= \Theta\left(\left(\frac{n}{m}\right)^3\left(m + \frac{m^2}{B}\right)\right) = \Theta\left(\frac{n^3}{m^2} + \frac{n^3}{Bm}\right) = \Theta\left(\frac{n^3}{M} + \frac{n^3}{B\sqrt{M}}\right) = \Theta\left(\frac{n^3}{B\sqrt{M}}\right)$$

Optimal for any algorithm that performs the operations given by the following definition of matrix multiplication:

$$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj}$$

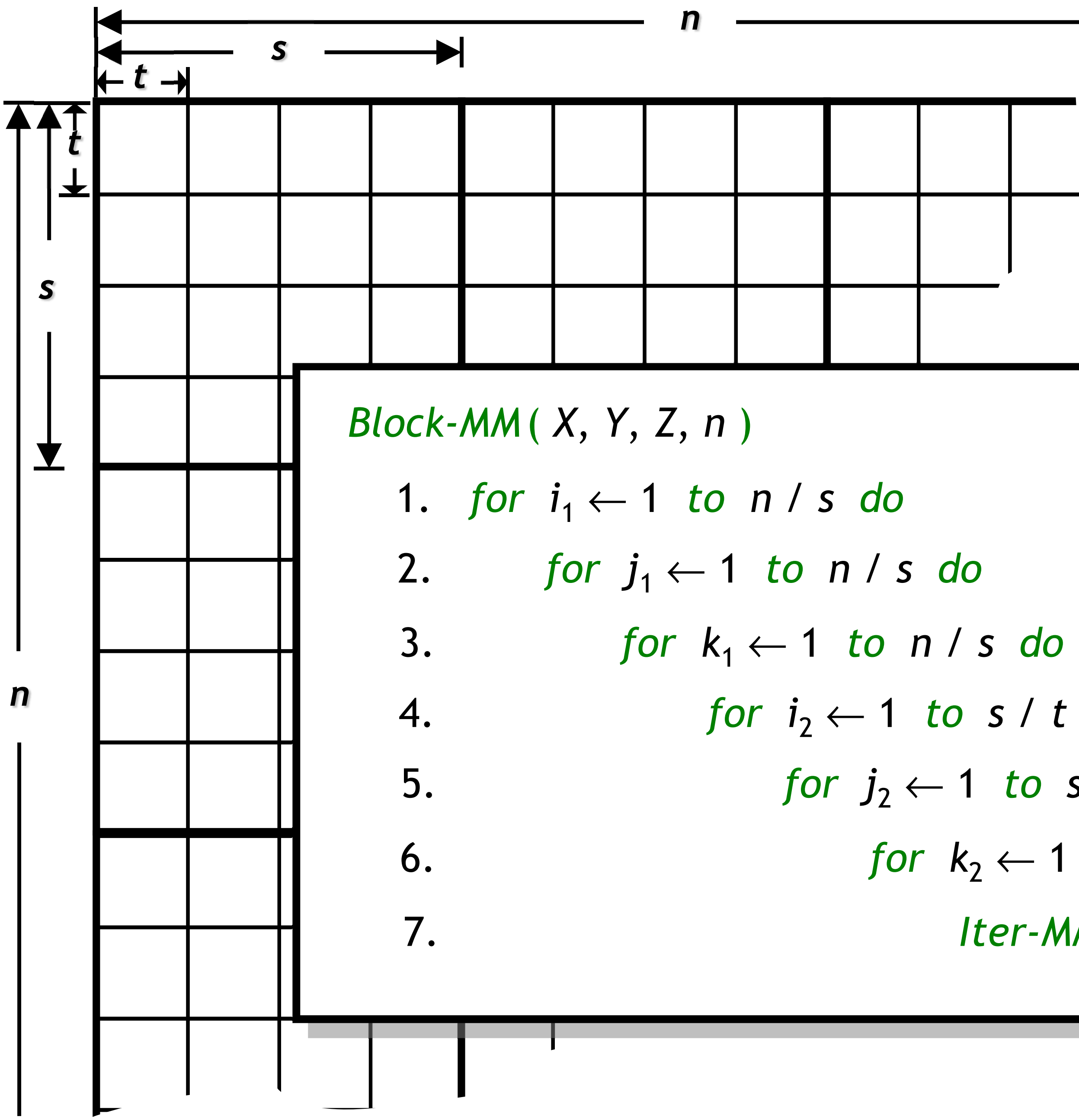
(Optimal: Hong & Kung, STOC'81)

Multiple Levels of Cache




```
Block-MM( X, Y, Z, n )
    for i ← 1 to n / s do
        for j ← 1 to n / s do
            for k ← 1 to n / s do
                Iter-MM( X_ik, Y_kj, Z_ij, s )
```

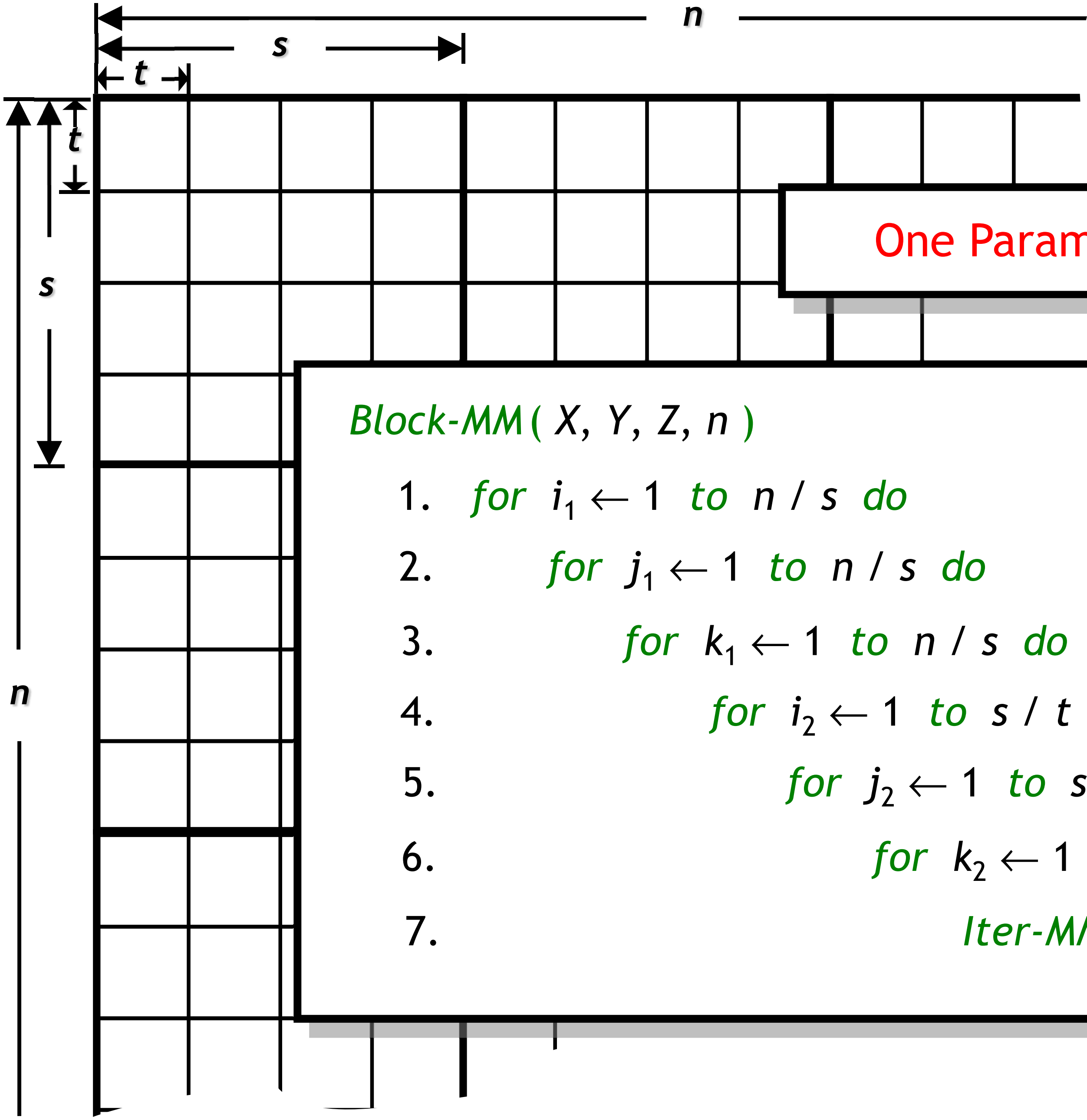
Multiple Levels of Cache



Block-MM (X, Y, Z, n)

1. *for* $i_1 \leftarrow 1$ *to* n / s *do*
2. *for* $j_1 \leftarrow 1$ *to* n / s *do*
3. *for* $k_1 \leftarrow 1$ *to* n / s *do*
4. *for* $i_2 \leftarrow 1$ *to* s / t *do*
5. *for* $j_2 \leftarrow 1$ *to* s / t *do*
6. *for* $k_2 \leftarrow 1$ *to* s / t *do*
7. *Iter-MM* ($(X_{i_1k_1})_{i_2k_2}$, $(Y_{k_1j_1})_{k_2j_2}$, $(X_{i_1j_1})_{i_2j_2}$, t)

Multiple Levels of Cache

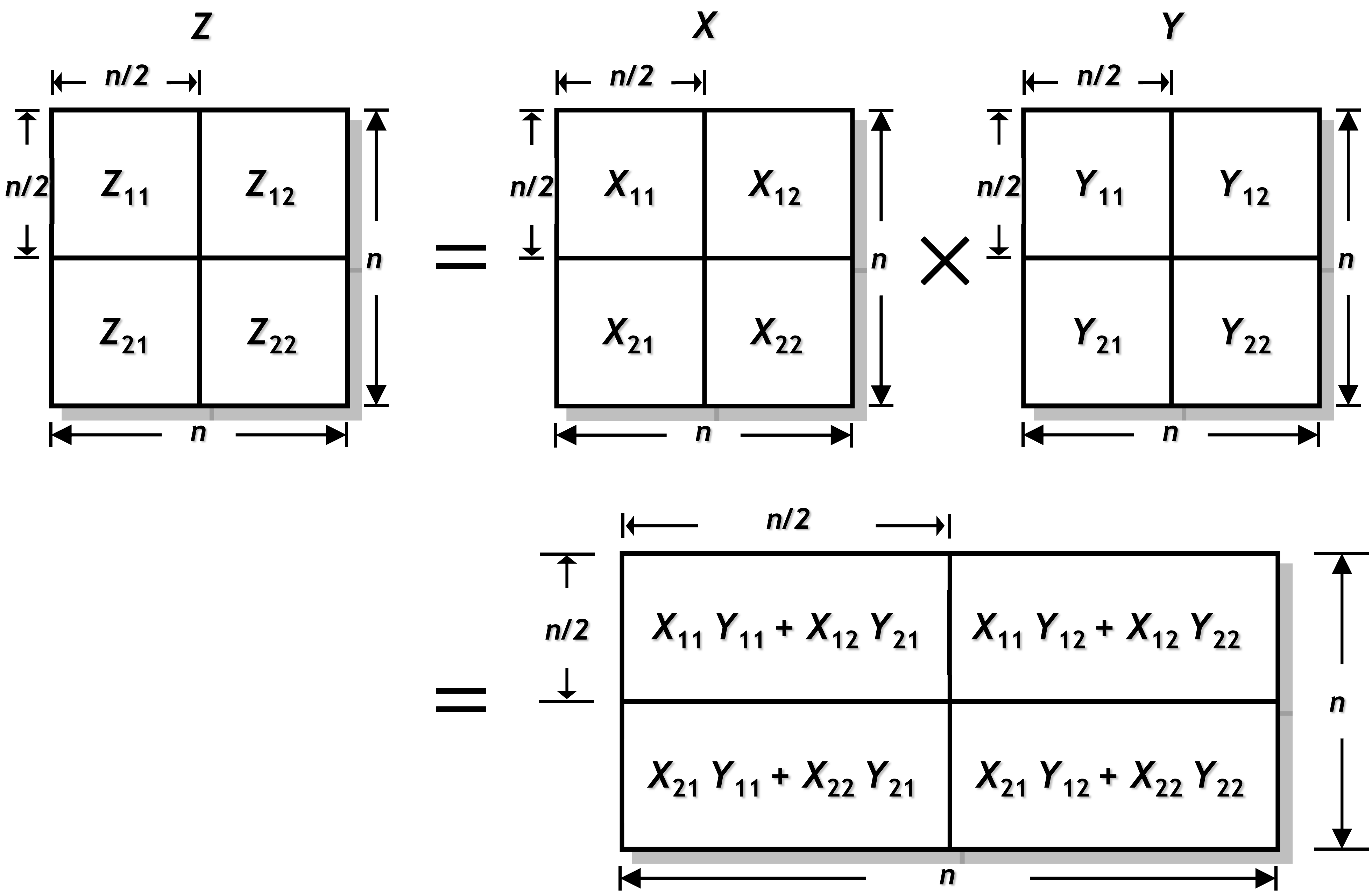


One Parameter Per Caching Level!

