CSE 548: Analysis of Algorithms

Lecture 2
(Divide-and-Conquer Algorithms: Integer Multiplication)

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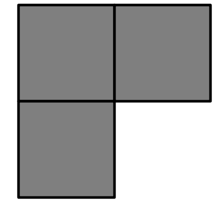
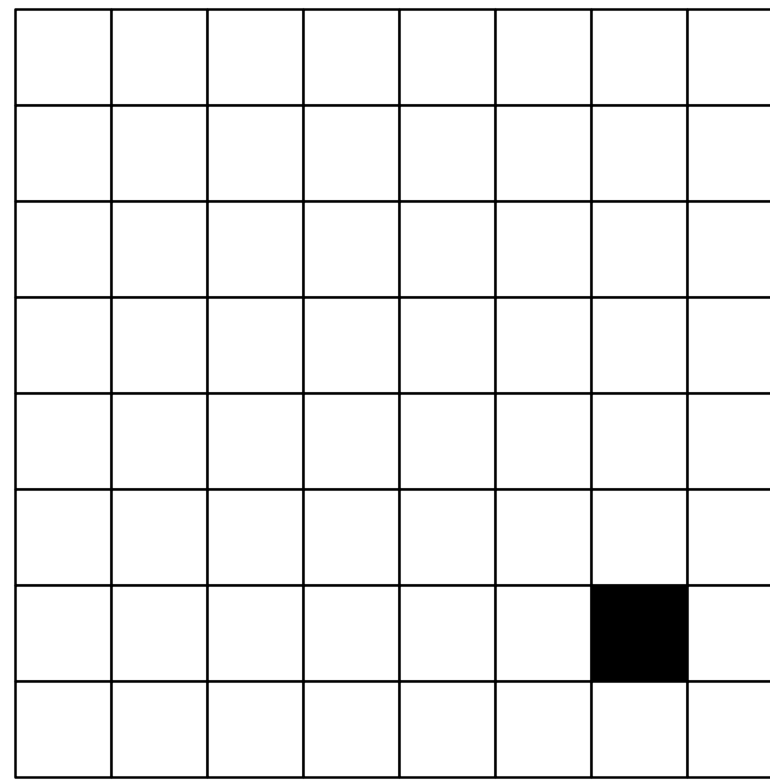
Tromino Cover

A *right tromino* is an L-shaped tile formed by three adjacent squares.



Puzzle: You are given a $2^n \times 2^n$ board with one missing square.

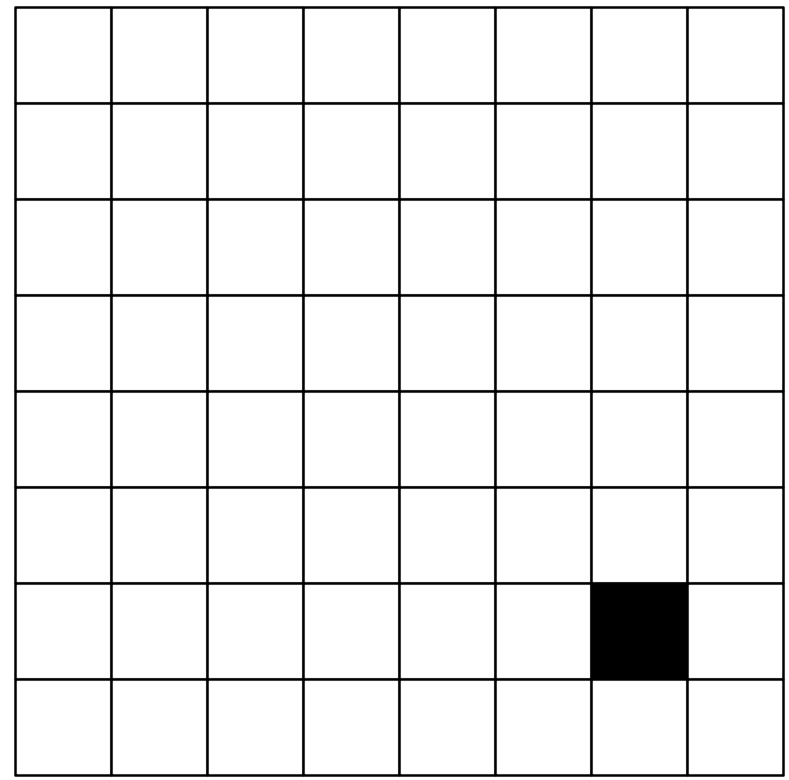
- you must cover all squares except the missing one exactly using right trominoes
- the trominoes must not overlap



