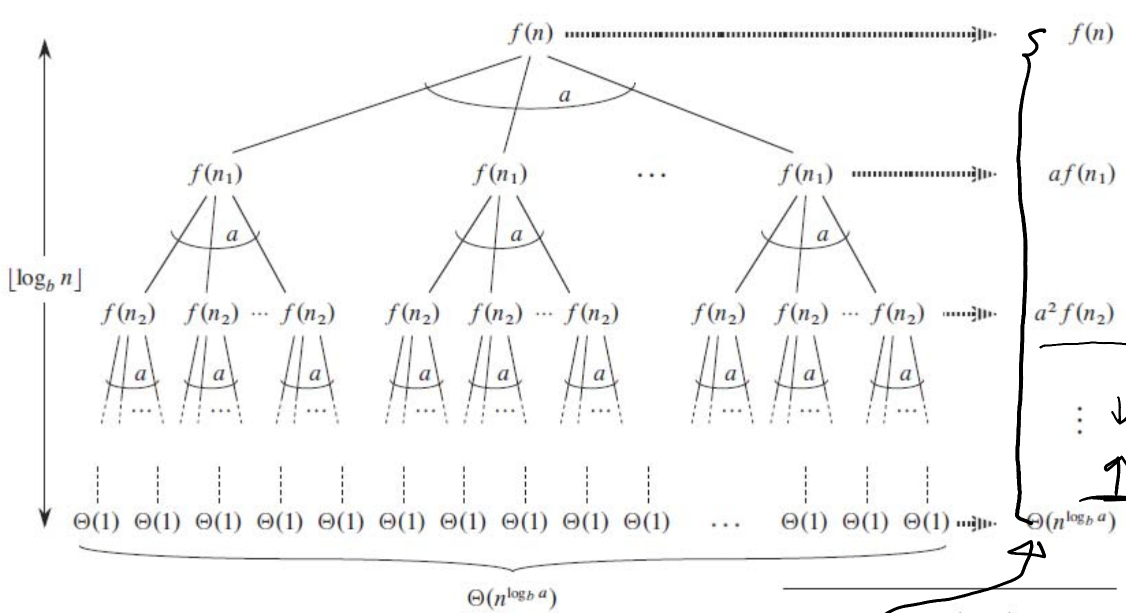
Master Theorem Cont.

2017년 9월 11일 월요일
오후 2:39

Recursion Tree for Master Theorem :
T(n) = aT(n/b) + f(n)



$n_k = n/b^k$

$$\text{Total: } \Theta(n^{\log_b a}) + \sum_{j=0}^{\lfloor \log_b n \rfloor - 1} a^j f(n_j)$$

Lemma 4.3

Let $a \geq 1$ and $b > 1$ be constants, and let $f(n)$ be a nonnegative function defined on exact powers of b. A function $g(n)$ defined over exact powers of b by

$$g(n) = \sum_{j=0}^{\log_b n - 1} a^j f(n/b^j) \tag{4.22}$$

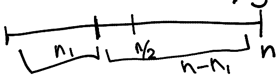
has the following asymptotic bounds for exact powers of b:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $\boxed{g(n) = O(n^{\log_b a})}$.
2. If $f(n) = \Theta(n^{\log_b a})$, then $g(n) = \Theta(n^{\log_b a} \lg n)$.
3. If $af(n/b) \leq cf(n)$ for some constant $c < 1$ and for all sufficiently large n, then $g(n) = \Theta(f(n))$.

1st part dominate

$$\begin{cases} T(n) \le c & \text{if } n = 1 \\ T(n) \le cn + T(n_1) + T(n-n_1) & \text{if } n > 1 \end{cases}$$

where $\lceil n/3 \rceil \le n_1 \le \lceil 2/3\, n \rceil$



Claim : $T(n) \le an\lg n + bn$. $\begin{cases} a = \dfrac{3c}{\lg(\frac{27}{4})} \\ b = c \end{cases}$

proof by Induction on n

Basis ($n=1$)

$$T(1) \le a\cdot 1 \cdot \underbrace{\lg 1}_{=0} + b\cdot 1 = b \;(= c)$$

Induction step ($n > 1$)

Assume by IH that claim holds for all $n' < n$

$$\begin{aligned} T(n) &\le cn + T(n_1) + T(n-n_1) \\ &\le cn + (an_1\lg n_1 + bn_1) + (a(n-n_1)\lg(n-n_1) + b(n-n_1)) \\ &= a[n_1\lg n_1 + (n-n_1)\lg(n-n_1)] + \cancel{bn_1} + bn - \cancel{bn_1} + cn \\ &\qquad (c+b)n \end{aligned}$$

assume $n_1 = \frac{1}{3}n$

$$\begin{aligned} &\le a\left[\frac{n}{3}\lg\frac{n}{3} + \left(\frac{2}{3}n\right)\lg\left(\frac{2}{3}n\right)\right] + (c+b)n \\ &= \frac{an}{3}\lg n - \frac{an}{3} + \frac{2}{3}an\lg n) - \frac{2}{3}an\lg\frac{3}{2} + (c+b)n \\ &= an\lg n + n\left\{\underbrace{\left[-\frac{a}{3}\lg\frac{3\times 9}{4} + b + c\right]}_{=b}\right\} \\ &= an\lg n + bn \end{aligned}$$

$$-\frac{a}{3}\lg\frac{27}{4} + \cancel{b} + c = \cancel{b}$$

$$a = \frac{3c}{\lg(\frac{27}{4})}$$

$\therefore O(n\lg n)$

Theorem (Case 1)

let c and $0 < \alpha \leq 1/2$ be const.

let $T(n) \leq c$ if $n = 1$

$T(n) \leq cn + T(n_1) + T(n - n_1)$

where $\lceil \alpha n \rceil \leq n_1 \leq \lceil (1-\alpha) n \rceil$

then $T(n)$ is $O(n \lg n)$

Claim the previous proof to prove the theorem