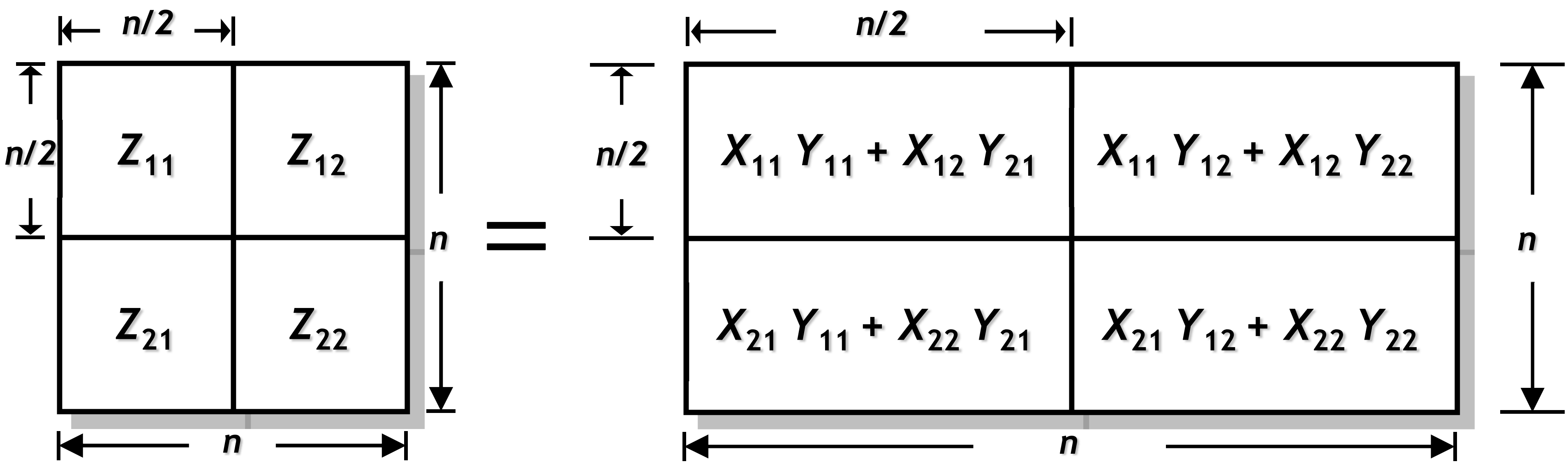
Block-MM (X, Y, Z, n)

for $i_1 \leftarrow 1$ *to* n / s *do*
  for $j_1 \leftarrow 1$ *to* n / s *do*
    for $k_1 \leftarrow 1$ *to* n / s *do*
      for $i_2 \leftarrow 1$ *to* s / t *do*
        for $j_2 \leftarrow 1$ *to* s / t *do*
          for $k_2 \leftarrow 1$ *to* s / t *do*
            Iter-MM ($(X_{i_1k_1})_{i_2k_2}$, $(Y_{k_1j_1})_{k_2j_2}$, $(X_{i_1j_1})_{i_2j_2}$, t)

Recursive Matrix Multiplication



Recursive Matrix Multiplication



Rec-MM (Z, X, Y)

if $Z \equiv 1 \times 1$ matrix *then* $Z \leftarrow Z + X \cdot Y$

else

Rec-MM (Z_{11}, X_{11}, Y_{11}), *Rec-MM* (Z_{11}, X_{12}, Y_{21})

Rec-MM (Z_{12}, X_{12}, Y_{12}), *Rec-MM* (Z_{12}, X_{12}, Y_{22})

Rec-MM (Z_{21}, X_{21}, Y_{11}), *Rec-MM* (Z_{21}, X_{22}, Y_{21})

Rec-MM (Z_{22}, X_{21}, Y_{12}), *Rec-MM* (Z_{22}, X_{22}, Y_{22})

Recursive Matrix Multiplication

Rec-MM (Z, X, Y)

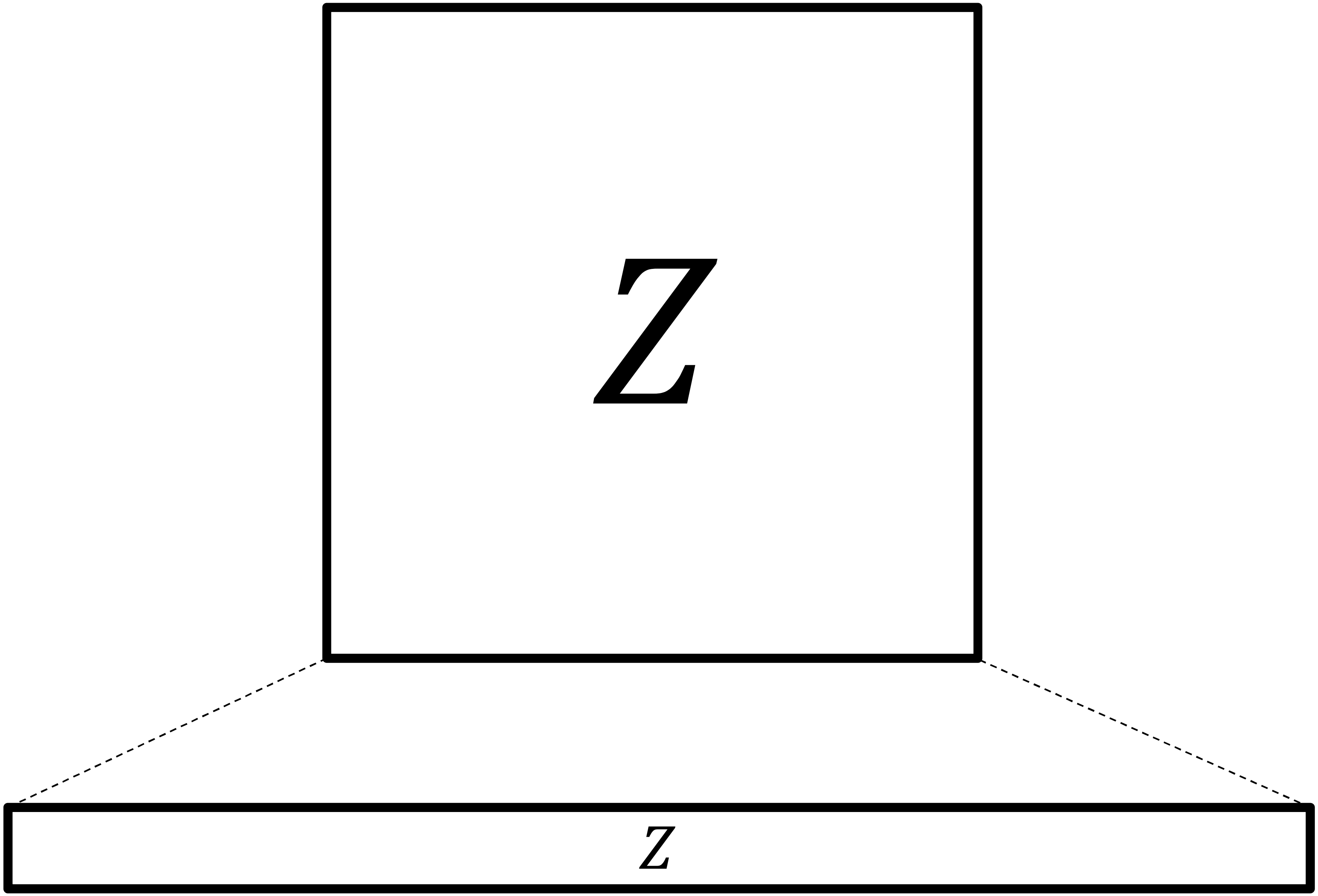
1. *if* $Z \equiv 1 \times 1$ matrix *then* $Z \leftarrow Z + X \cdot Y$
2. *else*
3. *Rec-MM* (Z_{11}, X_{11}, Y_{11}), *Rec-MM* (Z_{11}, X_{12}, Y_{21})
4. *Rec-MM* (Z_{12}, X_{12}, Y_{12}), *Rec-MM* (Z_{12}, X_{12}, Y_{22})
5. *Rec-MM* (Z_{21}, X_{21}, Y_{11}), *Rec-MM* (Z_{21}, X_{22}, Y_{21})
6. *Rec-MM* (Z_{22}, X_{21}, Y_{12}), *Rec-MM* (Z_{22}, X_{22}, Y_{22})

I/O-complexity (for $n > M$), $Q(n) = \begin{cases} O\left(n + \frac{n^2}{B}\right), & if\ n^2 \le \alpha M \\ 8Q\left(\frac{n}{2}\right) + O(1), & otherwise \end{cases}$

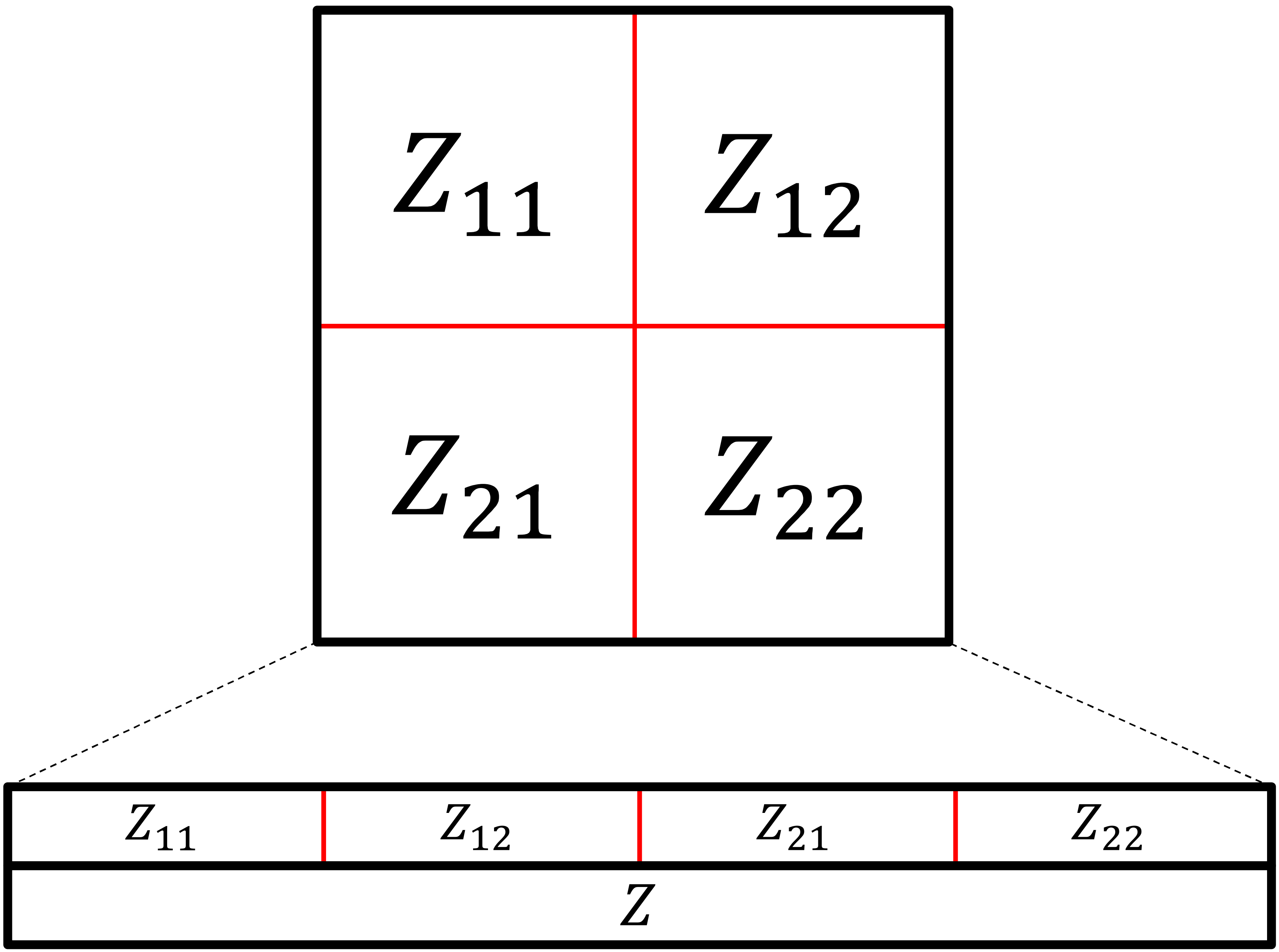
$$= O\left(\frac{n^3}{M} + \frac{n^3}{B\sqrt{M}}\right) = O\left(\frac{n^3}{B\sqrt{M}}\right), when\ M = \Omega(B^2)$$

I/O-complexity (for all n) $= O\left(\frac{n^3}{B\sqrt{M}} + \frac{n^2}{B} + 1\right)$ (why?)