$2^3 \times 2^3$ board

Tromino Cover

Steps

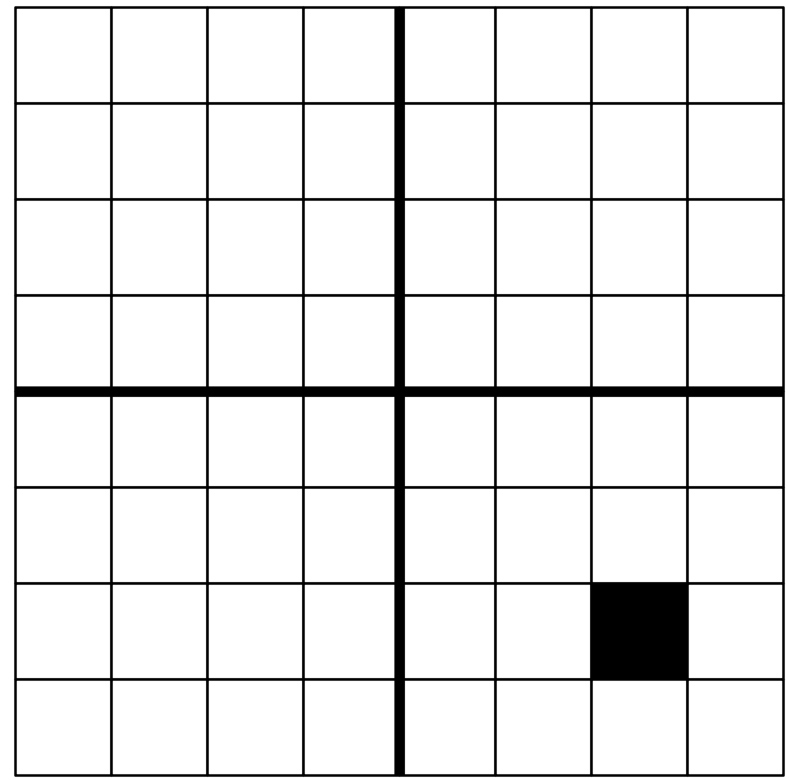


$2^3 \times 2^3$ board

Tromino Cover

Steps

– Divide the $2^n \times 2^n$ board into 4 disjoint $2^{n-1} \times 2^{n-1}$ subboards.

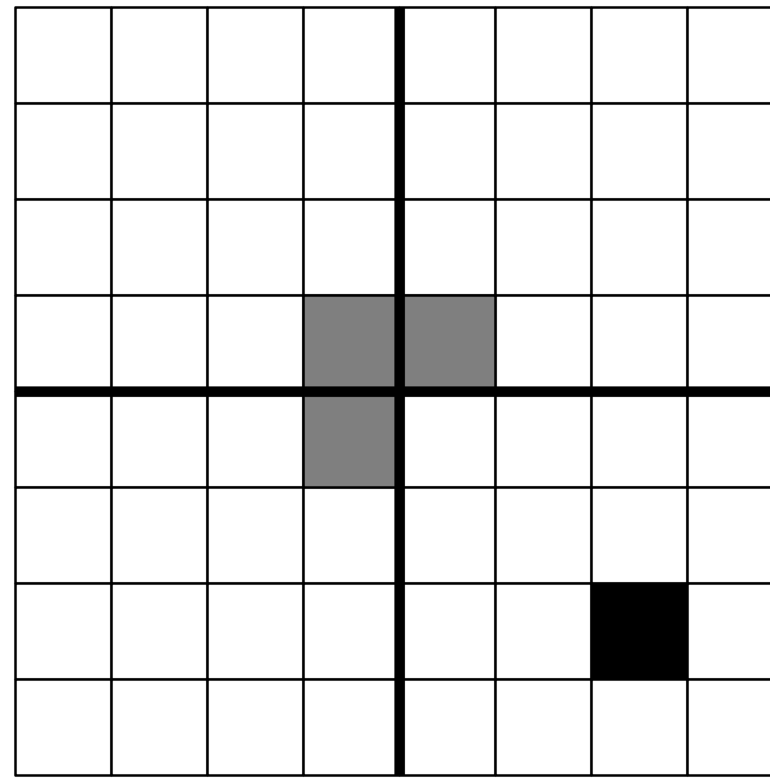


$2^3 \times 2^3$ board

Tromino Cover

Steps

- Divide the $2^n \times 2^n$ board into 4 disjoint $2^{n-1} \times 2^{n-1}$ subboards.
- Place a tromino at the center so that it fully covers one square from each of the three (3) subboards with no missing square, and misses the fourth subboard completely.