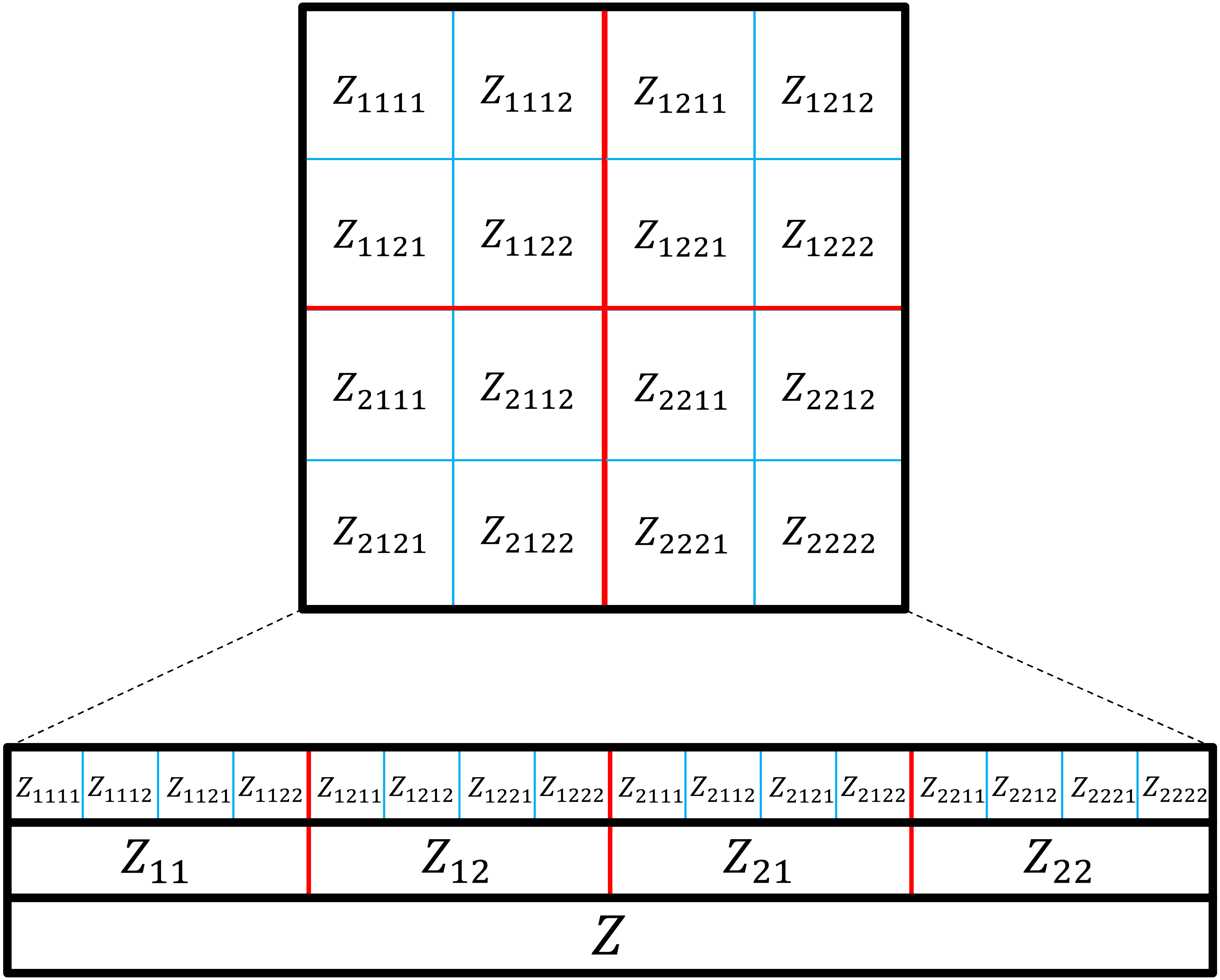
Recursive Matrix Multiplication with Z-Morton Layout



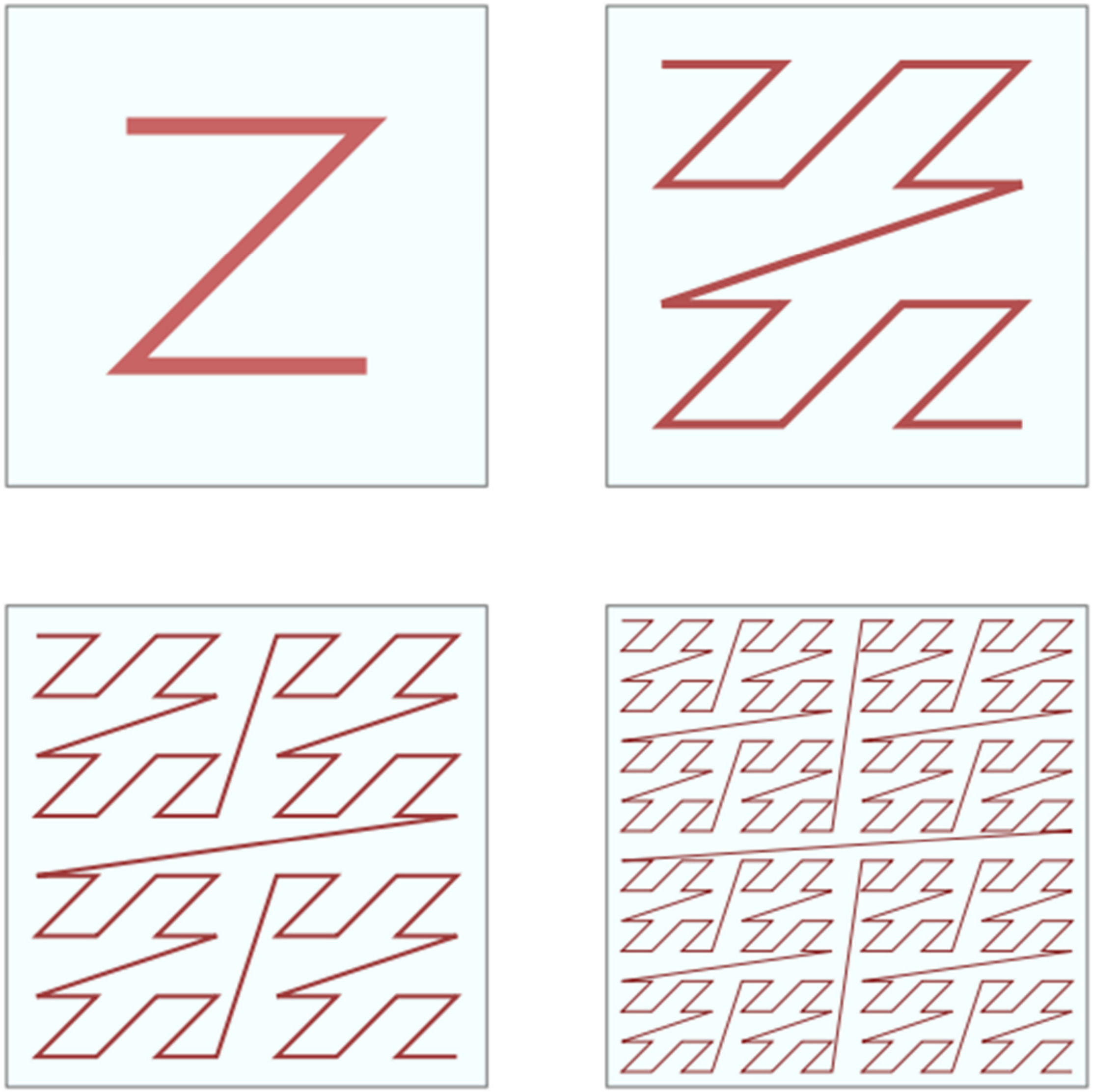
Recursive Matrix Multiplication with Z-Morton Layout



Recursive Matrix Multiplication with Z-Morton Layout



Recursive Matrix Multiplication with Z-Morton Layout



Source: wikipedia

Recursive Matrix Multiplication with Z-Morton Layout

Rec-MM (Z, X, Y)

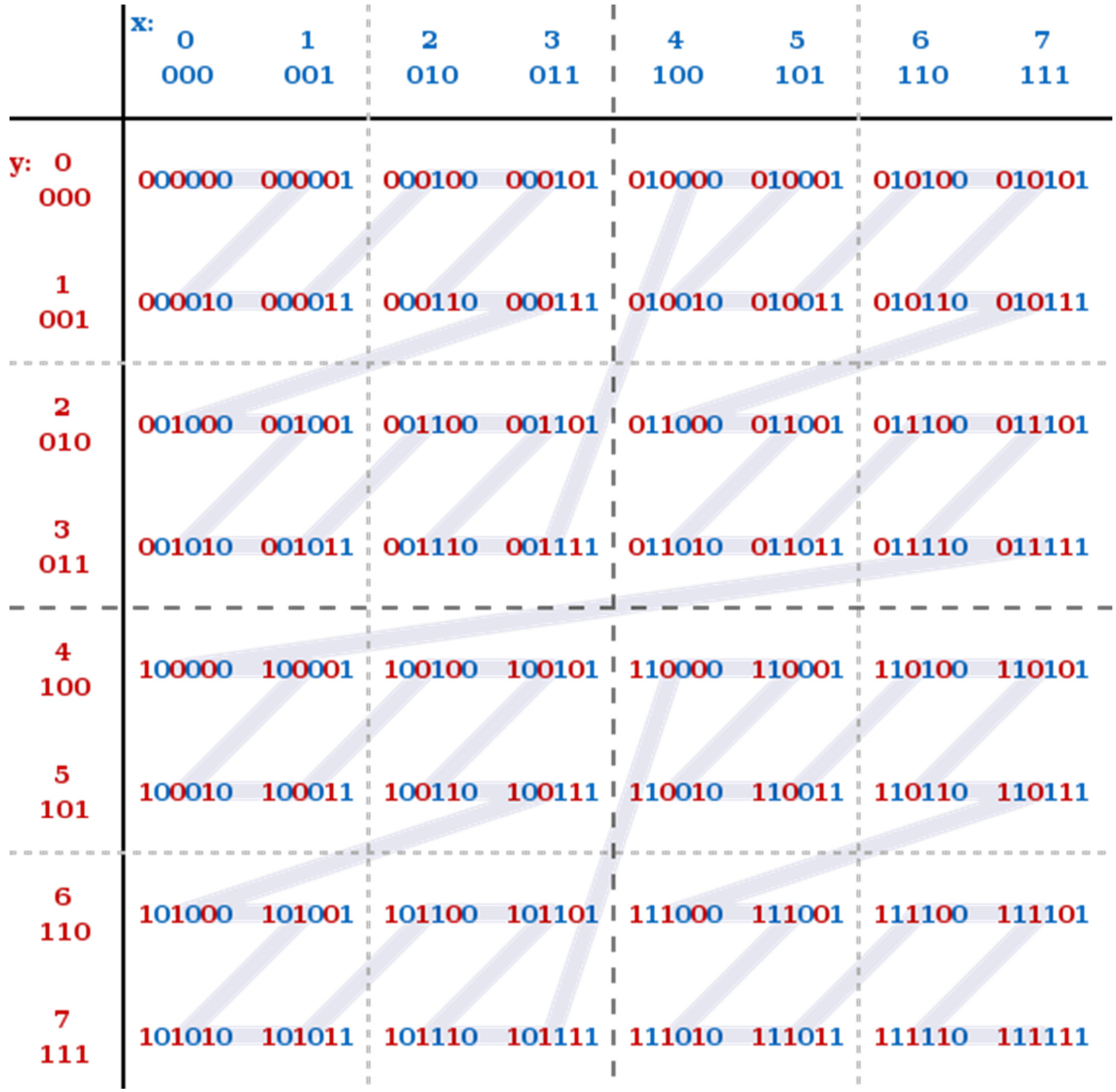
1. *if* $Z \equiv 1 \times 1$ matrix *then* $Z \leftarrow Z + X \cdot Y$
2. *else*
3. *Rec-MM* (Z_{11}, X_{11}, Y_{11}), *Rec-MM* (Z_{11}, X_{12}, Y_{21})
4. *Rec-MM* (Z_{12}, X_{12}, Y_{12}), *Rec-MM* (Z_{12}, X_{12}, Y_{22})
5. *Rec-MM* (Z_{21}, X_{21}, Y_{11}), *Rec-MM* (Z_{21}, X_{22}, Y_{21})
6. *Rec-MM* (Z_{22}, X_{21}, Y_{12}), *Rec-MM* (Z_{22}, X_{22}, Y_{22})

I/O-complexity (for $n > M$), $Q(n) = \begin{cases} O\left(1 + \frac{n^2}{B}\right), & if\ n^2 \leq \alpha M \\ 8Q\left(\frac{n}{2}\right) + O(1), & otherwise \end{cases}$

$$= O\left(\frac{n^3}{M\sqrt{M}} + \frac{n^3}{B\sqrt{M}}\right) = O\left(\frac{n^3}{B\sqrt{M}}\right), when\ M = \Omega(B)$$

I/O-complexity (for all n) $= O\left(\frac{n^3}{B\sqrt{M}} + \frac{n^2}{B} + 1\right)$

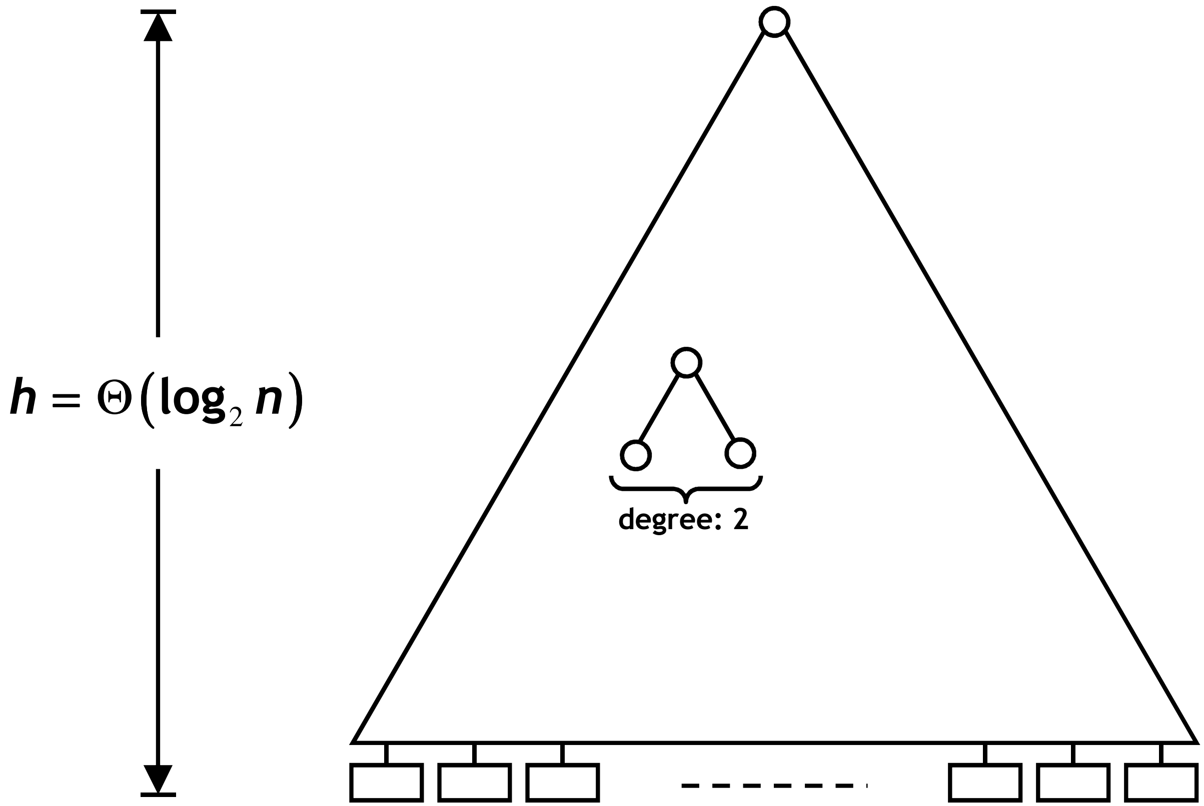
Recursive Matrix Multiplication with Z-Morton Layout



Source: wikipedia

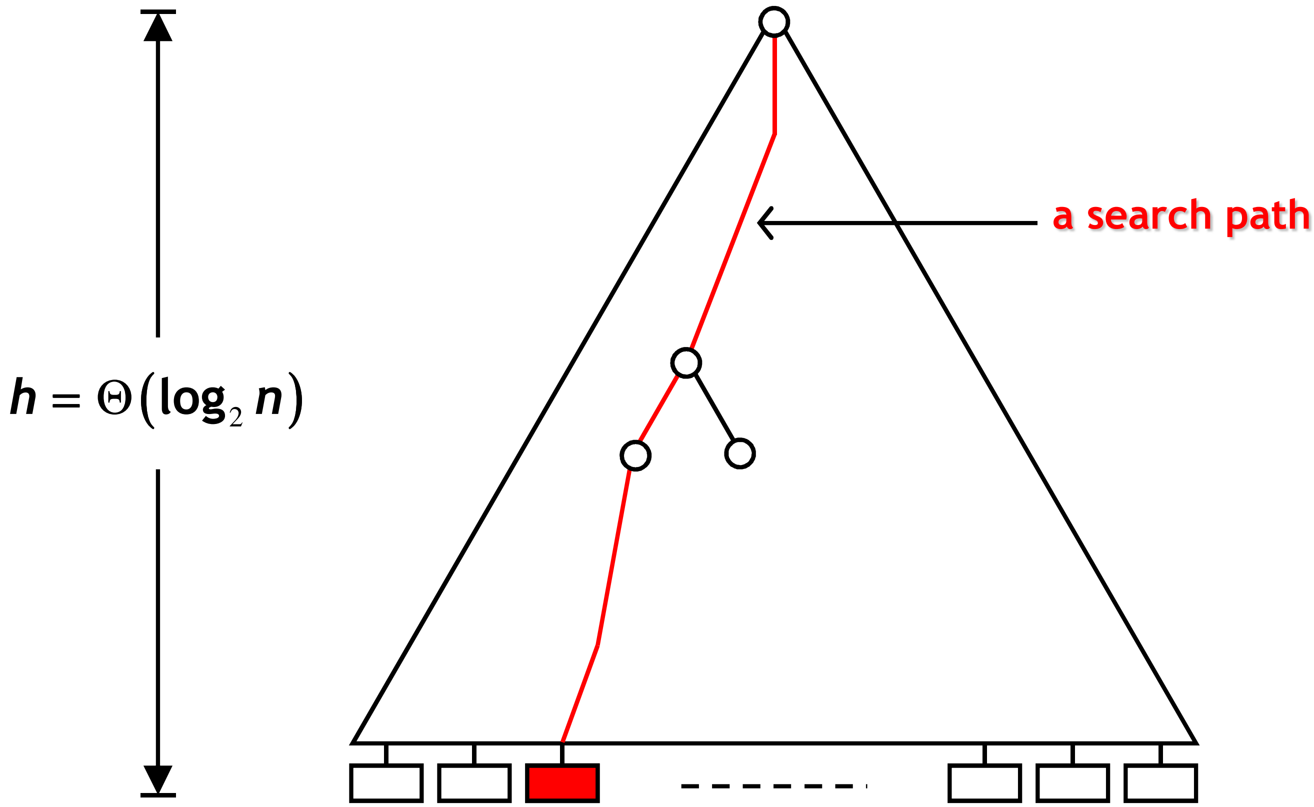
Searching
(Static B-Trees)

A Static Search Tree



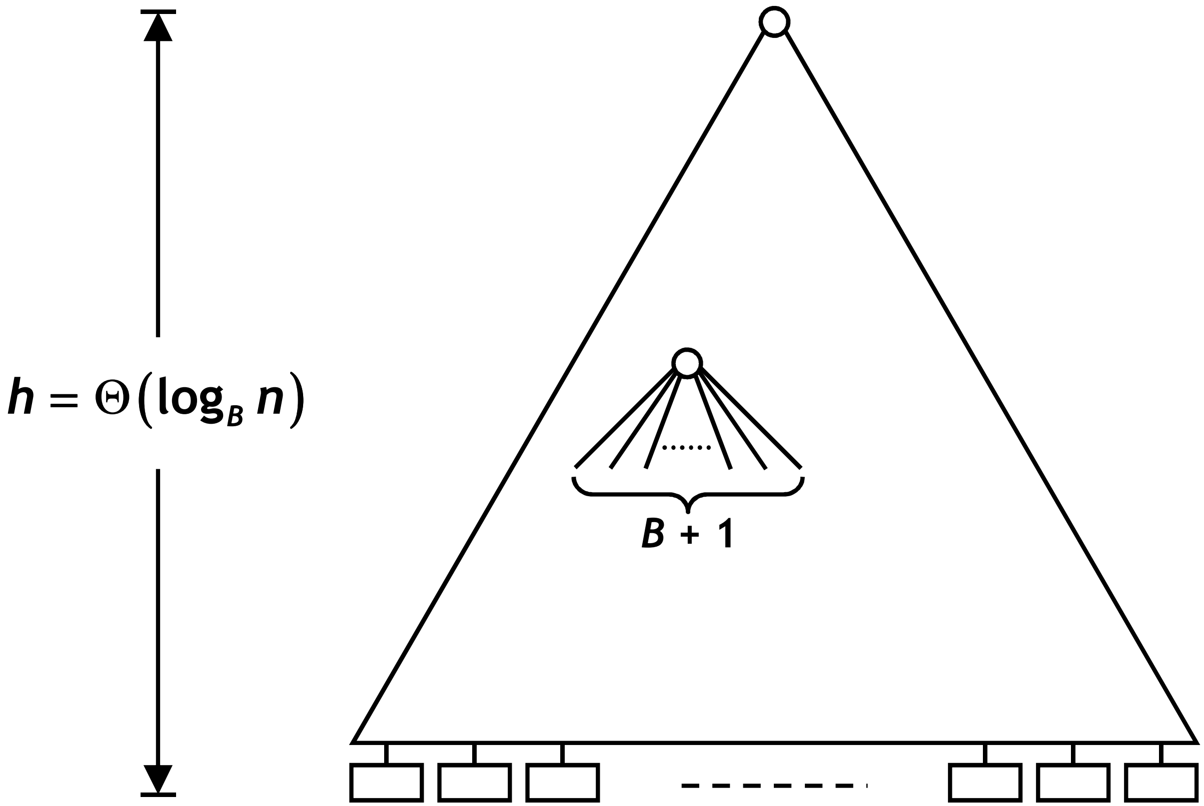
- A perfectly balanced binary search tree
- Static: no insertions or deletions
- Height of the tree, $h = \Theta(\log_2 n)$

A Static Search Tree



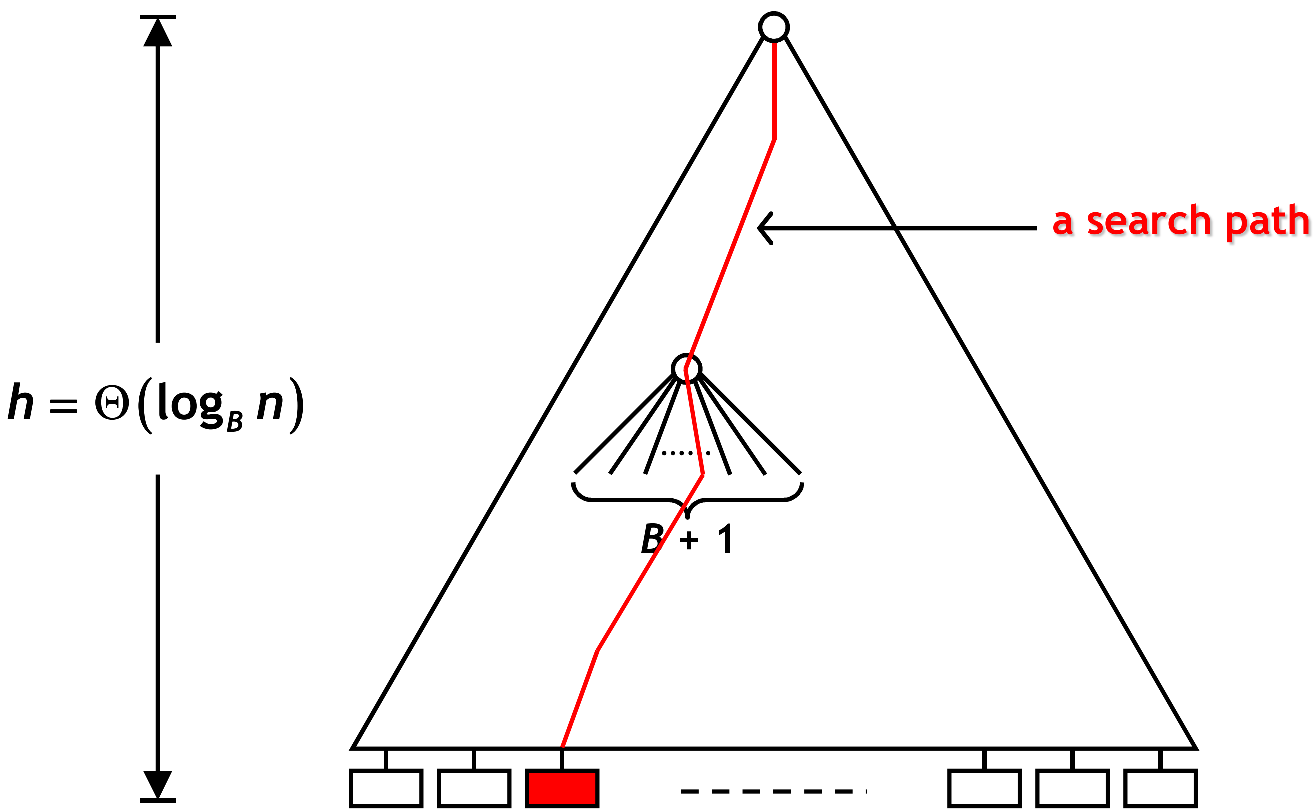
- A perfectly balanced binary search tree
- Static: no insertions or deletions
- Height of the tree, $h = \Theta(\log_2 n)$
- A search path visits $O(h)$ nodes, and incurs $O(h) = O(\log_2 n)$ I/Os

I/O-Efficient Static B-Trees



- Each node stores B keys, and has degree $B + 1$
- Height of the tree, $h = \Theta(\log_B n)$

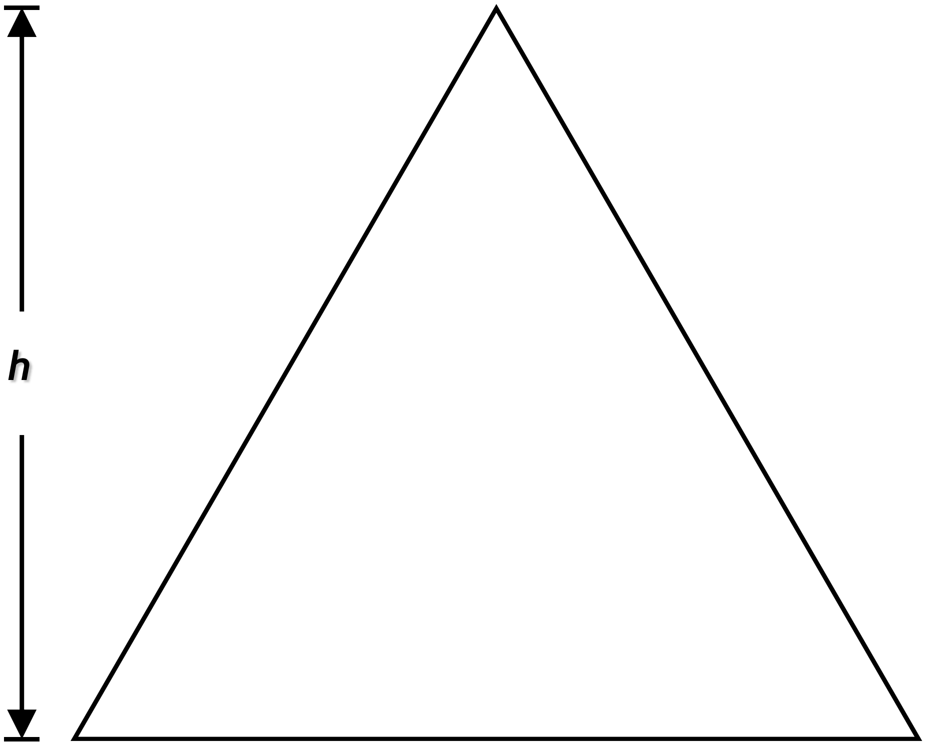
I/O-Efficient Static B-Trees



- Each node stores B keys, and has degree $B + 1$
- Height of the tree, $h = \Theta(\log_B n)$
- A search path visits $O(h)$ nodes, and incurs $O(h) = O(\log_B n)$ I/Os