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Tromino Cover

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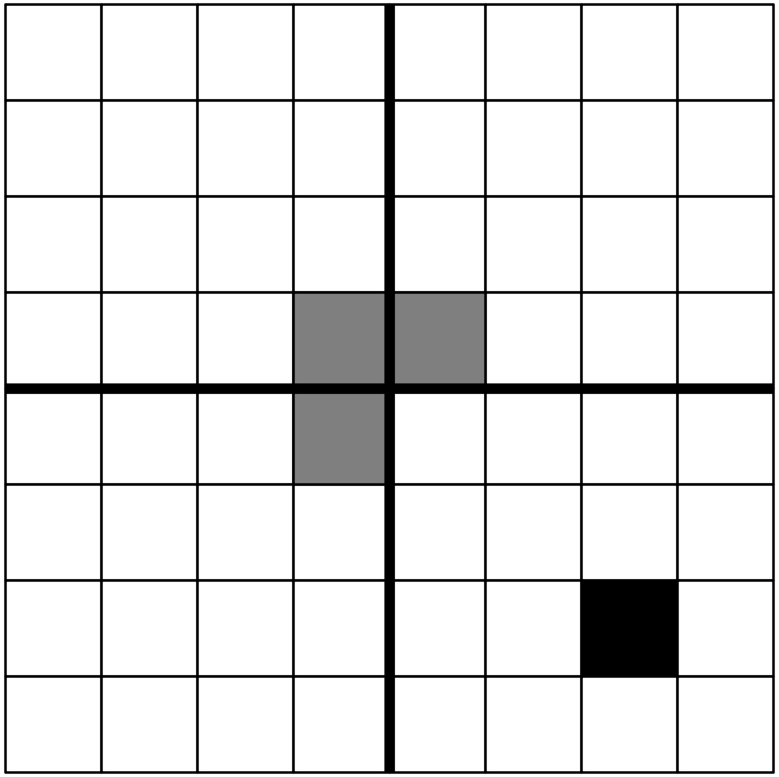
This reduces the original problem into 4 smaller instances of the same problem!



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Tromino Cover

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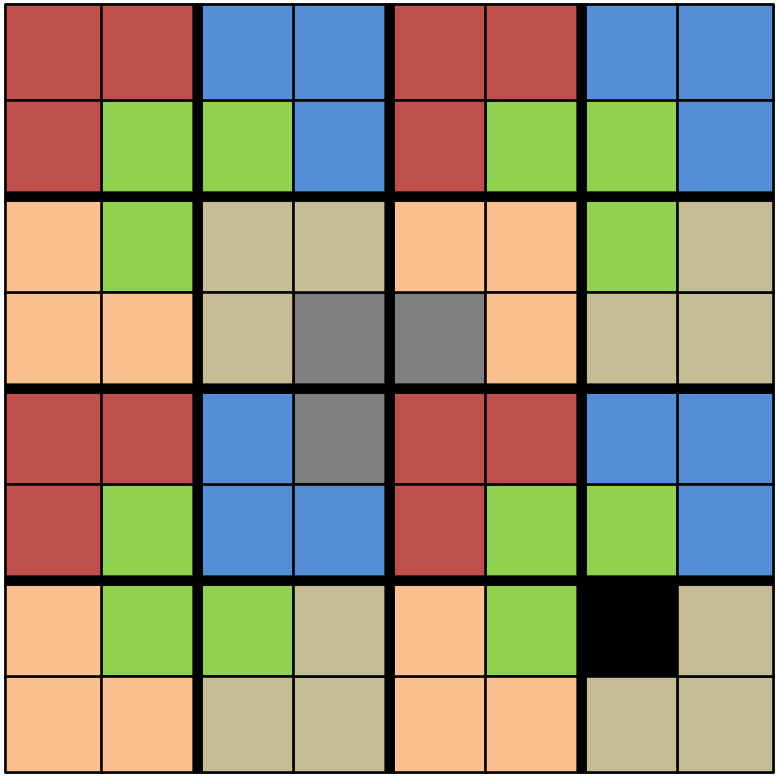
  This reduces the original problem into 4 smaller instances of the same problem!
- Solve each smaller subproblem recursively using the same technique.