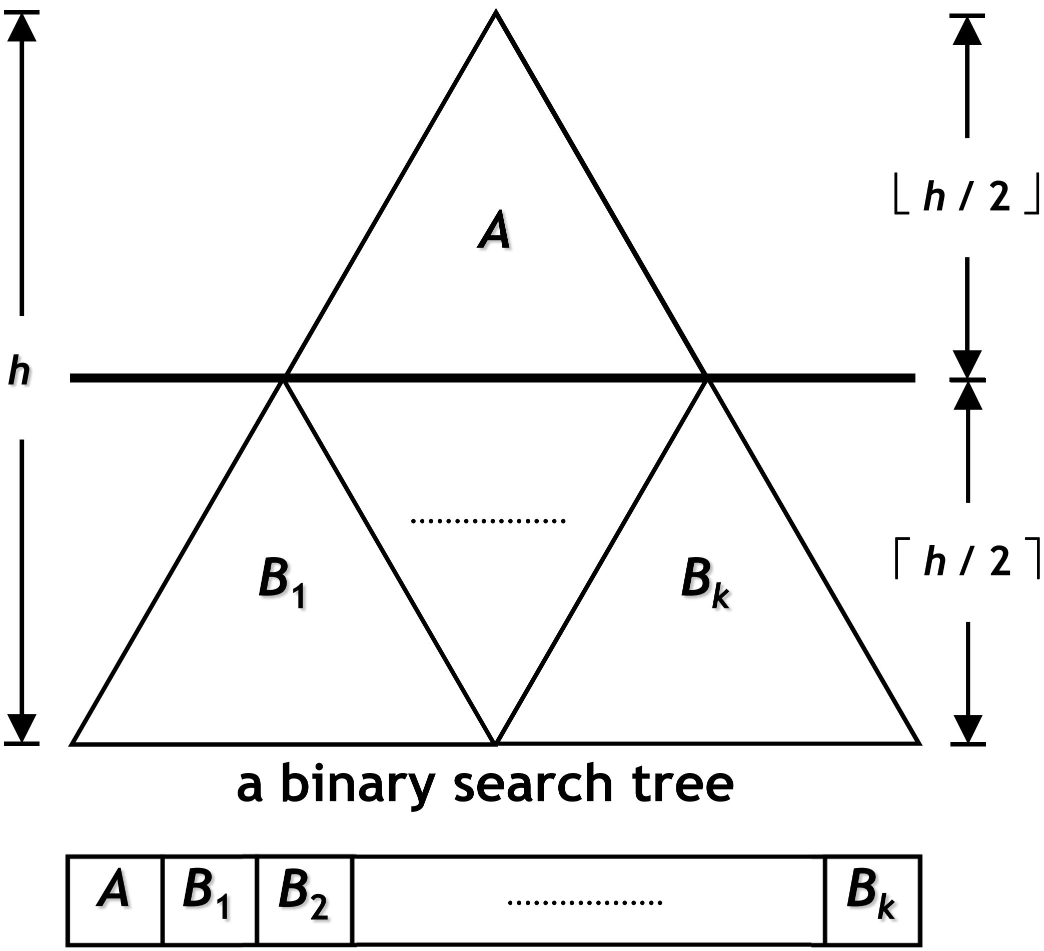
Cache-Oblivious Static B-Trees?

van Emde Boas Layout



a binary search tree

van Emde Boas Layout

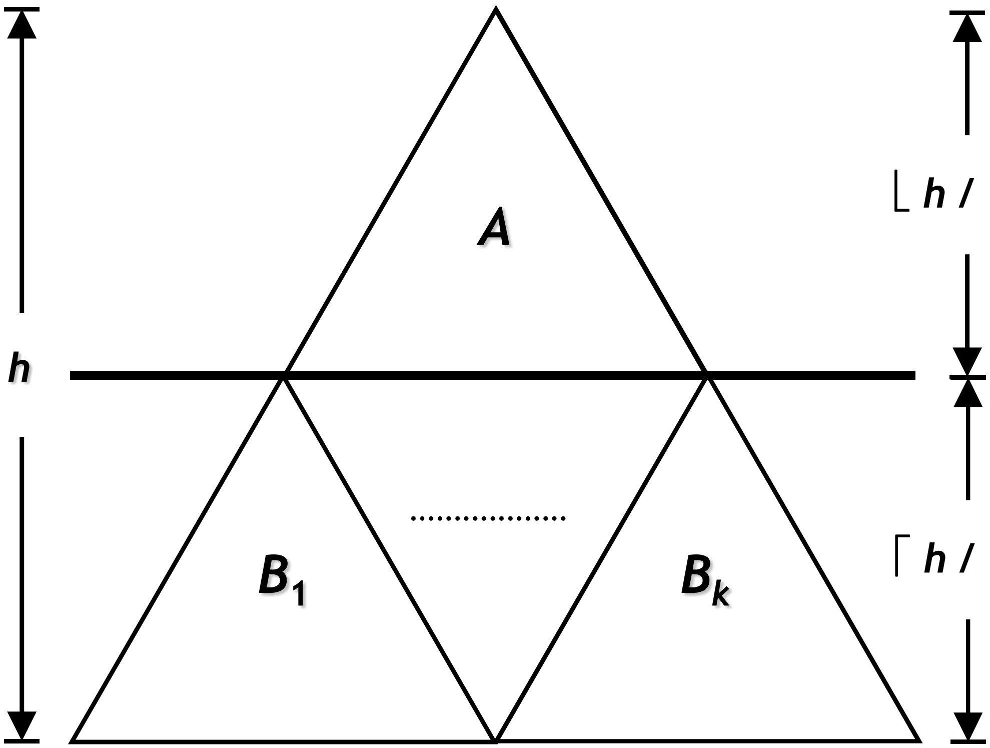


If the tree contains n nodes,

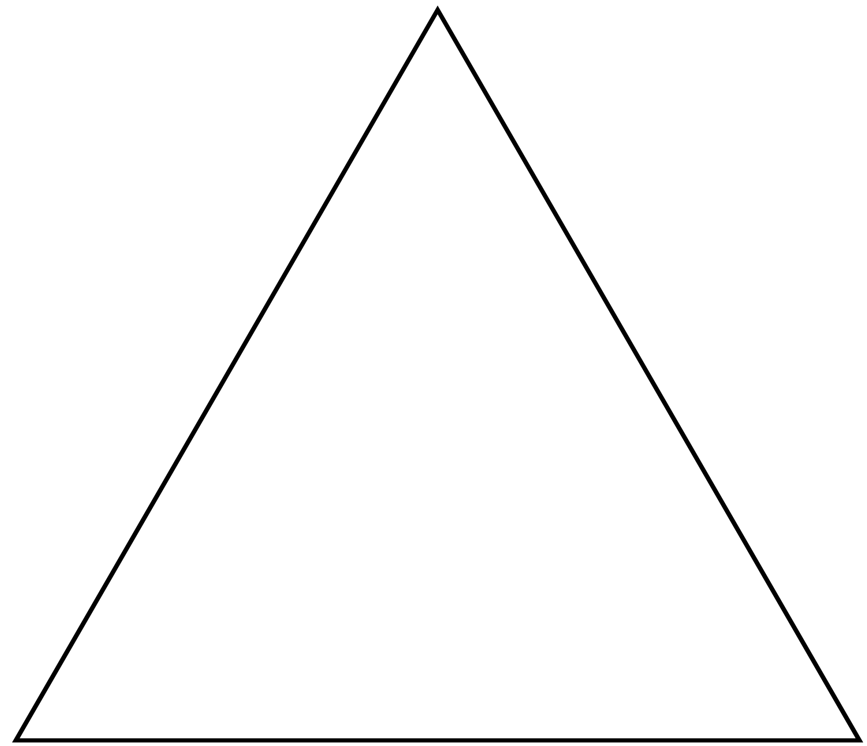
each subtree contains $\Theta\left(2^{h/2}\right) = \Theta(\sqrt{n})$ nodes,

and $k = \Theta(\sqrt{n})$.

van Emde Boas Layout



a binary search tree



Recursive Subdivision

If the tree contains n nodes,

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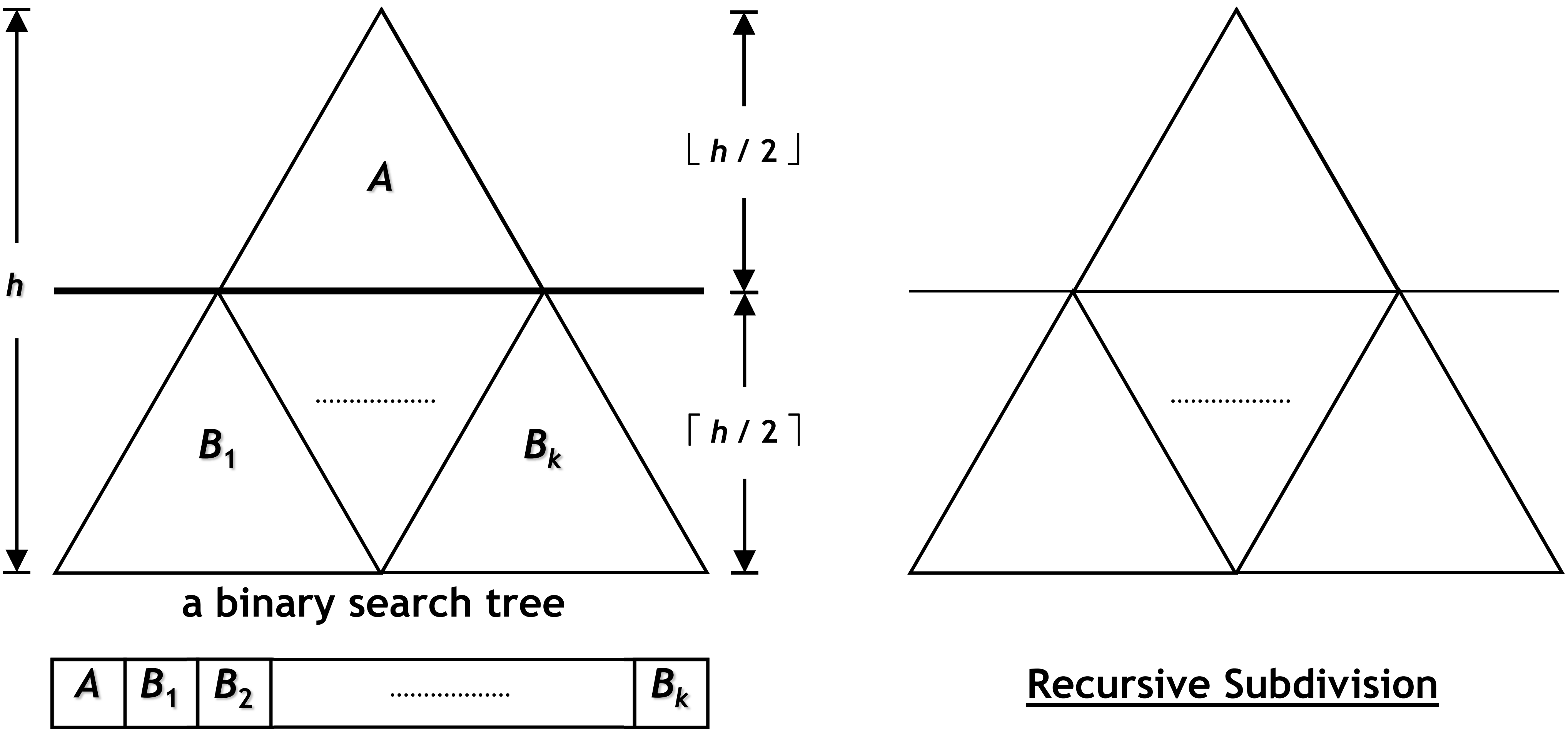
van Emde Boas Layout



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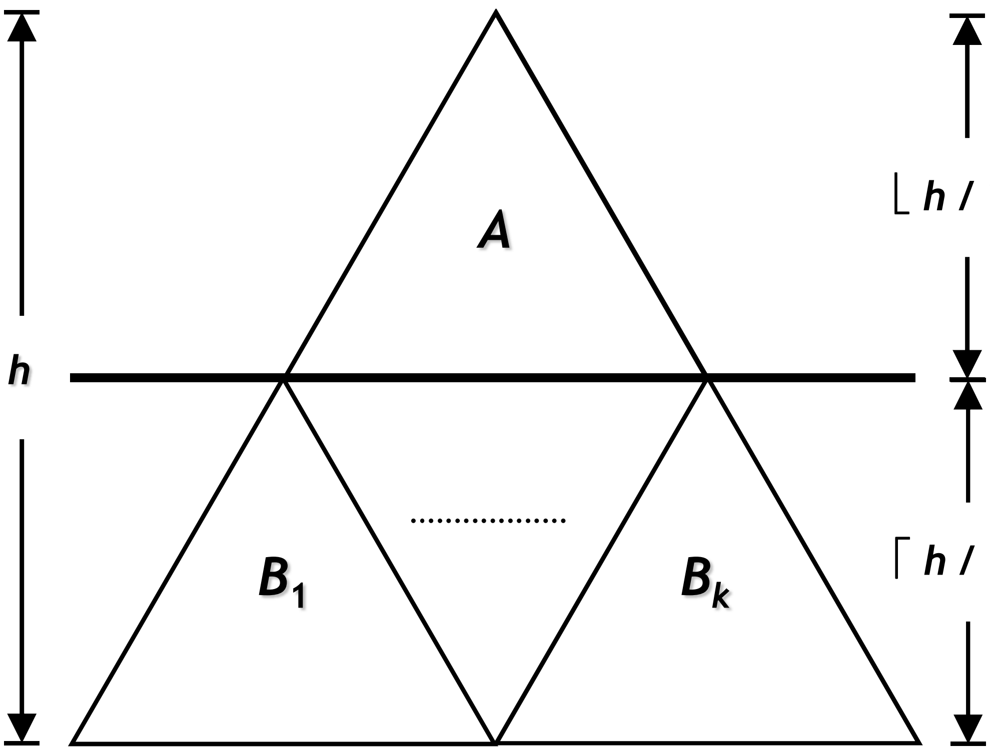
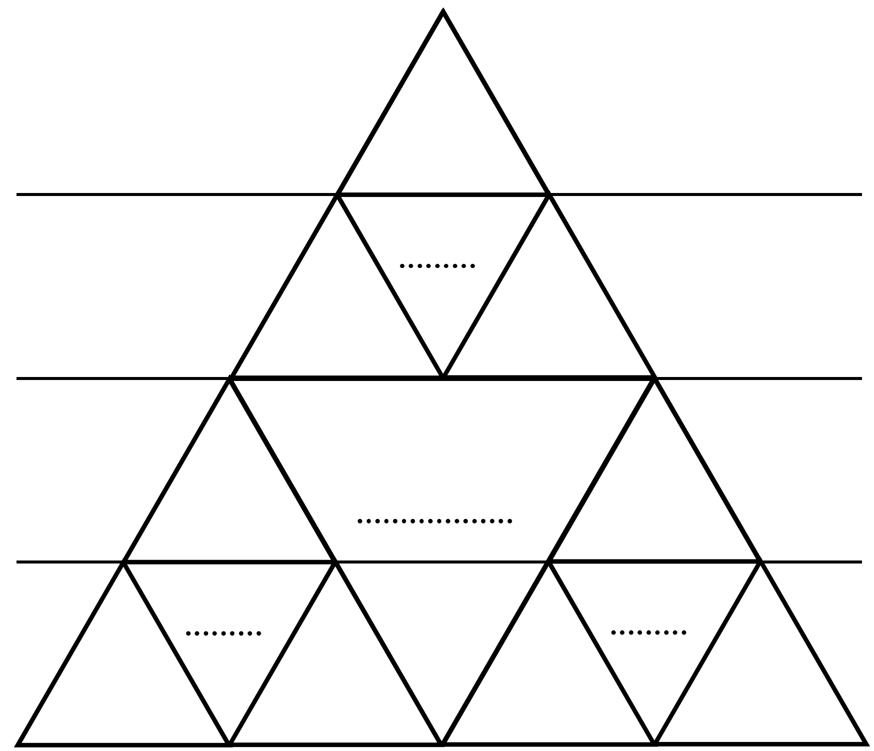
van Emde Boas Layout



a binary search tree



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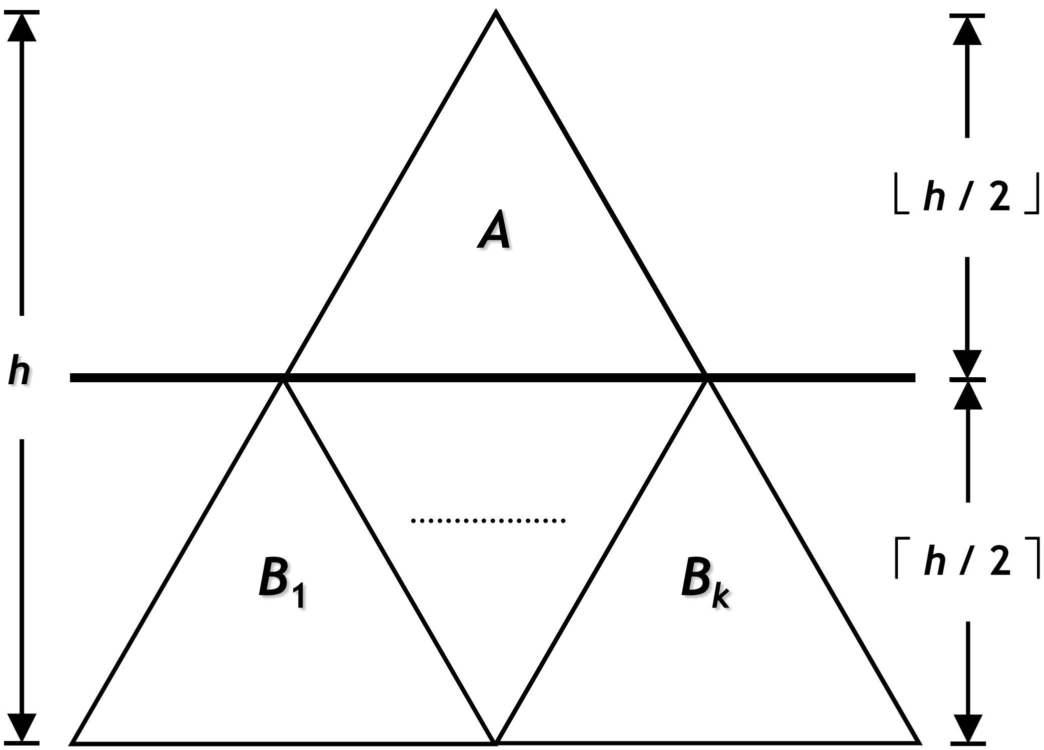

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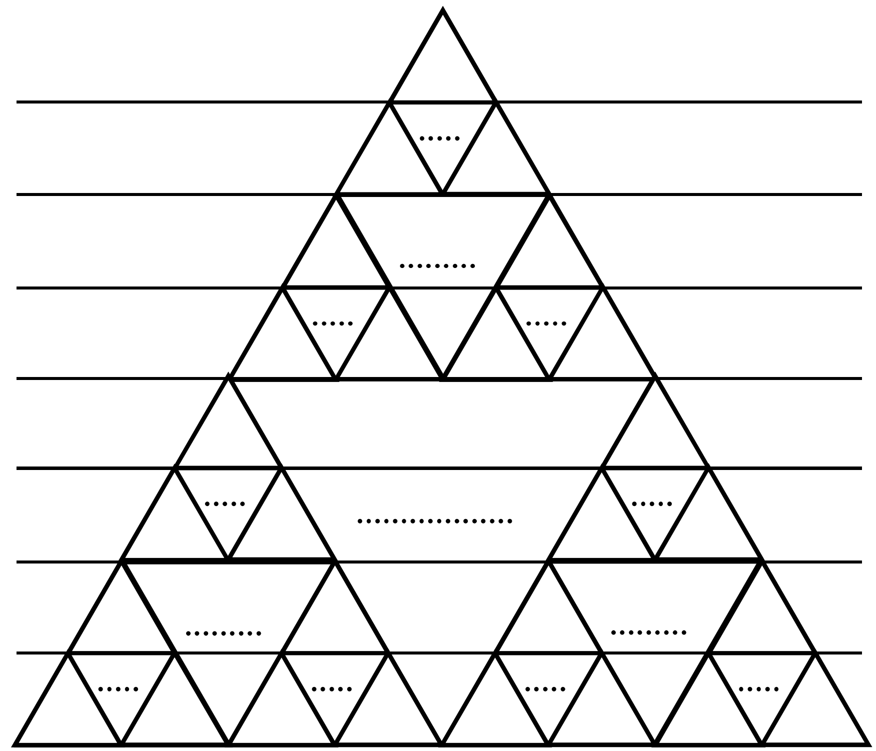

Recursive Subdivision

If the tree contains n nodes,

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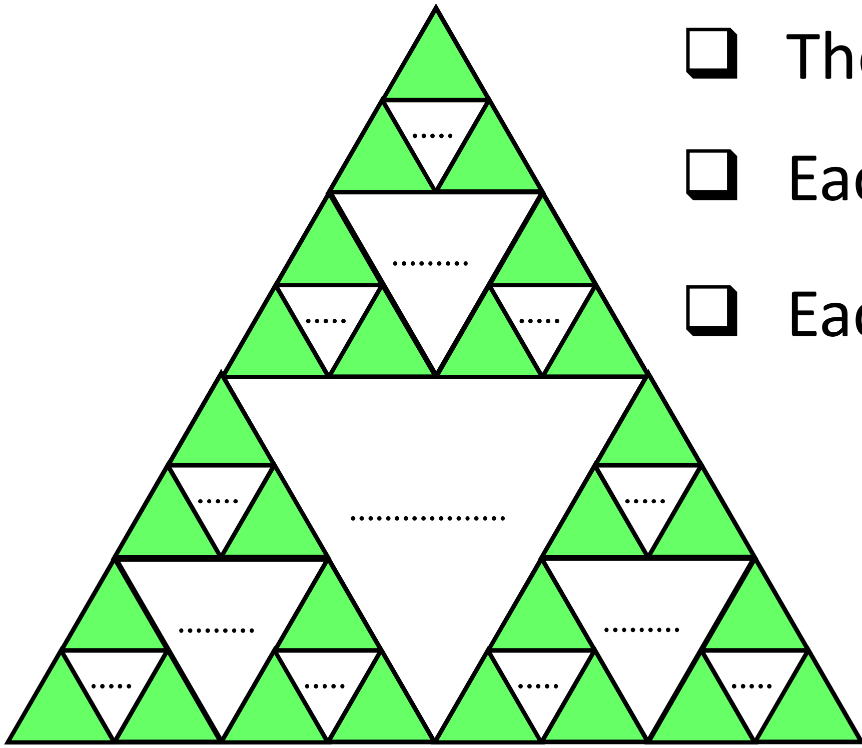
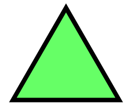
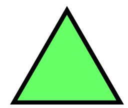
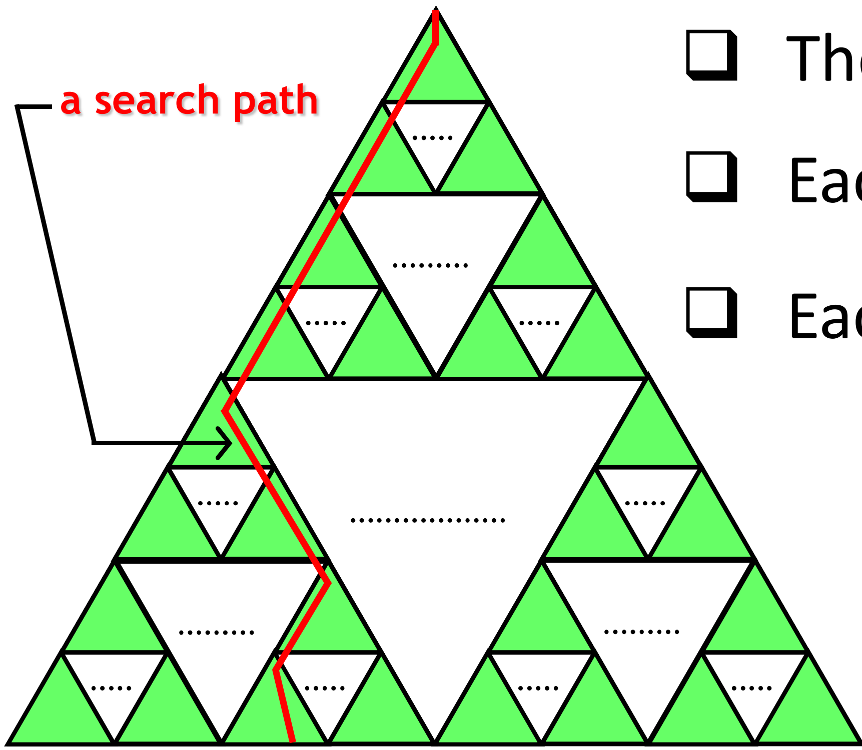
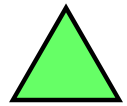
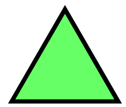
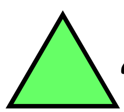
I/O-Complexity of a Search



- The height of the tree is $\log n$
- Each  has height between $\frac{1}{2}\log B$ & $\log B$.
- Each  spans at most 2 blocks of size B.

I/O-Complexity of a Search



- The height of the tree is $\log n$
- Each  has height between $\frac{1}{2}\log B$ & $\log B$.
- Each  spans at most 2 blocks of size B.
- p = number of 's visited by a search path
- Then $p \geq \frac{\log n}{\log B} = \log_B n$, and $p \leq \frac{\log n}{\frac{1}{2}\log B} = 2\log_B n$
- The number of blocks transferred is $\leq 2 \times 2\log_B n = 4\log_B n$

Sorting
(Mergesort)

Merge Sort

Merge-Sort (A, p, r) { *sort the elements in* $A[\, p \ldots r \,]$ }

1. *if* $p < r$ *then*
2. $q \leftarrow \lfloor (p + r) / 2 \rfloor$
3. *Merge-Sort* (A, p, q)
4. *Merge-Sort* ($A, q + 1, r$)
5. *Merge* (A, p, q, r)

Merging *k* Sorted Sequences

- $k \geq 2$ sorted sequences $S_1, S_2, \ldots, S_k$ stored in external memory
- $|S_i| = n_i$ for $1 \leq i \leq k$
- $n = n_1 + n_2 + \cdots + n_k$ is the length of the merged sequence S
- S (initially empty) will be stored in external memory
- Cache must be large enough to store
  - one block from each S_i
  - one block from S

Thus $M \geq (k+1)B$

Merging *k* Sorted Sequences

- Let $\mathcal{B}_i$ be the cache block associated with S_i, and let $\mathcal{B}$ be the block associated with S (initially all empty)
- Whenever a $\mathcal{B}_i$ is empty fill it up with the next block from S_i
- Keep transferring the next smallest element among all $\mathcal{B}_i$s to $\mathcal{B}$
- Whenever $\mathcal{B}$ becomes full, empty it by appending it to S
- In the *Ideal Cache Model* the block emptying and replacements will happen automatically $\Rightarrow$ cache-oblivious merging

I/O Complexity

- Reading S_i: #block transfers $\leq 2 + \frac{n_i}{B}$
- Writing S: #block transfers $\leq 1 + \frac{n}{B}$
- Total #block transfers $\leq 1 + \frac{n}{B} + \sum_{1 \leq i \leq k} \left(2 + \frac{n_i}{B} \right) = O\left(k + \frac{n}{B} \right)$

Cache-Oblivious 2-Way Merge Sort

```
Merge-Sort ( A, p, r )        { sort the elements in A[ p … r ] }

1. if p < r then
2.     q ← ⌊ ( p + r ) / 2 ⌋
3.     Merge-Sort ( A, p, q )
4.     Merge-Sort ( A, q + 1, r )
5.     Merge ( A, p, q, r )
```

I/O Complexity: $$Q(n) = \begin{cases} O\left(1 + \frac{n}{B}\right), & if\ n \leq M, \\ 2Q\left(\frac{n}{2}\right) + O\left(1 + \frac{n}{B}\right), & otherwise. \end{cases}$$

$$= O\left(\frac{n}{B} \log \frac{n}{M}\right)$$

How to improve this bound?