$2^3 \times 2^3$ board

Tromino Cover

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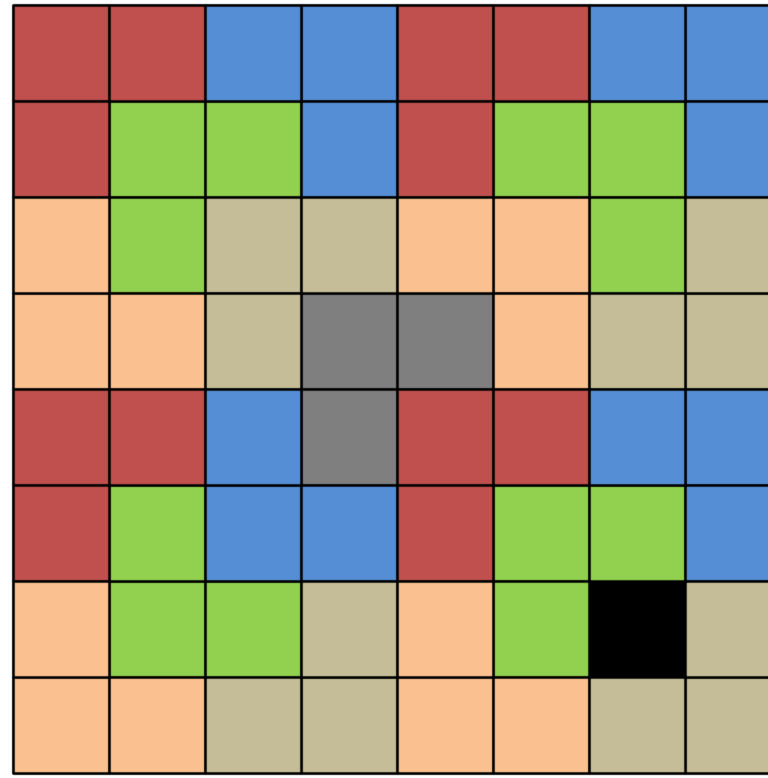


$2^3 \times 2^3$ board

– Solve each smaller subproblem recursively using the same technique.

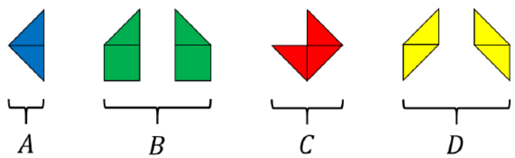
– This algorithm design technique is called *recursive divide & conquer.*

An Old Homework Problem

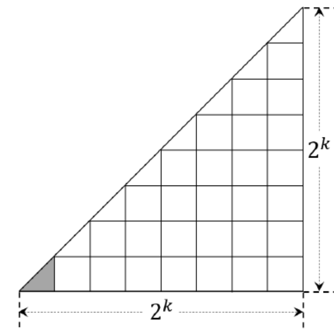
CSE548, AMS542: Analysis of Algorithms, Spring 2015 — **Date:** Feb 16

Homework #1

(Due: Mar 5)



(a) Four types of tiles



(b) A right triangular grid to tile ($k = 3$)

Figure 1: Tiling a right triangular grid.

Task 1. [70 Points] Tiling a Triangular Grid

Given an isosceles right triangular grid for some $k \geq 2$ as shown in Figure 1(b), this problem asks you to completely cover it using the tiles given in Figure 1(a). The bottom-left corner of the grid must not be covered. No two tiles can overlap and all tiles must remain completely inside the given triangular grid. You must use all four types of tiles shown in Figure 1(a), and no tile type can be used to cover more than 40% of the total grid area. You are allowed to rotate the tiles, as needed, before putting them on the grid.

(a) [**25 Points**] Design and explain a recursive divide-and-conquer algorithm for tiling the grid under the constraints given above. Include pseudocode.