Cache-Oblivious *k*-Way Merge Sort

I/O Complexity: $Q(n) = \begin{cases} O\left(1 + \frac{n}{B}\right), & if\ n \leq M, \\ k \cdot Q\left(\frac{n}{k}\right) + O\left(k + \frac{n}{B}\right), & otherwise. \end{cases}$

$$= O\left(k \cdot \frac{n}{M} + \frac{n}{B} \log_k \frac{n}{M}\right)$$

How large can k be?

Recall that for k-way merging, we must ensure

$$M \geq (k+1)B \Rightarrow k \leq \frac{M}{B} - 1$$

Cache-Aware $\left(\frac{M}{B}-1\right)$-Way Merge Sort

I/O Complexity: $Q(n) = \begin{cases} O\left(1+\frac{n}{B}\right), & if\ n \le M, \\ k \cdot Q\left(\frac{n}{k}\right) + O\left(k+\frac{n}{B}\right), & otherwise. \end{cases}$

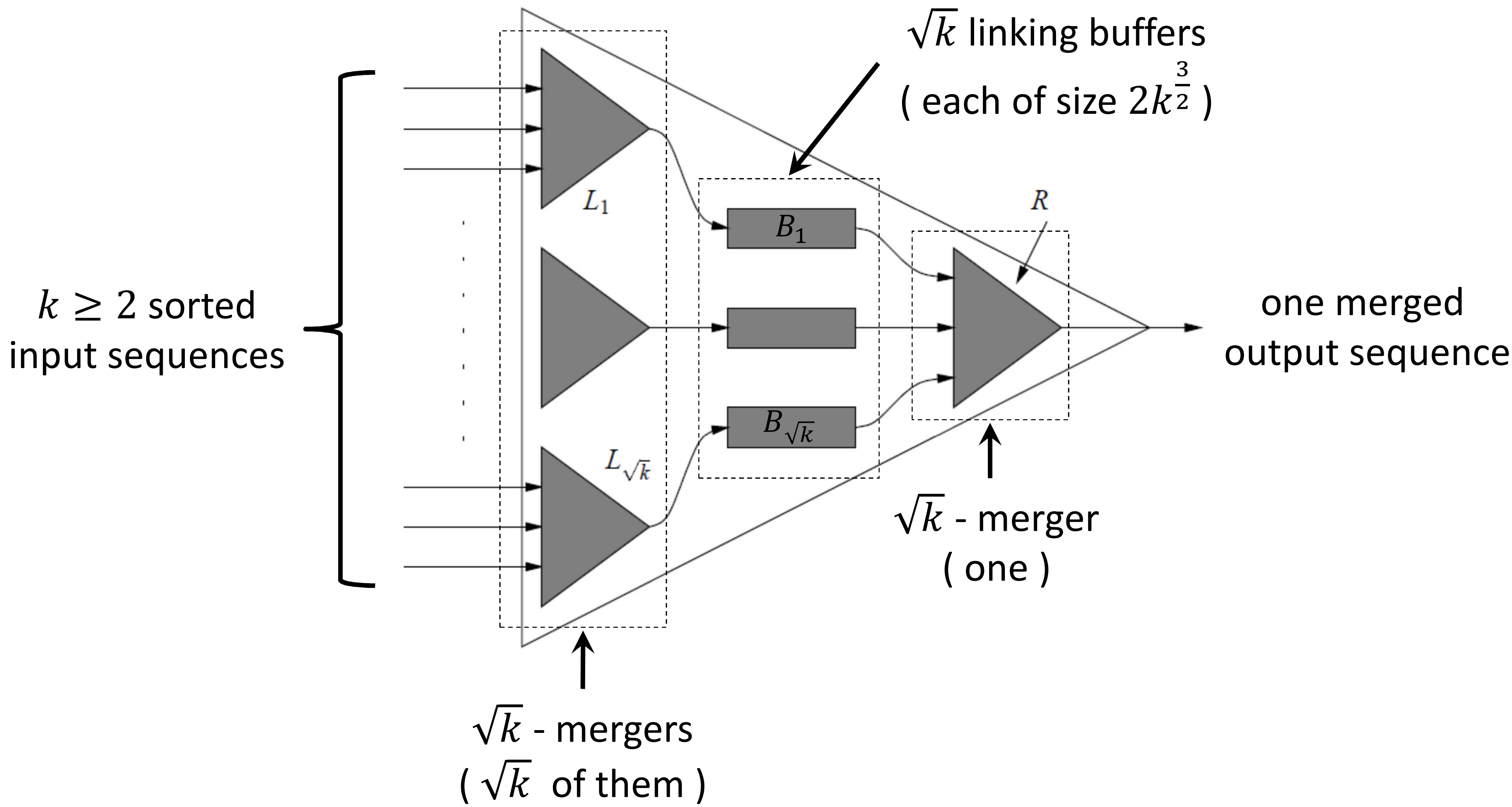
$$= O\left(k \cdot \frac{n}{M} + \frac{n}{B}\log_k \frac{n}{M}\right)$$

Using $k = \frac{M}{B} - 1$, we get:

$$Q(n) = O\left(\left(\frac{M}{B}-1\right)\frac{n}{M} + \frac{n}{B}\log_{\frac{M}{B}}\left(\frac{n}{M}\right)\right) = O\left(\frac{n}{B}\log_{\frac{M}{B}}\left(\frac{n}{M}\right)\right)$$

Sorting
(Funnelsort)

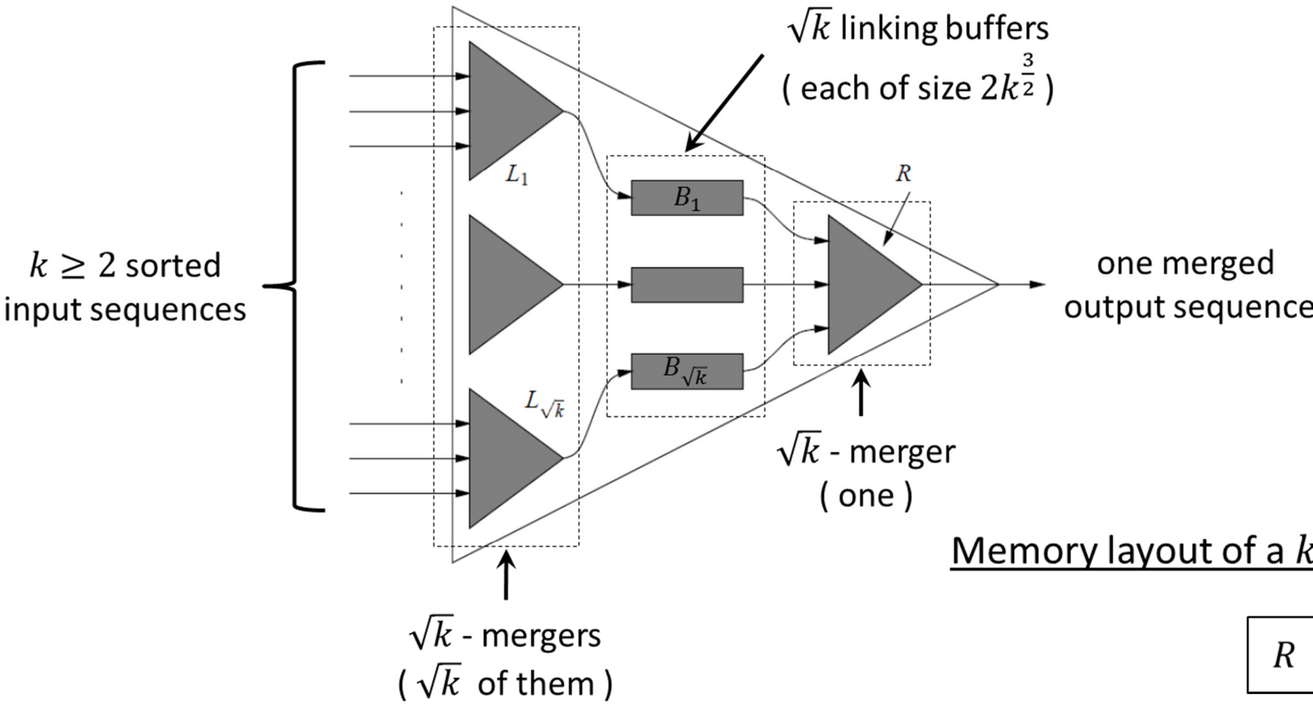
k-Merger (*k*-Funnel)



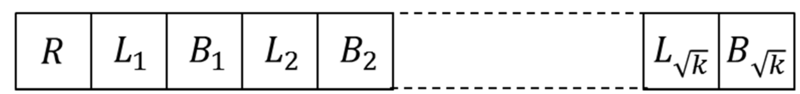
Memory layout of a k-merger:

R	L_1	B_1	L_2	B_2		$L_{\sqrt{k}}$	$B_{\sqrt{k}}$

k-Merger (k-Funnel)



Memory layout of a k-merger:

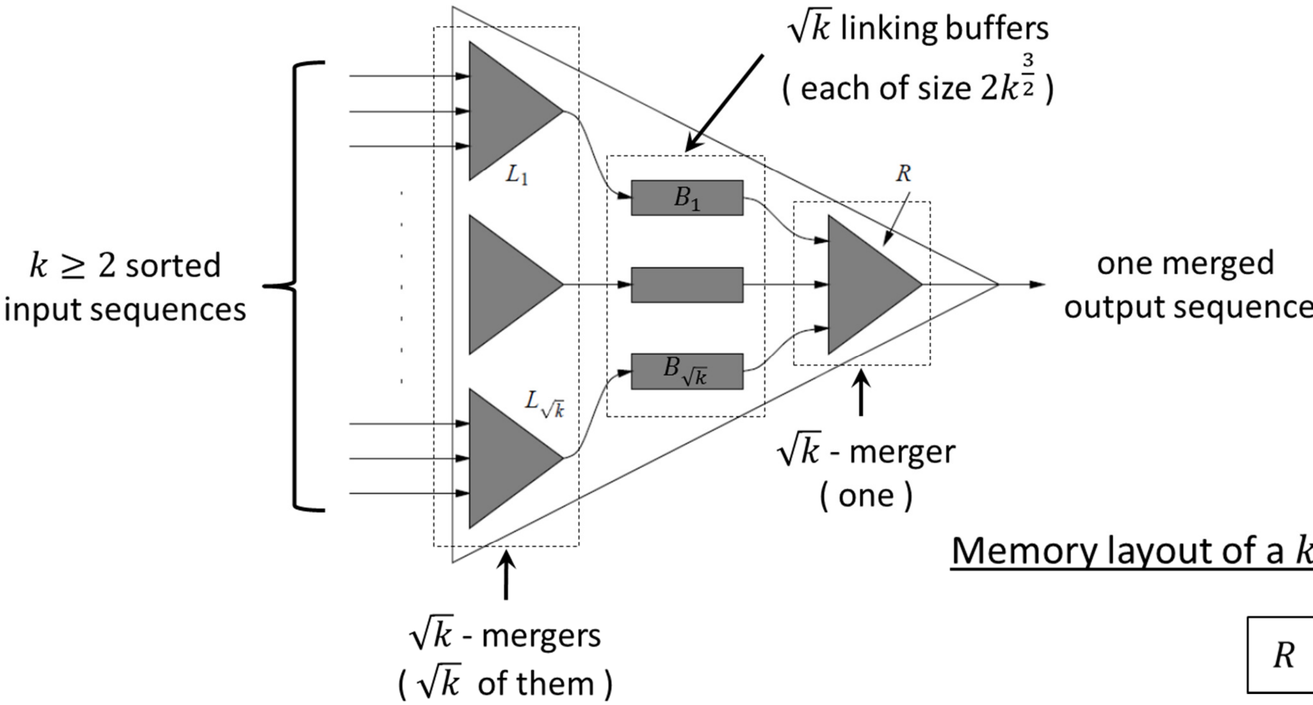


Space usage of a k-merger: $S(k) = \begin{cases} \Theta(1), & if\ k \leq 2, \\ (\sqrt{k}+1)S(\sqrt{k}) + \Theta(k^2), & otherwise. \end{cases}$

$$= \Theta(k^2)$$

A k-merger occupies $\Theta(k^2)$ contiguous locations.

k-Merger (*k*-Funnel)