(b) [**25 Points**] Write down recurrences describing the running time of your algorithm from part (a), and solve them.

(c) [**20 Points**] Write down recurrences for counting the number of tiles of each type used by your algorithm, and solve them to show that no tile type covers more than 40% of the total grid area.

A Latin Phrase

"Divide et impera"

(meaning: "divide and rule" or "divide and conquer")

— Philip II, king of Macedon (382-336 BC),
describing his policy toward the Greek city-states
(some say the Roman emperor Julius Caesar,
100-44 BC, is the source of this phrase)

The strategy is to break large power alliances into smaller ones that are easier to manage (or subdue).

This is a combination of political, military and economic strategy of gaining and maintaining power.

Unsurprisingly, this is also a very powerful problem solving strategy in computer science.

Divide-and-Conquer

1. **Divide:** divide the original problem into smaller subproblems that are easier are to solve
2. **Conquer:** solve the smaller subproblems (perhaps recursively)
3. **Merge:** combine the solutions to the smaller subproblems to obtain a solution for the original problem

Integer Multiplication

Multiplying Two n-bit Numbers

$$\begin{array}{rcc|cl} & \overbrace{}^{\frac{n}{2}\,bits} & & \overbrace{}^{\frac{n}{2}\,bits} & \\ x = & \boxed{\quad x_L \quad} & & \boxed{\quad x_R \quad} & = 2^{n/2}x_L + x_R \\ y = & \boxed{\quad y_L \quad} & & \boxed{\quad y_R \quad} & = 2^{n/2}y_L + y_R \end{array}$$

$$\underbrace{}_{n\ bits}$$

$$xy = \left(2^{n/2}x_L + x_R\right)\left(2^{n/2}y_L + y_R\right) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R$$

So # $\frac{n}{2}$-bit products: 4

bit shifts (by n or $\frac{n}{2}$ bits): 2

additions (at most $2n$ bits long) : 3

We can compute the $\frac{n}{2}$-bit products recursively.

Let $T(n)$ be the overall running time for n-bit inputs. Then

$$T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 4T\left(\frac{n}{2}\right) + \Theta(n) & otherwise. \end{cases} = \Theta\left(n^2\right) \quad \text{(how? derive)}$$

Multiplying Two n-bit Numbers Faster (Karatsuba's Algorithm)

$$
\begin{array}{rcl}
 & \overbrace{\hspace{5em}}^{\frac{n}{2}\,bits}\overbrace{\hspace{5em}}^{\frac{n}{2}\,bits} & \\
x = & \boxed{\quad x_L \quad}\boxed{\quad x_R \quad} & = 2^{n/2}x_L + x_R \\
y = & \boxed{\quad y_L \quad}\boxed{\quad y_R \quad} & = 2^{n/2}y_L + y_R \\
 & \underbrace{\hspace{10em}}_{n\,bits} &
\end{array}
$$

$$
\begin{aligned}
xy &= \left(2^{n/2}x_L + x_R\right)\left(2^{n/2}y_L + y_R\right) \\
&= 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R \\
&= 2^n x_L y_L + 2^{n/2}\left((x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R\right) + x_R y_R
\end{aligned}
$$

So # $\frac{n}{2}$- or $\left(\frac{n}{2} + 1\right)$ -bit products: 3

Then the overall running time for n-bit inputs:

$$
T(n) = \begin{cases} \Theta(1) & if\ n = 1, \\ 3T\left(\dfrac{n}{2}\right) + \Theta(n) & otherwise. \end{cases}
$$

$= \Theta\left(n^{\log_2 3}\right) = O\left(n^{1.59}\right)$(how? derive)

Algorithms for Multiplying Two *n*-bit Numbers

Inventor	Year	Complexity
Classical	—	$\Theta(n^2)$
Anatolii Karatsuba	1960	$\Theta(n^{\log_2 3})$
Andrei Toom & Stephen Cook (generalization of Karatsuba's algorithm)	1963 – 66	$\Theta\left(n 2^{\sqrt{2 \log_2 n}} \log n\right)$
Arnold Schönhage & Volker Strassen (Fast Fourier Transform)	1971	$\Theta(n \log n \log \log n)$
Martin Fürer (Fast Fourier Transform)	2005	$n \log n \, 2^{O(\log^* n)}$

Lower bound: $\Omega(n)$ (why?)