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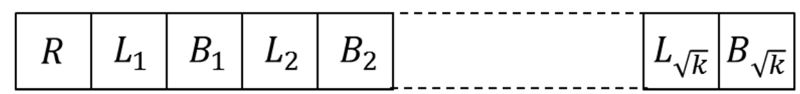
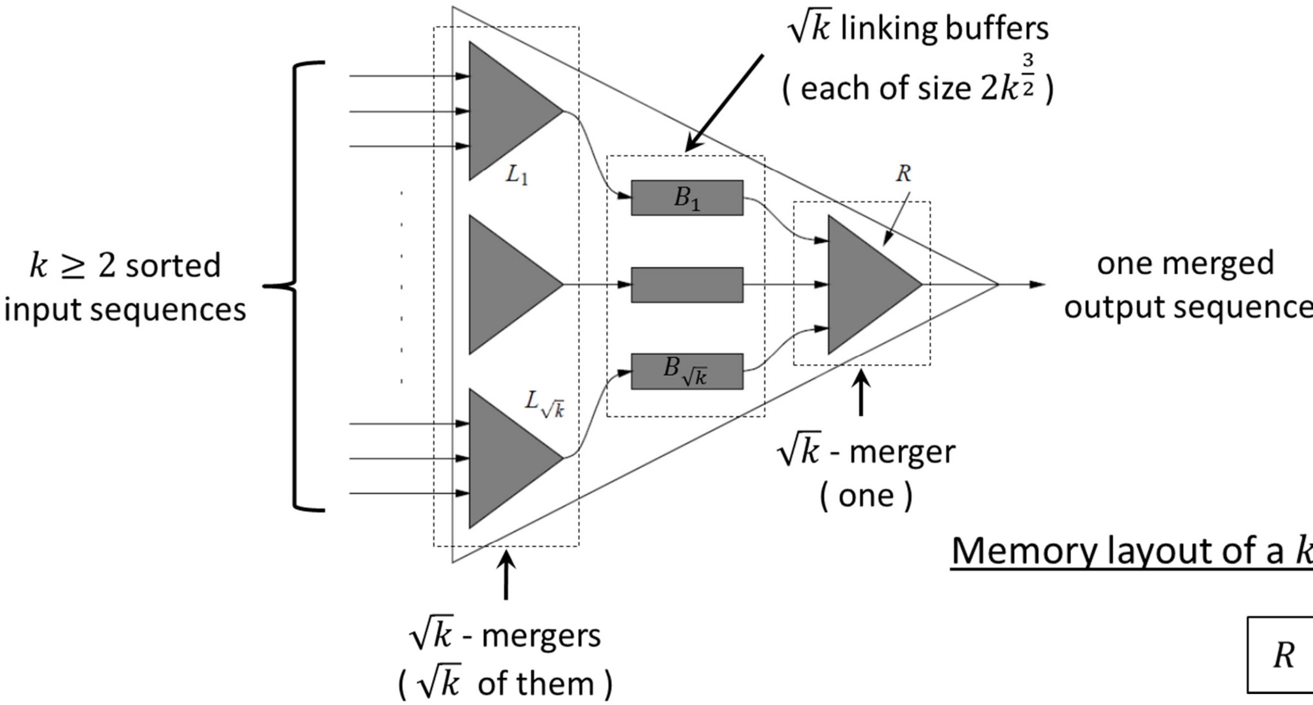


Each invocation of a k-merger

– produces a sorted sequence of length k^3

– incurs $\mathrm{O}\left(1 + k + \frac{k^3}{B} + \frac{k^3}{B}\log_M\left(\frac{k}{B}\right)\right)$ cache misses provided $M = \Omega(B^2)$

k-Merger (*k*-Funnel)



Memory layout of a k-merger:

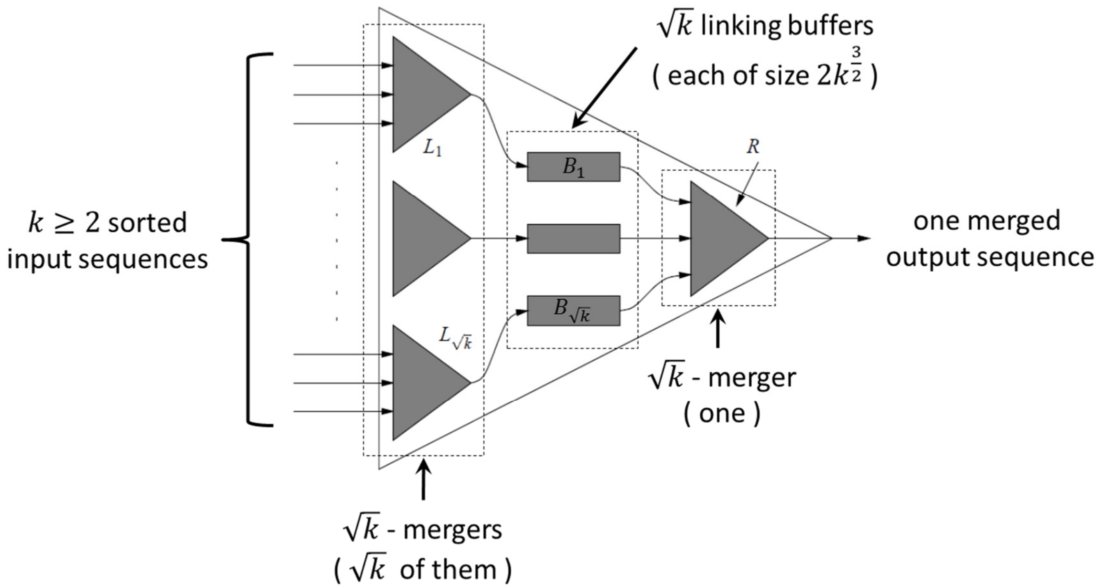
R	L_1	B_1	L_2	B_2	…	$L_{\sqrt{k}}$	$B_{\sqrt{k}}$

Cache-complexity:

$$Q'(k) = \begin{cases} O\left(1 + k + \frac{k^3}{B}\right), & if\ k < \alpha\sqrt{M}, \\ \left(2k^{\frac{3}{2}} + 2\sqrt{k}\right)Q'(\sqrt{k}) + \Theta(k^2), & otherwise. \end{cases}$$

$$= O\left(\frac{k^3}{B}\log_M\left(\frac{k}{B}\right)\right), \qquad \text{provided } M = \Omega(B^2)$$

k-Merger (k-Funnel)



Memory layout of a k-merger:

R	L_1	B_1	L_2	B_2		$L_{\sqrt{k}}$	$B_{\sqrt{k}}$

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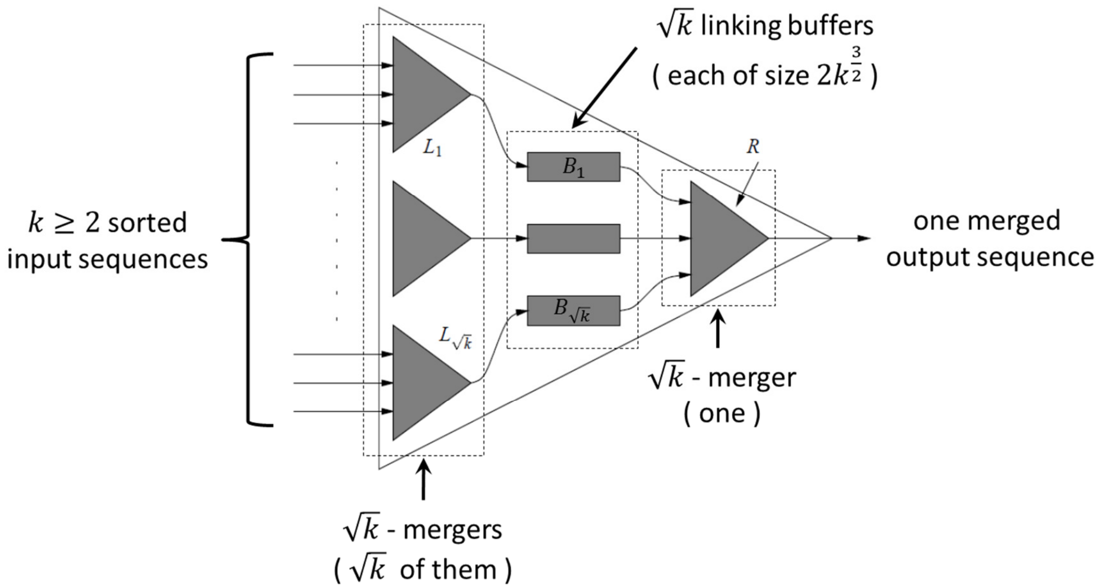
$$= O\left(\frac{k^3}{B}\log_M\left(\frac{k}{B}\right)\right), \quad \text{provided } M = \Omega(B^2)$$

$$k < \alpha\sqrt{M}:\ Q'(k) = O\left(1 + k + \frac{k^3}{B}\right)$$

- Let r_i be #items extracted the i-th input queue. Then $\sum_{i=1}^{k} r_i = O(k^3)$.
- Since $k < \alpha\sqrt{M}$ and $M = \Omega(B^2)$, at least $\frac{M}{B} = \Omega(k)$ cache blocks are available for the input buffers.
- Hence, #cache-misses for accessing the input queues (assuming circular buffers) $= \sum_{i=1}^{k} O\left(1 + \frac{r_i}{B}\right) = O\left(k + \frac{k^3}{B}\right)$

k-Merger (k-Funnel)



Memory layout of a k-merger:

R	L_1	B_1	L_2	B_2	...	$L_{\sqrt{k}}$	$B_{\sqrt{k}}$

Cache-complexity:

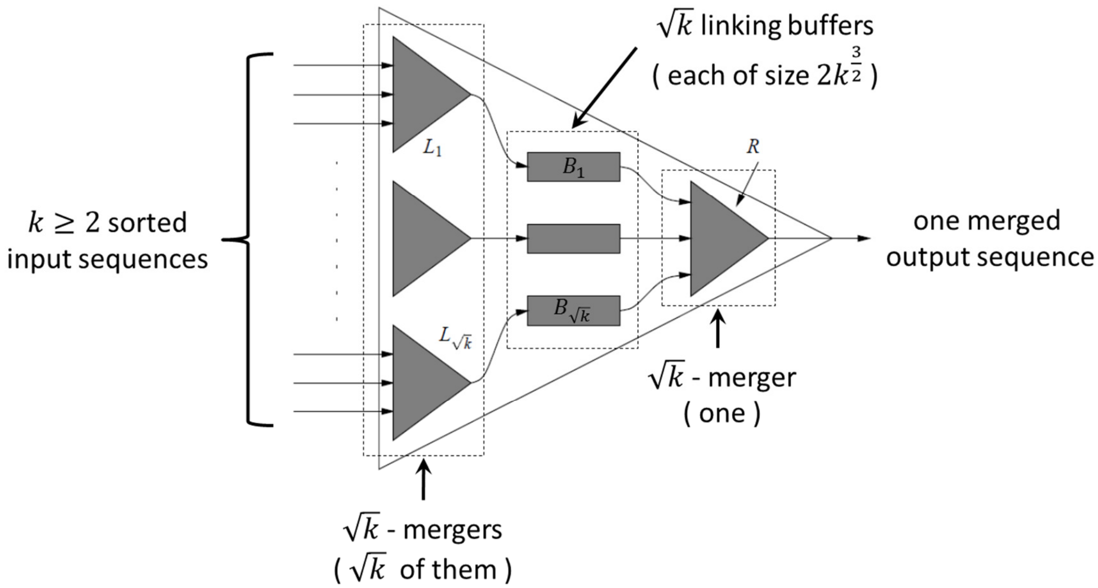
$$Q'(k) = \begin{cases} O\left(1 + k + \frac{k^3}{B}\right), & if\ k < \alpha\sqrt{M}, \\ \left(2k^{\frac{3}{2}} + 2\sqrt{k}\right)Q'(\sqrt{k}) + \Theta(k^2), & otherwise. \end{cases}$$

$$= O\left(\frac{k^3}{B}\log_M\left(\frac{k}{B}\right)\right), \quad \text{provided } M = \Omega(B^2)$$

$$k < \alpha\sqrt{M}\text{: } Q'(k) = O\left(1 + k + \frac{k^3}{B}\right)$$

- #cache-misses for accessing the input queues $= O\left(k + \frac{k^3}{B}\right)$
- #cache-misses for writing the output queue $= O\left(1 + \frac{k^3}{B}\right)$
- #cache-misses for touching the internal data structures $= O\left(1 + \frac{k^2}{B}\right)$
- Hence, total #cache-misses $= O\left(1 + k + \frac{k^3}{B}\right)$

k-Merger (k-Funnel)



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Cache-complexity:

$$Q'(k) = \begin{cases} O\left(1 + k + \frac{k^3}{B}\right), & if\ k < \alpha\sqrt{M}, \\ \left(2k^{\frac{3}{2}} + 2\sqrt{k}\right)Q'\left(\sqrt{k}\right) + \Theta(k^2), & otherwise. \end{cases}$$

$$= O\left(\frac{k^3}{B}\log_M\left(\frac{k}{B}\right)\right), \quad \text{provided } M = \Omega(B^2)$$

$$k \geq \alpha\sqrt{M}:\ Q'^{(k)} = \left(2k^{\frac{3}{2}} + 2\sqrt{k}\right)Q'\left(\sqrt{k}\right) + \Theta(k^2)$$

- Each call to R outputs $k^{\frac{3}{2}}$ items. So, #times merger R is called $= \frac{k^3}{k^{\frac{3}{2}}} = k^{\frac{3}{2}}$
- Each call to an L_i puts $k^{\frac{3}{2}}$ items into B_i. Since k^3 items are output, and the buffer space is $\sqrt{k} \times 2k^{\frac{3}{2}} = 2k^2$, #times the L_i's are called $\leq k^{\frac{3}{2}} + 2\sqrt{k}$
- Before each call to R, the merger must check each L_i for emptiness, and thus incurring $O\left(\sqrt{k}\right)$ cache-misses. So, #such cache-misses $= k^{\frac{3}{2}} \times O\left(\sqrt{k}\right) = O(k^2)$

Funnelsort

- Split the input sequence A of length n into $n^{\frac{1}{3}}$ contiguous subsequences $A_1, A_2, \ldots, A_{n^{\frac{1}{3}}}$ of length $n^{\frac{2}{3}}$ each
- Recursively sort each subsequence
- Merge the $n^{\frac{1}{3}}$ sorted subsequences using a $n^{\frac{1}{3}}$-merger

Cache-complexity:

$$Q(n) = \begin{cases} \mathrm{O}\left(1 + \frac{n}{B}\right), & \textit{if } n \le M, \\ n^{\frac{1}{3}} Q\left(n^{\frac{2}{3}}\right) + Q'\left(n^{\frac{1}{3}}\right), & \textit{otherwise}. \end{cases}$$

$$= \begin{cases} \mathrm{O}\left(1 + \frac{n}{B}\right), & \textit{if } n \le M, \\ n^{\frac{1}{3}} Q\left(n^{\frac{2}{3}}\right) + \mathrm{O}\left(\frac{n}{B} \log_M \left(\frac{n}{B}\right)\right), & \textit{otherwise}. \end{cases}$$

$$= \mathrm{O}\left(1 + \frac{n}{B} \log_M n\right)$$